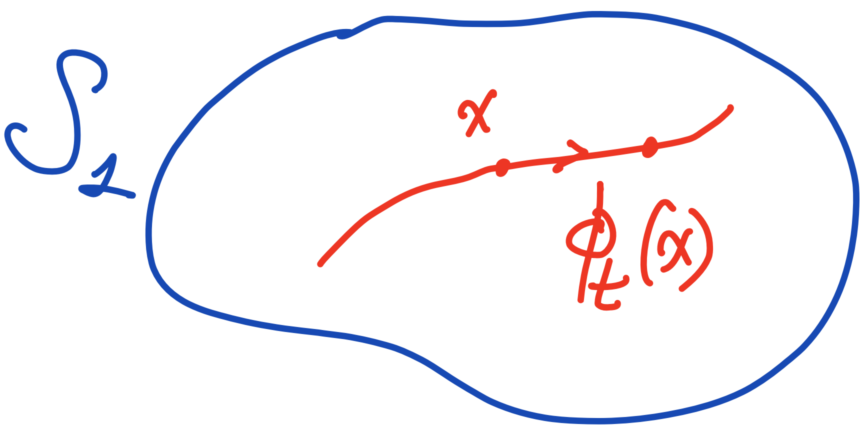
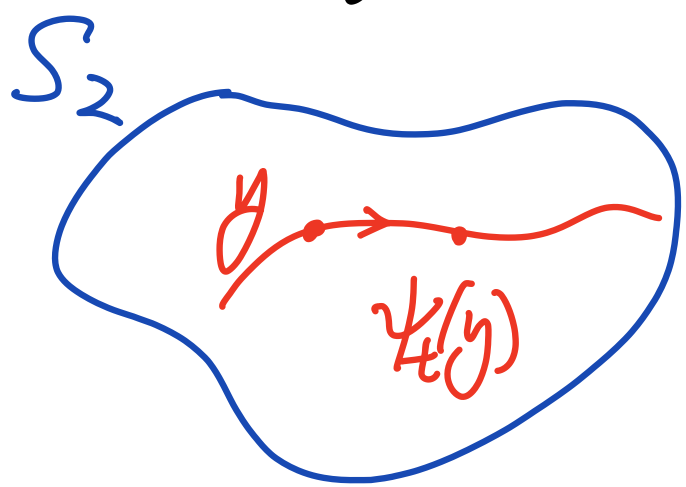
Comparison between Two Dynamics

Topological Conjugacy

Consider

(1) $$\boxed{\begin{aligned} \frac{dX}{dt} &= F(X) \\ X(0) &= x \end{aligned}}$$ vs $$\boxed{\begin{aligned} \frac{dY}{dt} &= G(Y) \\ Y(0) &= y \end{aligned}}$$ (2)





Comparison between Two Dynamics

Topological Conjugacy

The dynamics of (1) (in S_1) and (2) (in S_2) are <u>conjugate</u> if there is a <u>homeomorphism</u> H between S_1 & S_2 such that

