Second Order Linear Equations

$$\boxed{a\ddot{x} + b\dot{x} + cx = 0, \quad x(0) = x_0, \quad \dot{x}(0) = y_0}$$

$(x = x(t))$

Characteristic polynomial:

$$ar^2 + br + c = 0$$

$$\boxed{r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Second Order Linear Equations

$$\boxed{a\ddot{x} + b\dot{x} + cx = 0, \quad x(0) = x_0, \quad \dot{x}(0) = y_0}$$

$(x = x(t))$

(1) $b^2 - 4ac > 0 \implies$ 2 real roots

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$\boxed{x(t) = Ae^{r_1 t} + Be^{r_2 t}}$$

(A, B determined by x_0, y_0)

Second Order Linear Equations

$$\boxed{a\ddot{x} + b\dot{x} + cx = 0, \quad x(0) = x_0, \quad \dot{x}(0) = y_0}$$

$(x = x(t))$

(2) $b^2 - 4ac = 0 \implies$ 2 repeated real roots

$$r_1 = r_2 = \frac{-b}{2a} \quad (= r)$$

$$\boxed{x(t) = Ae^{rt} + B\underline{t}e^{rt}}$$

(A, B determined by x_0, y_0)

Second Order Linear Equations

$$\boxed{a\ddot{x} + b\dot{x} + cx = 0, \quad x(0) = x_0, \quad \dot{x}(0) = y_0}$$

$(x = x(t))$

(3) $b^2 - 4ac < 0 \implies$ 2 complex roots

$r_1 = \alpha + i\beta, \quad r_2 = \alpha - \beta i, \quad \alpha = \dfrac{-b}{2a}, \quad \beta = \dfrac{\sqrt{4ac - b^2}}{2a}$

$$\boxed{x(t) = Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t}$$

(A, B determined by x_0, y_0)

Second Order Linear Equations

$$\boxed{a\ddot{x} + b\dot{x} + cx = 0, \quad x(0) = x_0, \quad \dot{x}(0) = y_0}$$

$$(x = x(t))$$

In all cases,

$$\boxed{x(t) = A\varphi_1(t) + B\varphi_2(t)}$$

2 linearly independent (homogeneous) solutions

$$a\ddot{\varphi}_i + b\dot{\varphi}_i + c\varphi_i = 0, \quad i = 1,2$$

Inhomogeneous Version

$$\boxed{a\ddot{x}+b\dot{x}+cx=f(t),\qquad x(0)=x_0,\quad \dot{x}(0)=y_0}$$

"Variational of Parameters"

Set $x(t)=\underline{\overset{=?}{u_1(t)}}\,\varphi_1(t)+\underline{\overset{=?}{u_2(t)}}\,\varphi_2(t)$

$$(*)\left\{\begin{aligned}
&a\left(\ddot{u}_1\varphi_1+2\dot{u}_1\dot{\varphi}_1+\cancel{u_1\ddot{\varphi}_1}+\ddot{u}_2\varphi_2+2\dot{u}_2\dot{\varphi}_2+\cancel{u_2\ddot{\varphi}_2}\right)\\
&+b\left(\underline{\dot{u}_1\varphi_1}+\cancel{u_1\dot{\varphi}_1}+\underline{\dot{u}_2\varphi_2}+\cancel{u_2\dot{\varphi}_2}\right)\\
&+c\left(\cancel{u_1\varphi_1}+\cancel{u_2\varphi_2}\right)\qquad\qquad = f
\end{aligned}\right.$$

Inhomogeneous Version

$$\boxed{a\ddot{x} + b\dot{x} + cx = f(t), \quad x(0) = x_0, \quad \dot{x}(0) = y_0}$$

"Variational of Parameters"

Set $\quad \dot{u}_1 \varphi_1 + \dot{u}_2 \varphi_2 = 0$ $\longleftarrow$ Eqn ①

$\dot{①} \Longrightarrow \ddot{u}_1 \varphi_1 + \dot{u}_1 \dot{\varphi}_1 + \ddot{u}_2 \varphi_2 + \dot{u}_2 \dot{\varphi}_2 = 0$

$\Longrightarrow$ (*) becomes

$$\dot{u}_1 \dot{\varphi}_1 + \dot{u}_2 \dot{\varphi}_2 = \frac{f}{a}$$ $\longleftarrow$ Eqn ②

Inhomogeneous Version

$$\boxed{a\ddot{x} + b\dot{x} + cx = f(t), \quad x(0) = x_0, \quad \dot{x}(0) = y_0}$$

"Variational of Parameters"

Eqns 1 & 2 $\Rightarrow$ solve for $\dot{u}_1$, $\dot{u}_2$

$$\dot{u}_1 = \frac{\begin{vmatrix} 0 & \varphi_2 \\ \frac{f}{a} & \dot{\varphi}_2 \end{vmatrix}}{\begin{vmatrix} \varphi_1 & \varphi_2 \\ \dot{\varphi}_1 & \dot{\varphi}_2 \end{vmatrix}} = \frac{-\frac{f}{a}\varphi_2}{\varphi_1\dot{\varphi}_2 - \dot{\varphi}_1\varphi_2}$$

$$\varphi_1\dot{\varphi}_2 - \dot{\varphi}_1\varphi_2 = W(\varphi_1, \varphi_2) \quad \text{Wronskian}$$

$\left(u_1 = \int \dot{u}_1 \, dt + A \right)$

Inhomogeneous Version

$$\boxed{a\ddot{x} + b\dot{x} + cx = f(t), \quad x(0) = x_0, \quad \dot{x}(0) = y_0}$$

"Variational of Parameters"

Eqns ① & ② $\Rightarrow$

$$\dot{u}_2 = \frac{\begin{vmatrix} \varphi_1 & 0 \\ \dot{\varphi}_1 & \frac{f}{a} \end{vmatrix}}{\begin{vmatrix} \varphi_1 & \varphi_2 \\ \dot{\varphi}_1 & \dot{\varphi}_2 \end{vmatrix}} = \frac{\frac{f}{a}\varphi_1}{\varphi_1\dot{\varphi}_2 - \dot{\varphi}_1\varphi_2}$$

$\left(u_2 = \int \dot{u}_2 \, dt + B\right)$

$W(\varphi_1, \varphi_2)$ Wronskian

Inhomogeneous Version

$$\boxed{a\ddot{x} + b\dot{x} + cx = f(t), \quad x(0) = x_0, \quad \dot{x}(0) = y_0}$$

"Variational of Parameters"

Eqns 1 & 2 $\Rightarrow$

$$x(t) = u_1(t)\varphi_1(t) + u_2(t)\varphi_2(t)$$

$$= \left(\int \frac{-\frac{f}{a}\varphi_2}{W(\varphi_1, \varphi_2)}\,dt + A\right)\varphi_1(t) + \left(\int \frac{\frac{f}{a}\varphi_1}{W(\varphi_1, \varphi_2)}\,dt + B\right)\varphi_2(t)$$

Inhomogeneous Version

$$\boxed{a\ddot{x} + b\dot{x} + cx = f(t), \quad x(0) = x_0, \quad \dot{x}(0) = y_0}$$

"Variational of Parameters"

Eqns ① & ② $\Rightarrow$

$$\boxed{x(t) = \left(\int \frac{-\frac{f}{a}\varphi_2}{W(\varphi_1, \varphi_2)}\,dt \right)\varphi_1(t) + \left(\int \frac{\frac{f}{a}\varphi_1}{W(\varphi_1, \varphi_2)}\,dt \right)\varphi_2(t) + A\varphi_1(t) + B\varphi_2(t)}$$

(A, B determined by x_0, y_0)

Euler Equation

$t > 0$

$$\boxed{a t^2 \ddot{x}(t) + b t \dot{x}(t) + c x(t) = 0}$$

$x(t_0) = x_0$

$\dot{x}(t_0) = y_0$

$(t_0 > 0)$

"Characteristic Polynomial"

Let $\underline{x(t) = t^r \quad (r = ?)}$

$$a t^2 r(r-1) t^{r-2} + b t r t^{r-1} + c t^r = 0$$

$$a r(r-1) + b r + c = 0$$

$$\boxed{a r^2 + (b-a) r + c = 0 \Rightarrow r_1, r_2}$$

Euler Equation

$$\boxed{a t^2 \ddot{x}(t) + b t \dot{x}(t) + c x(t) = 0}$$

$x(t_0) = x_0$

$\dot{x}(t_0) = y_0$

$(t_0 > 0)$

"Characteristic Polynomial"

$r_1 \neq r_2$, real, $\quad x(t) = \underline{A t^{r_1} + B t^{r_2}}$

$r_1 = r_2 \ (= r)$, real, $\quad x(t) = \underline{A t^{r} + B (\log t) t^{r}}$

$r_1, r_2 = \alpha \pm i\beta$ (complex) $\quad x(t) = A t^{(\alpha + i\beta)} + B t^{(\alpha - i\beta)}$

$$= \underline{\tilde{A} t^{\alpha} \cos(\beta \ln t) + \tilde{B} t^{\alpha} \sin(\beta \ln t)}$$