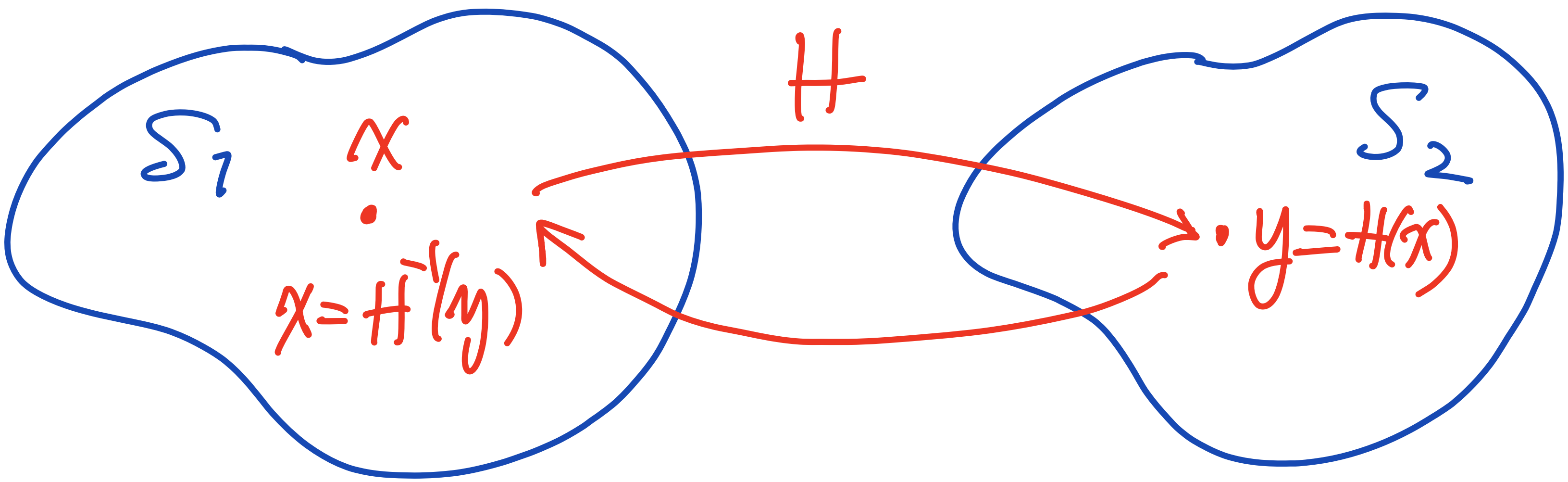
$$(*) \quad \boxed{H(\phi_t(x)) = \psi_t(H(x))}$$

[M, p. 126]

Comparison between Two Dynamics

Topological Conjugacy

Homeomorphism:



(1) H is one-to-one and onto

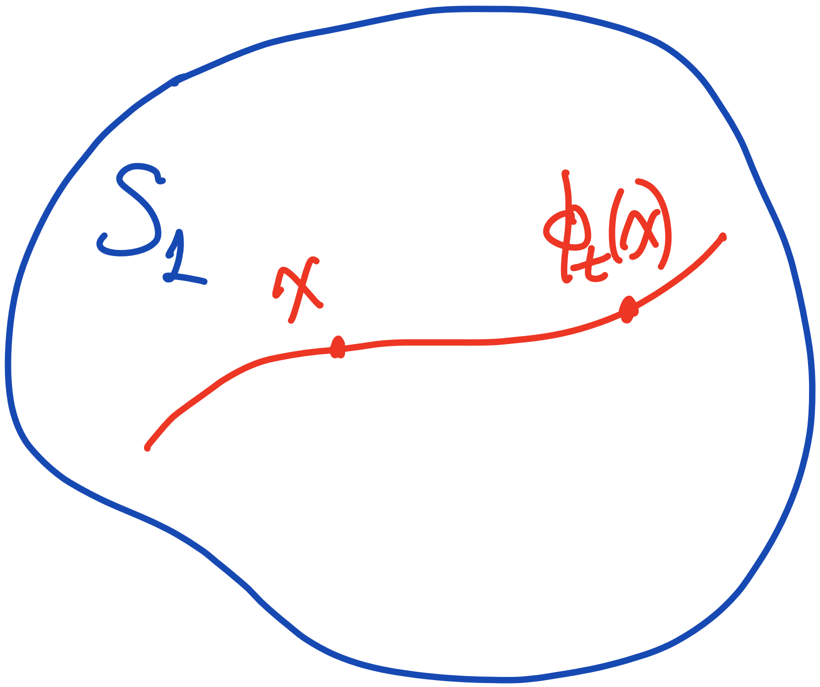
($\Longrightarrow$ H^{-1} exists)