Euler Equation

$$\boxed{a t^2 \ddot{x}(t) + b t \dot{x}(t) + c x(t) = 0}$$

$x(t_0) = x_0$

$\dot{x}(t_0) = y_0$

$(t_0 > 0)$

Alternative method: change of time

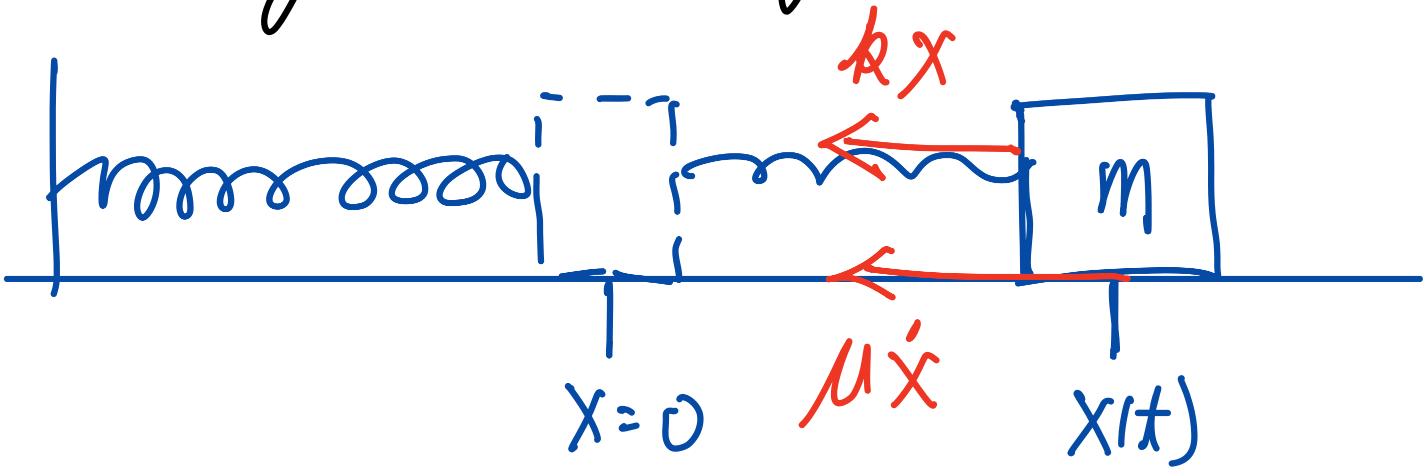
$$t = e^s, \quad s = \log t \Rightarrow \frac{dx}{dt} = \frac{dy}{ds}\frac{ds}{dt} = \frac{1}{t}\frac{dy}{ds} = e^{-s}\frac{dy}{ds}$$

$x(t) = y(s)$

$$a e^{2s} e^{-s} \frac{d}{ds}\left(e^{-s}\frac{d}{ds} y(s)\right) + b e^{s} e^{-s} \frac{d}{ds}\left(y(s)\right) + c y(s) = 0$$

$$\Rightarrow \boxed{a \frac{d^2}{ds^2} y + (b-a)\frac{dy}{ds} + c y = 0}$$

Spring Mass System



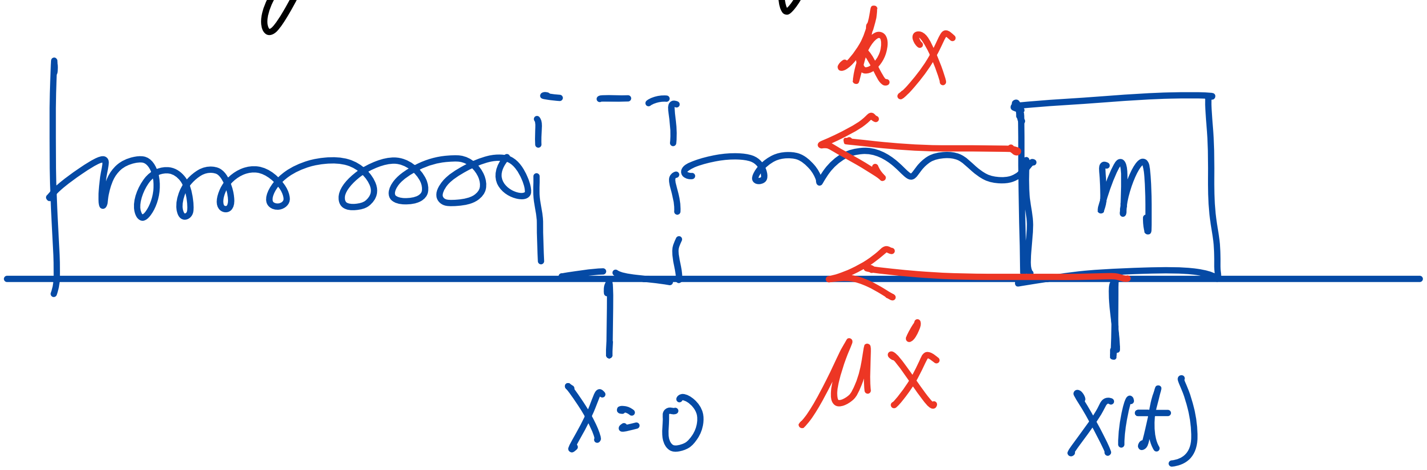
$$m\ddot{x} = \underbrace{-kx}_{\text{Hooke's Law}} - \mu\dot{x}$$

friction

$$m\ddot{x} + \mu\dot{x} + kx = 0$$

Char. poly: $mr^2 + \mu r + k = 0$

Spring Mass System



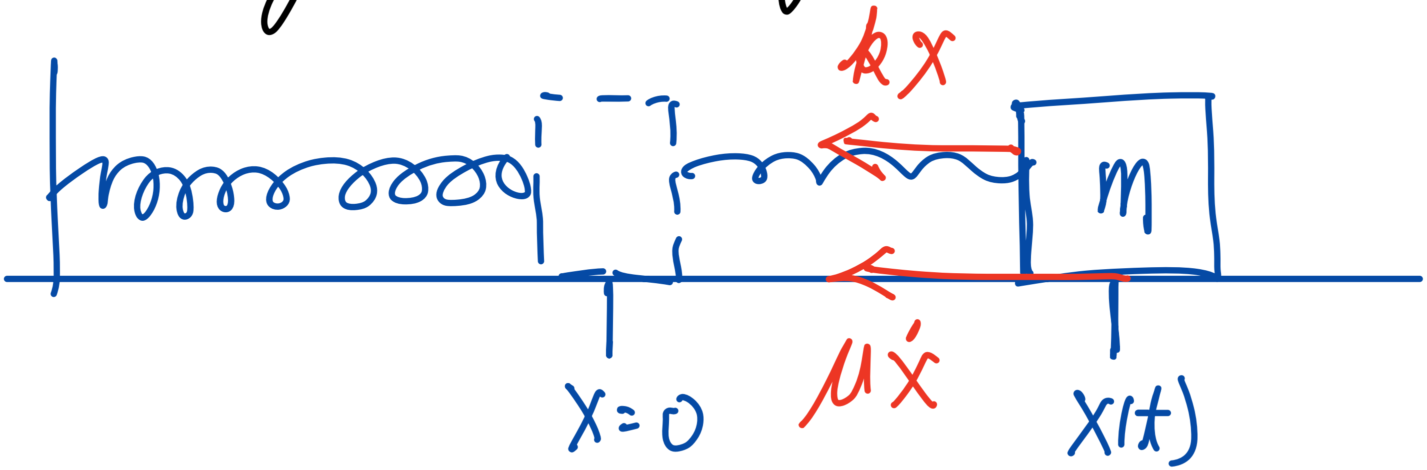
char. poly: $mr^2 + \mu r + k = 0$

$$r_1, r_2 = \frac{-\mu \pm \sqrt{\mu^2 - 4mk}}{2m}$$

1. $\mu^2 > 4mk \Longrightarrow$ overdamped, $r_1, r_2 < 0$

$$x(t) = Ae^{r_1 t} + Be^{r_2 t}$$

Spring Mass System



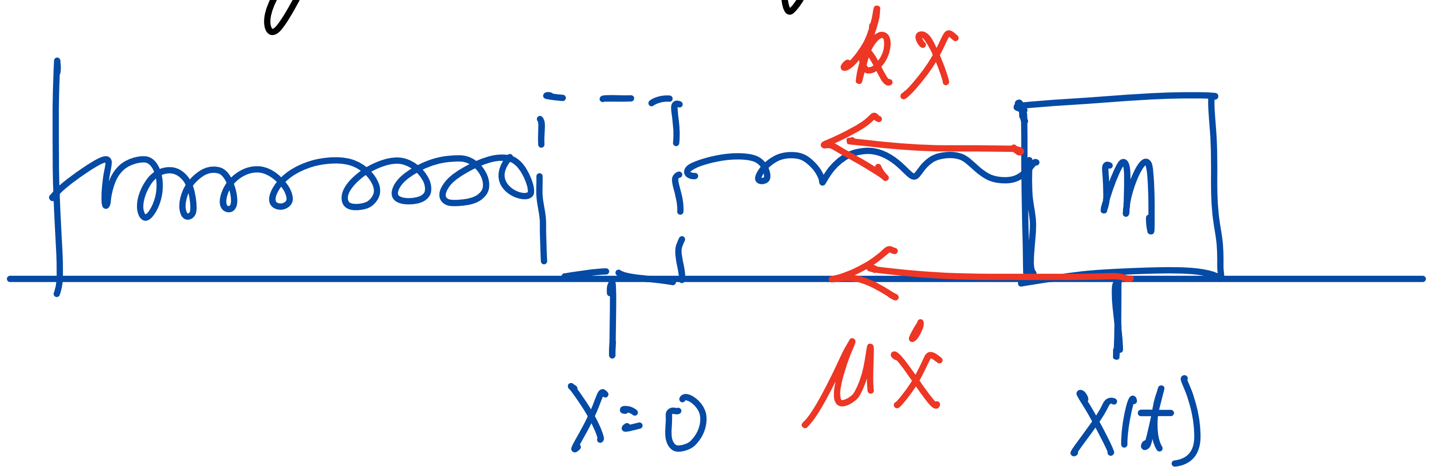
$$m r^2 + \mu r + k = 0$$

$$r_1, r_2 = \frac{-\mu \pm \sqrt{\mu^2 - 4mk}}{2m}$$

② $\mu^2 = 4mk \implies$ critically damped, $r_1 = r_2$
$= -\frac{\mu}{2m}$

$$x(t) = Ae^{-\frac{\mu}{2m}t} + Bte^{-\frac{\mu}{2m}t}$$

Spring Mass System



$$mr^2 + \mu r + k = 0$$

$$r_1, r_2 = \frac{-\mu \pm \sqrt{\mu^2 - 4mk}}{2m}$$

③ $\mu^2 < 4mk \Rightarrow$ under-damped, $r_1, r_2 = \alpha \pm i\beta$,

$$x(t) = Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t$$

$\alpha = -\frac{\mu}{2m} < 0$

2x2 Linear System

$$\boxed{\frac{dX}{dt} = AX, \quad X(0) = X_0}$$

$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, $X_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