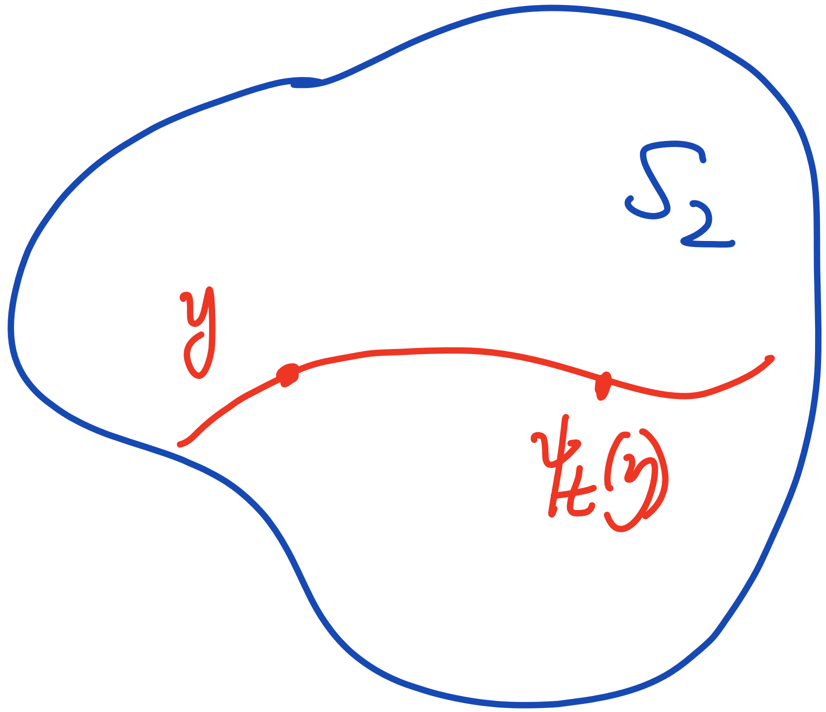
(2) Both H and H^{-1} are continuous.