(Solution: $X(t) = e^{At} X_0$)

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Characteristic polynomial

$$p(\lambda) = \det(A - \lambda I) = (a-\lambda)(d-\lambda) - bc$$

$$= \lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

$\Longrightarrow \lambda_1, \lambda_2$ two eigenvalues

2x2 Linear System

(i) 2 distinct, real, eigenvalues, $\lambda_1 \neq \lambda_2$

$\Rightarrow$ 2 eigenvectors: $AV_1 = \lambda_1 V_1$, $AV_2 = \lambda_2 V_2$

$$X(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2$$

(i)' Repeating eigenvalues, $\lambda_1 = \lambda_2$,

but still with 2 eigenvectors V_1, V_2, eg. $A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$$\Longrightarrow X(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2$$

2x2 Linear System

② Repeating eigenvalues $\lambda_1 = \lambda_2 \; (= \lambda)$ but with only 1 eigenvector V: $AV = \lambda V$

eg $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

Find U s.t. $(A - \lambda I)U = V$

generalized eigenvector

$$\left((A - \lambda I)^2 U = 0 \right)$$

$$\boxed{X(t) = c_1 e^{\lambda t} V + c_2 e^{\lambda t}(tV + U)}$$

2x2 Linear System

③ λ_1, λ_2, 2 complex eigenvalues

(Still $AV_1 = \lambda_1 V_1$, $AV_2 = \lambda_2 V_2$, V_1, V_2 – complex

$X(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2$)

Express using real vectors

real vec.

$\lambda_1, \lambda_2 = \alpha \pm i\beta$, $V_1, V_2 = R \pm iS$

$$\boxed{\begin{aligned} X(t) &= c_1 e^{(\alpha+i\beta)t}(R+iS) + c_2 e^{(\alpha-i\beta)t}(R-iS) \\ &= \tilde{c}_1 e^{\alpha t}(\cos\beta t\, R - \sin\beta t\, S) \\ &\quad + \tilde{c}_2 e^{\alpha t}(\sin\beta t\, R + \cos\beta t\, S) \end{aligned}}$$

2x2 Linear System

3 λ_1, λ_2, 2 complex eigenvalues

(Still $AV_1 = \lambda_1 V_1$, $AV_2 = \lambda_2 V_2$, V_1, V_2 – complex

$X(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2$)

Express using real vectors

real vec.

$\lambda_1, \lambda_2 = \alpha \pm i\beta$, $V_1, V_2 = R + iS$

$$\boxed{X(t) = e^{\alpha t} \begin{bmatrix} R & S \end{bmatrix} \begin{bmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{bmatrix} \begin{bmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{bmatrix}}$$

[M, p.36 (2.22)]

[M]

2.2 ▪ Two-Dimensional Linear Systems

The properties of the eigenvalues of the arbitrary real, 2×2 matrix, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, can be easily obtained in general. Its eigenvalues are roots of the characteristic polynomial

$$\begin{aligned} &p(\lambda) = \lambda^2 - \tau\lambda + \delta = 0, \\ &\tau \equiv \mathrm{tr}(A) = a + d, \\ &\delta \equiv \det(A) = ad - bc. \end{aligned} \tag{2.18}$$

Thus, the eigenvalues depend only on the values of τ, the *trace* of A, and δ, the *determinant* of A. The roots of p are

$$\lambda_\pm = \frac{\tau \pm \sqrt{\Delta}}{2}, \quad \Delta \equiv \tau^2 - 4\delta = (a-d)^2 + 4bc. \tag{2.19}$$

Here, Δ is the *discriminant* of p. There are five different eigenvalue regions in the (τ, δ)-plane, as shown in Figure 2.1. The insets in the figure show complex λ-planes

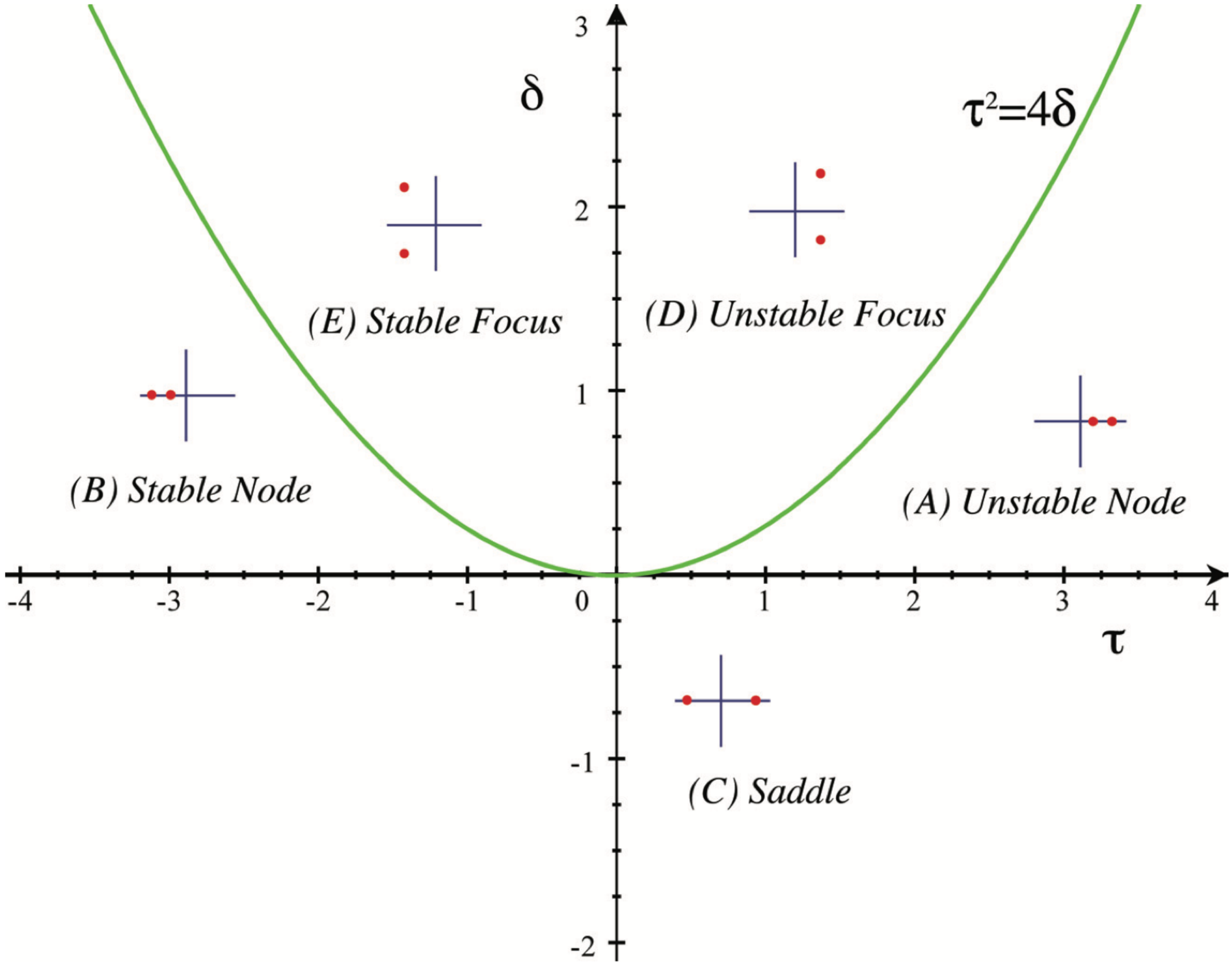


Figure 2.1. *Classification of the eigenvalues for a* 2×2 *linear system in the parameter space of the trace,* τ, *and determinant,* δ.