Comparison between Two Dynamics

Topological Conjugacy

$$(*)\quad \boxed{H(\phi_t(x)) = \psi_t(H(x))}$$

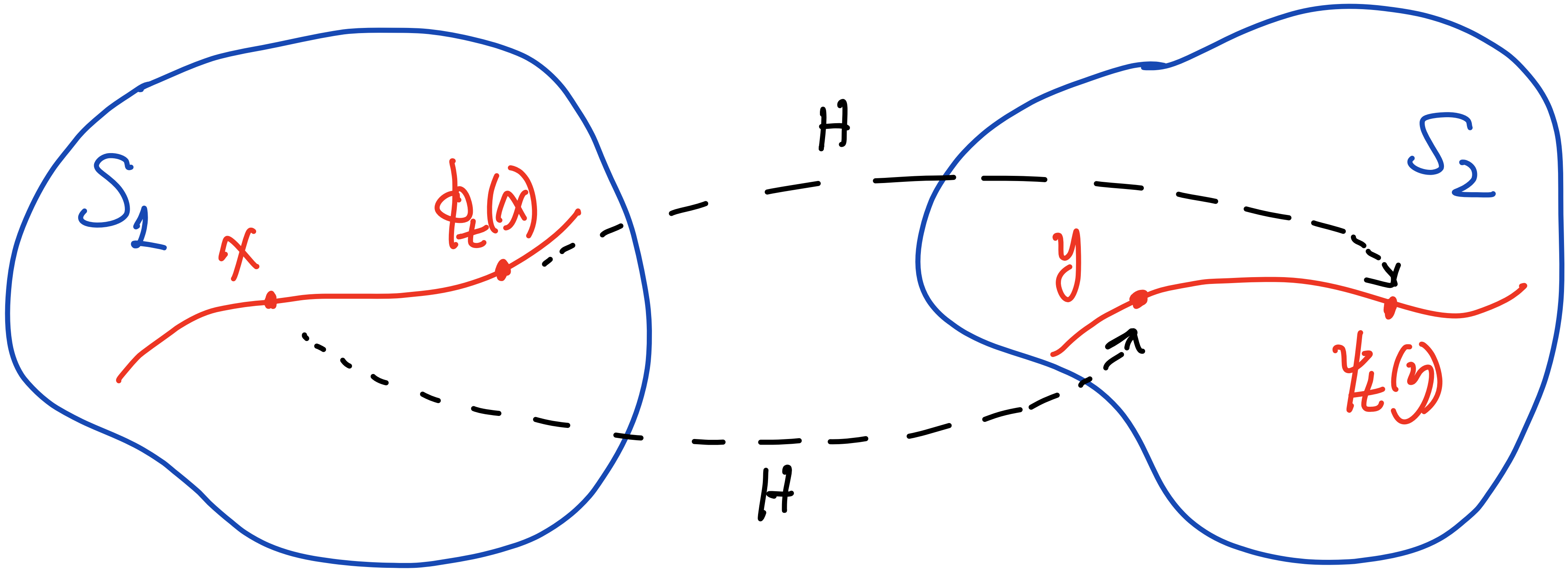




Comparison between Two Dynamics

Topological Conjugacy

(*) $$\boxed{H(\phi_t(x)) = \psi_t(H(x))}$$



Comparison between Two Dynamics

Topological Conjugacy

$$(*)\quad \boxed{H(\phi_t(x)) = \psi_t(H(x))}$$

$$\begin{tikzcd}
x \arrow[r, "\phi_t(\cdot)"] \arrow[d, "H"'] & \phi_t(x) \arrow[d, "H"] & \text{in } S_1 \\
y = H(x) \arrow[r, "\psi_t(\cdot)"] & \psi_t(y) & \text{in } S_2
\end{tikzcd}$$

Comparison between Two Dynamics

Topological Conjugacy

The dynamics of (1) (in S_1) and (2) (in S_2) are <u>diffeomorphic</u> if there is a <u>diffeomorphism</u> H between S_1 & S_2 such that

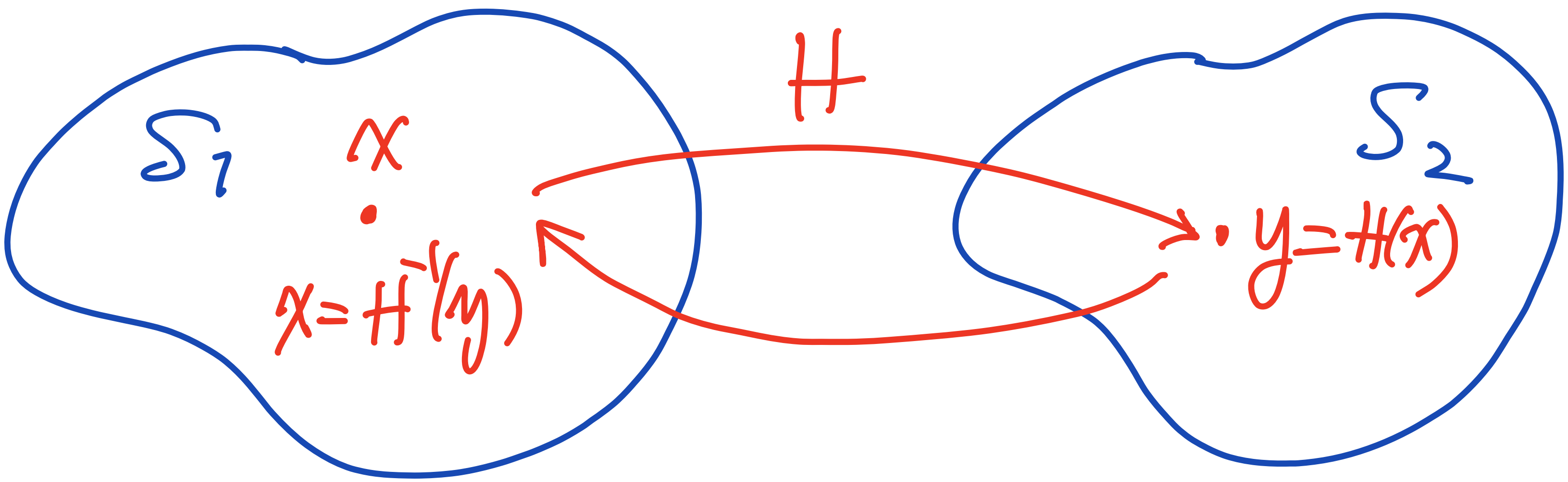
$$(*) \quad \boxed{H(\phi_t(x)) = \psi_t(H(x))}$$

[M, p. 129]

Comparison between Two Dynamics

Topological Conjugacy

Diffeomorphism



(1) H is one-to-one and onto ($\Rightarrow$ H^{-1} exists)

(2) Both H and H^{-1} are continuous.

(3) Both DH & DH^{-1} exist.

Comparison between Two Dynamics

Topological Equivalence

The dynamics of (1) (in S_1) and (2) (in S_2) are <u>equivalent</u> if there is a <u>homeomorphism</u> H between S_1 & S_2 such that

$$(*) \quad \boxed{H(\{\phi_t(x)\}_t) = \{\psi_t(H(x))\}_t}$$

[M, p. 127]

Comparison between Two Dynamics

Topological Equivalence

(*) $$\boxed{H(\{\phi_t(x)\}_t) = \{\psi_t(H(x))\}_t}$$

(1) The "paths" are the same but the "time" can be different.

(2) There is a time change $t \longrightarrow \tau = \tau(t)$ such that

$$\boxed{H(\phi_t(x)) = \psi_{\tau(t)}(H(x))}$$

Comparison between Two Dynamics

Topological Equivalence

An Example of Time Change:

$$\boxed{\frac{dX}{dt} = F(X)}$$

[M, Thm 4.7, p.106]

$$\downarrow$$

$$\boxed{\frac{dY}{d\tau} = \frac{F(Y)}{1 + \|F(Y)\|}}$$

Comparison between Two Dynamics

Topological Equivalence

An Example of Time Change:

$$\boxed{\frac{dX}{dt} = F(X)}$$

[M, Thm 4.7, p.106]

$$\downarrow$$

$$\boxed{\frac{dY}{d\tau} = \frac{F(Y)}{1 + \|F(Y)\|}}$$

(1) $\tau(t) = \int_0^t \left(1 + \|F(X(s))\|\right) ds$

(2) Y has global solution as R.H.S. is bounded.

Some Results Concerning Conjugacy & Diffeomorphism

(1) If $\frac{dX}{dt} = F(X)$ & $\frac{dY}{dt} = G(Y)$ are diffeomorphic (H), then $[D_X F(X_*)]$ & $[D_Y G(Y_*)]$ are similar at equilibrium pts x_* & $y_* = H(x_*)$

[$A^{n\times n}$ & $B^{n\times n}$ are similar if there is an invertible $P^{n\times n}$ s.t. $\boxed{A = PBP^{-1}}$]

[M. p.130]

Some Results Concerning Conjugacy & Diffeomorphism

(2) [M. Thm 4.33, p.130] (Linear Diffeomorphism)

$$\boxed{\frac{dX}{dt} = AX} \quad \& \quad \boxed{\frac{dY}{dt} = BY}$$

are diffeomorphic if and only if

A is similar to B

Some Results Concerning Conjugacy & Diffeomorphism

A is similar to B $\iff$ Change of Basis

$$A = PBP^{-1} \iff P^{-1}A = BP^{-1}$$

$$\begin{tikzcd}
X \in \mathbb{R}^n \arrow[r, "A"] \arrow[d, "P^{-1}"'] & AX \in \mathbb{R}^n \arrow[d, "P^{-1}"] \\
Y \in \mathbb{R}^n \arrow[r, "B"] & BY \in \mathbb{R}^n
\end{tikzcd}$$