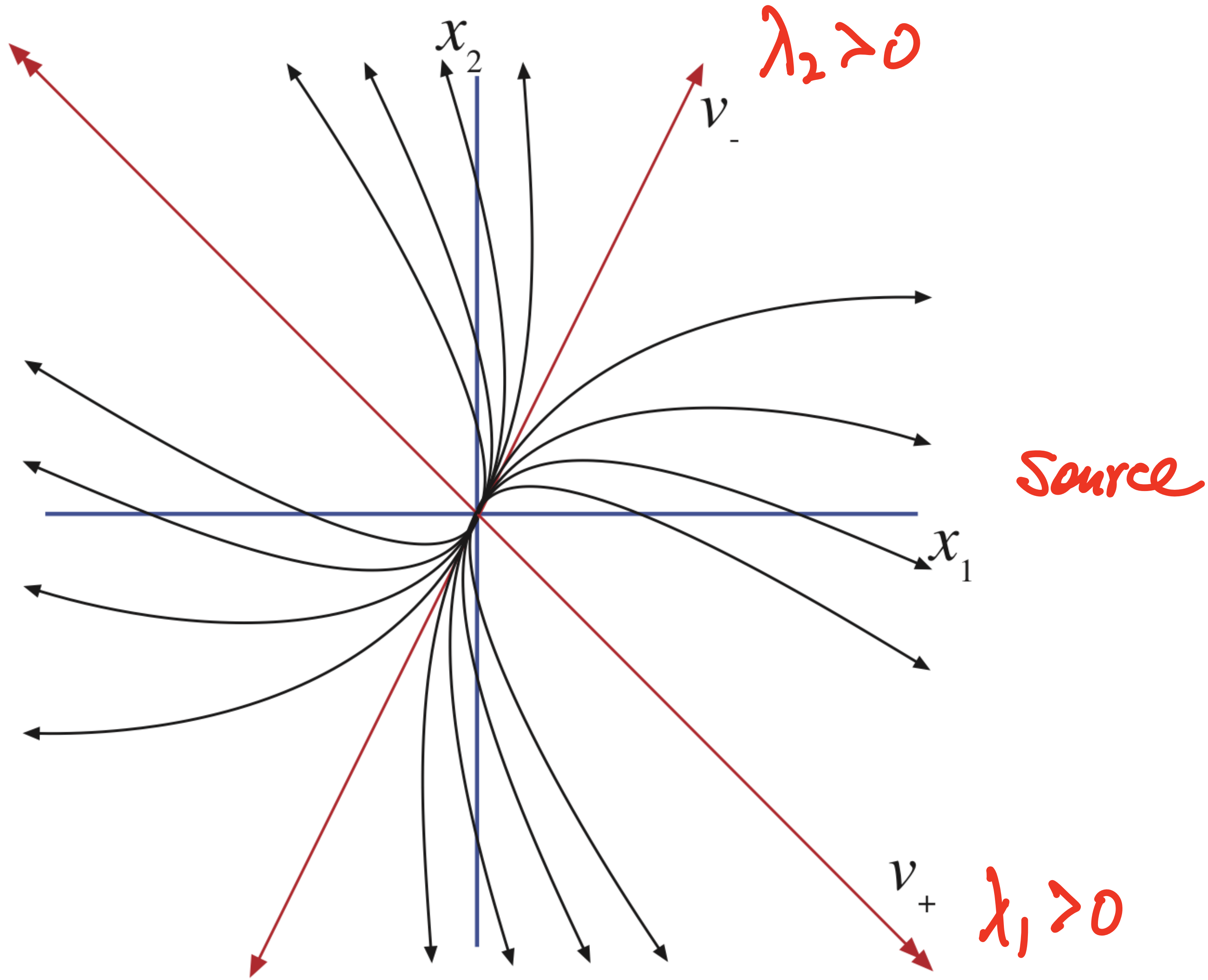


Figure 2.2. *Phase portrait of an unstable node with* $v_+ = (1,-1)^T$, *and* $v_- = (1,2)^T$ *and* [redacted]λ_-. *The arrows denote the direction of motion.*