Some Results Concerning Conjugacy & Diffeomorphism

A is similar to B $\iff$ Change of Basis

$$A = PBP^{-1} \iff P^{-1}A = BP^{-1}$$

$$
\begin{tikzcd}
X \in \mathbb{R}^n \arrow[r, "A"] \arrow[d, "P^{-1}"'] \arrow[dr, bend right, "BP^{-1}X"] \arrow[dr, bend left, "P^{-1}AX"] & AX \in \mathbb{R}^n \arrow[d, "P^{-1}"] \\
Y \in \mathbb{R}^n \arrow[r, "B"] & BY \in \mathbb{R}^n
\end{tikzcd}
$$

$Y = P^{-1}X$

$\Updownarrow$

$X = PY$

Some Results Concerning Conjugacy & Diffeomorphism

(3) [M. Thm 4.35, p.131] (Linear Conjugacy)

$$\boxed{\frac{dX}{dt} = AX} \quad (1) \qquad \& \qquad \boxed{\frac{dY}{dt} = BY} \quad (2)$$

Let A & B are hyperbolic.

Then (1) & (2) are conjugate if and only if

$\dim(E^s(A)) = \dim(E^s(B))$ & $\dim(E^u(A)) = \dim(E^u(B))$

Some Results Concerning Conjugacy & Diffeomorphism

(4) [M. Thm 4.36, p.132] Hartman–Grobman Thm

$$(1)\quad \boxed{\frac{dX}{dt} = AX} \quad \text{vs} \quad \boxed{\frac{dY}{dt} = AY + g(Y)} \quad (2)$$

A – hyperbolic, $\quad |g(Y)| \lesssim O(|Y|^2) \ll |Y|$

Then (1) & (2) are conjugate

Some Results Concerning Conjugacy & Diffeomorphism

(4) [M. Thm 4.36, p.132] Hartman-Grobman Thm

$$(1)\ \boxed{\frac{dX}{dt} = AX} \quad vs \quad \boxed{\frac{dY}{dt} = AY + g(Y)}\ (2)$$

A - hyperbolic, $|g(Y)| \lesssim O(|Y|^2) \ll |Y|$

Then (1) & (2) are conjugate

(In general, H is less smooth than g)

Some Results Concerning Conjugacy & Diffeomorphism

(4) [M. Thm 4.36, p.132] Hartman-Grobman Thm

$$\boxed{\frac{dX}{dt} = AX} \quad (1) \qquad \text{vs} \qquad \boxed{\frac{dY}{dt} = AY + g(Y)} \quad (2)$$

A - hyperbolic, $|g(Y)| \lesssim O(|Y|^2) \ll |Y|$

Then (1) & (2) are conjugate

(If g is C^2, then H might only be C^1.)

Non-hyperbolic Case: Center Manifold Thm. (M, Thm 5.21)

p. 179

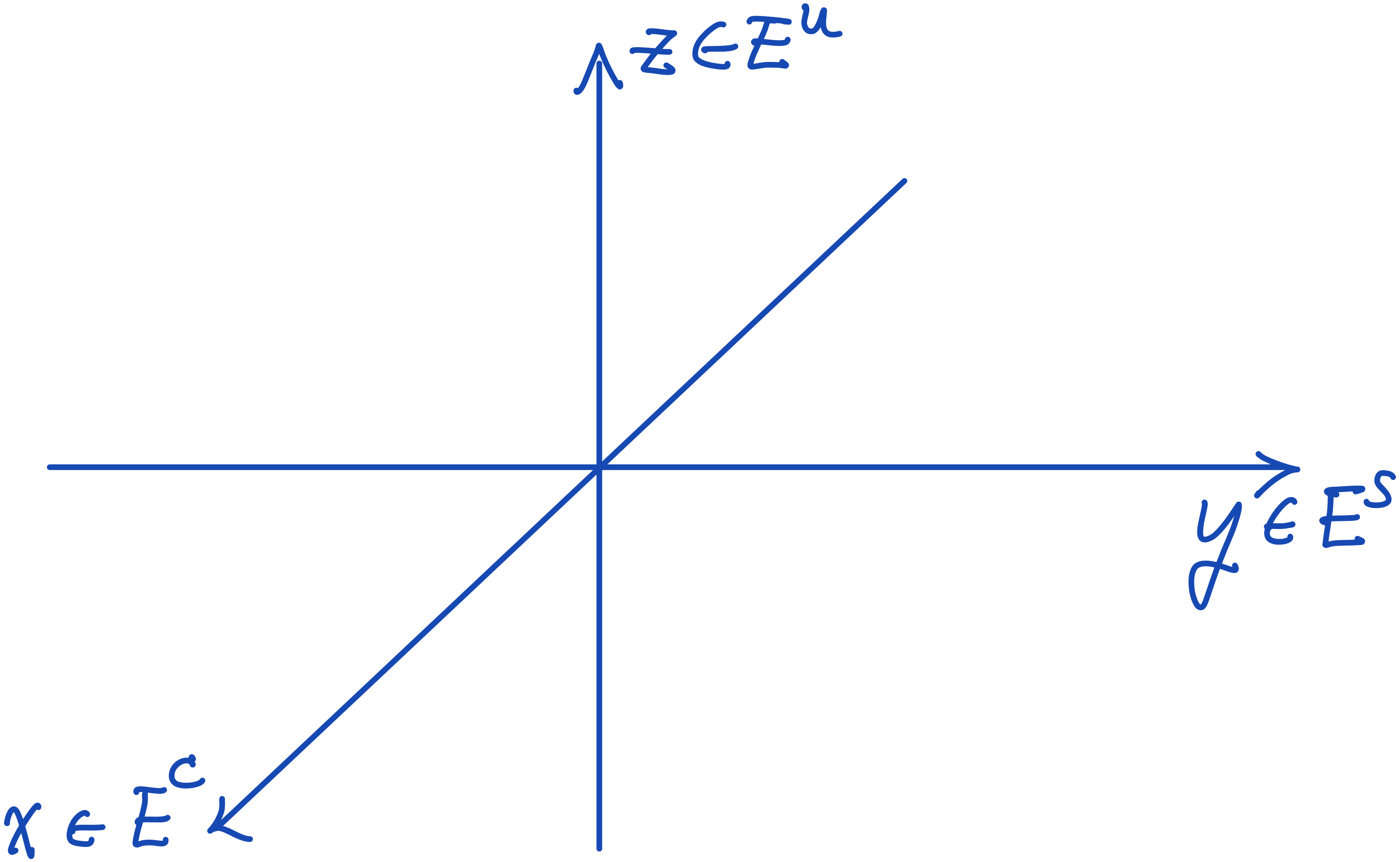
$$\boxed{\frac{dX}{dt} = AX + g(X), \quad |g(X)| \leq C|X|^2}$$

$$\begin{cases} \dfrac{dx}{dt} = Cx + P(x,y,z), & Re(\lambda(C)) = 0 \\[2ex] \dfrac{dy}{dt} = Sy + Q(x,y,z), & Re(\lambda(S)) < 0 \\[2ex] \dfrac{dz}{dt} = Uz + R(x,y,z), & Re(\lambda(U)) > 0 \end{cases}$$