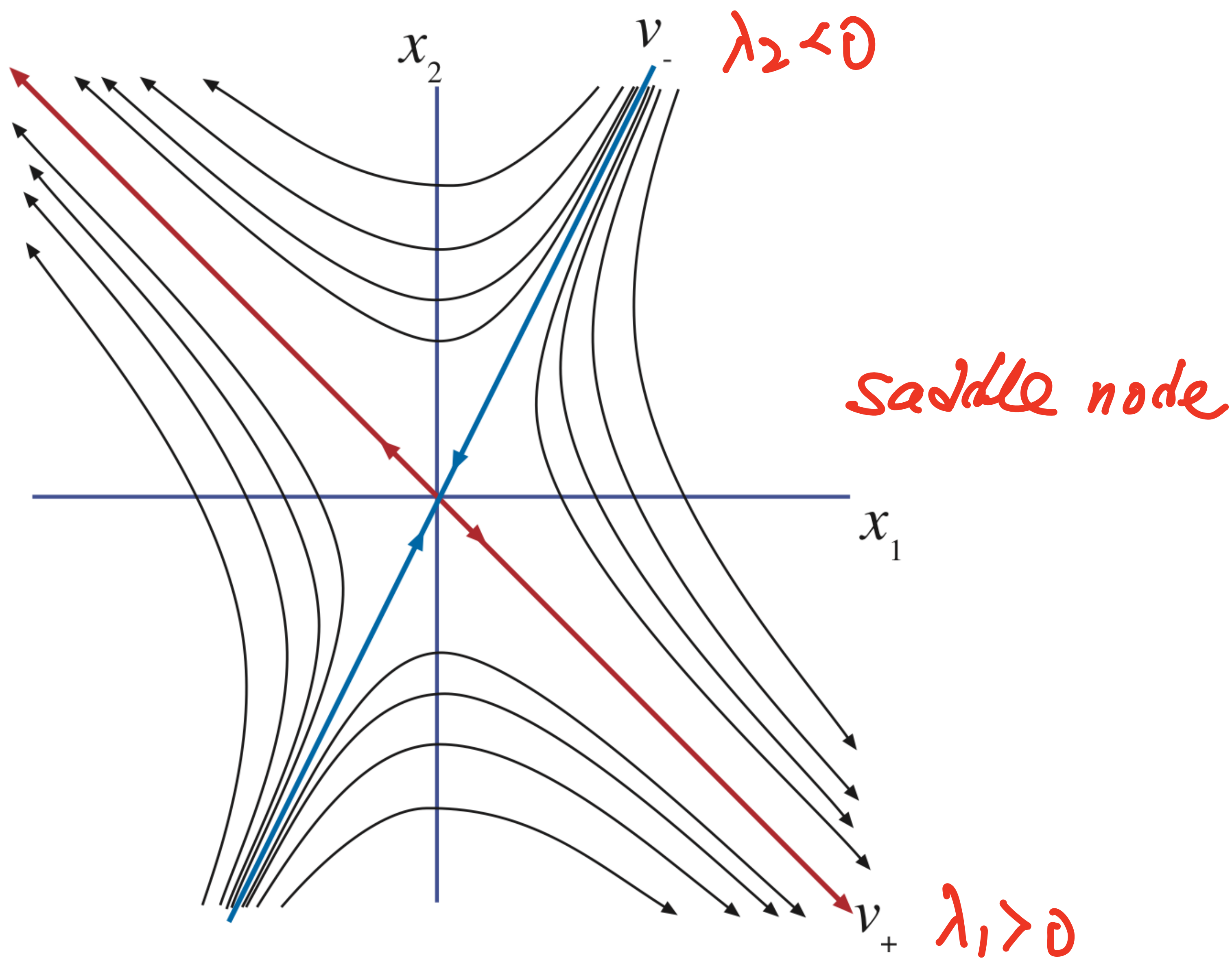
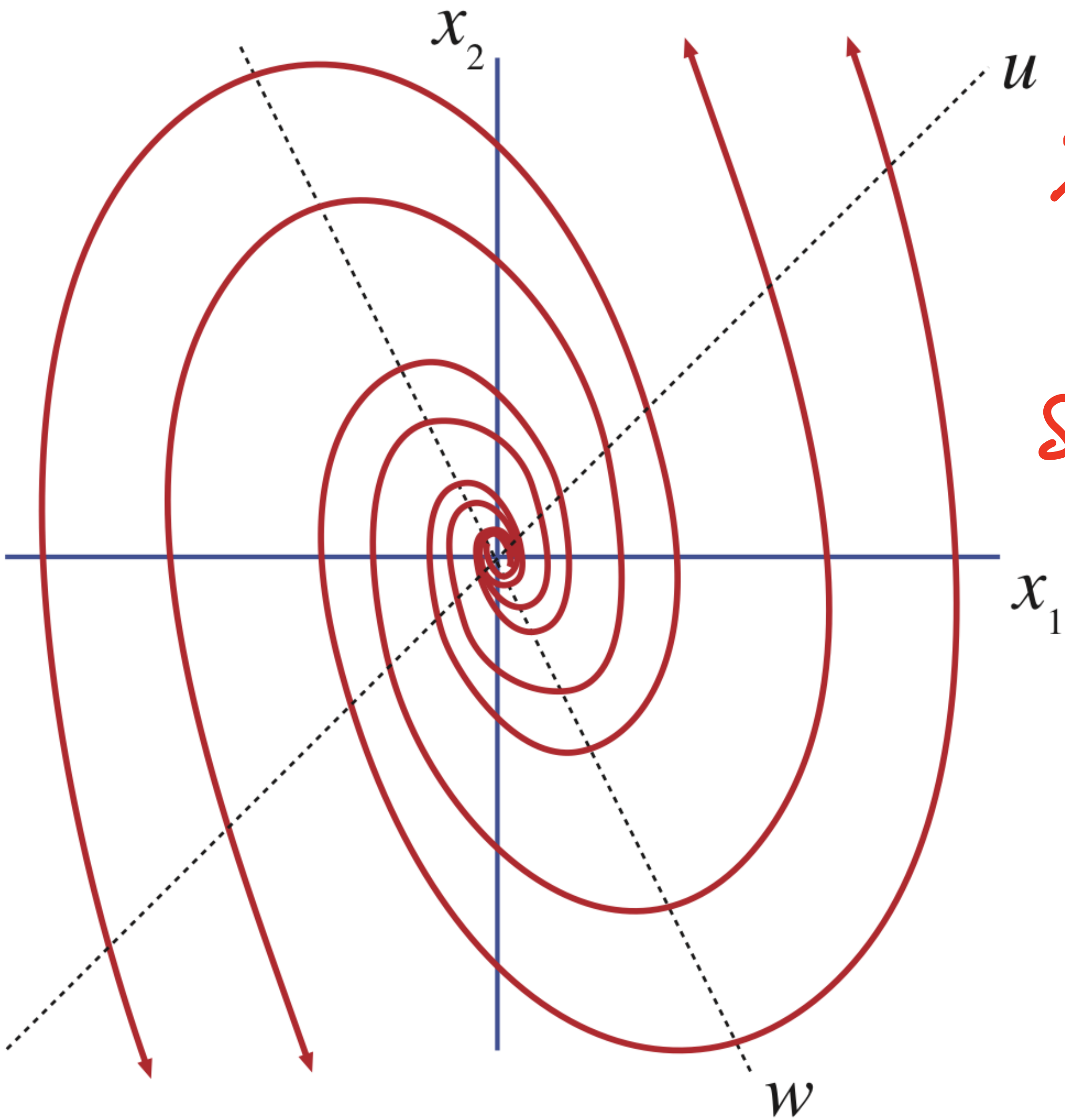


Figure 2.3. *Phase portrait of a saddle with* $v_+ = (1, -1)^T$, *and* $v_- = (1, 2)^T$ *and* [redacted]. *The arrows denote the direction of motion.*



$\lambda_1, \lambda_2 = \alpha \pm i\beta$

$\alpha > 0$

Spiral out

Figure 2.4. *Phase portrait of an unstable focus with* $u = (1,1)^T$, *and* $w = (1,-2)^T$ *and* [redacted]. *Here the motion is counterclockwise since* $\det[u, w] < 0$.