Non-hyperbolic Case: Center Manifold Thm. (M, Thm 5.21)

p. 179

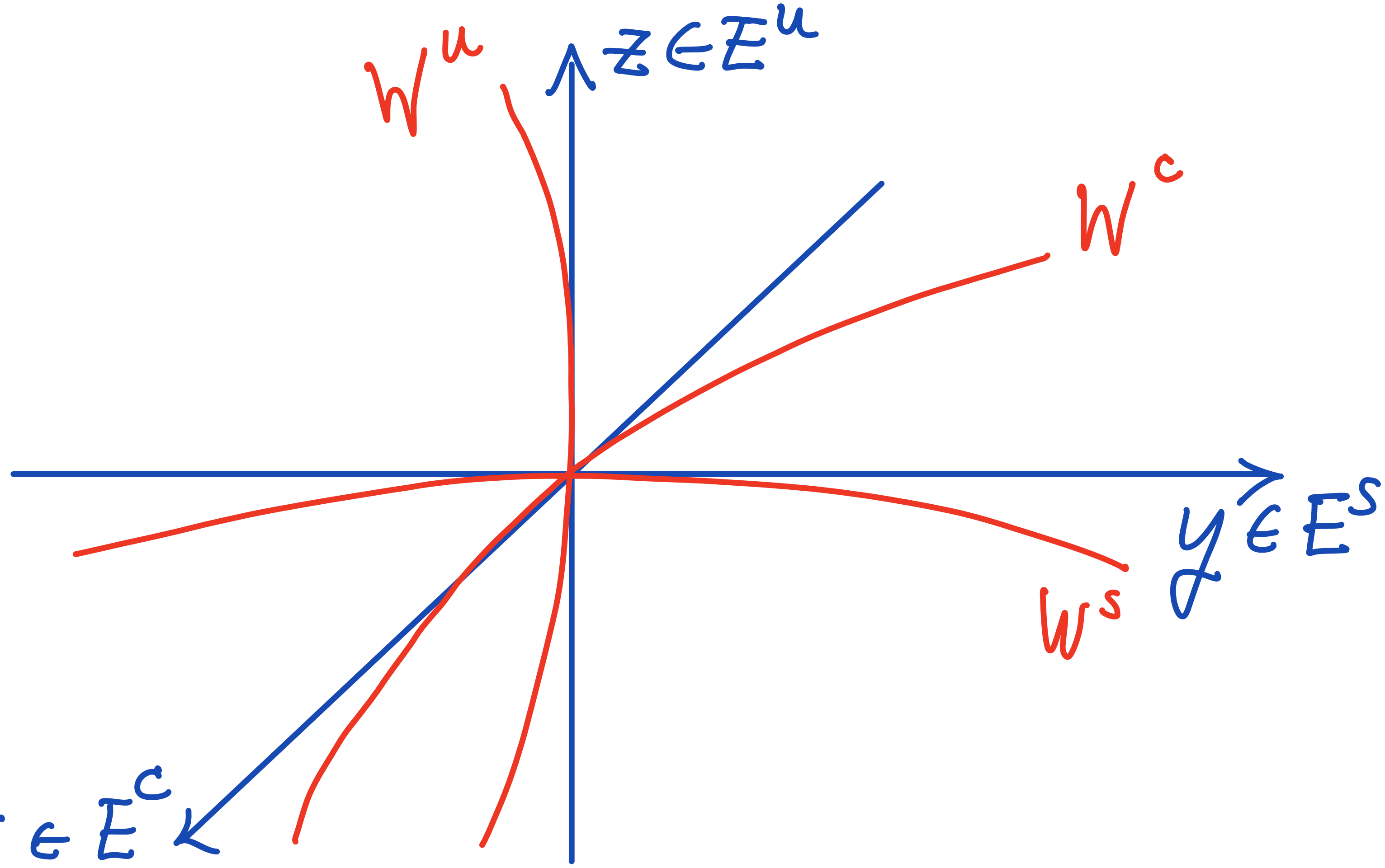
$$\boxed{\frac{dX}{dt} = AX + g(X), \quad |g(X)| \leq C|X|^2}$$



Non-hyperbolic Case: Center Manifold Thm. (M, Thm 5.21)

p. 179

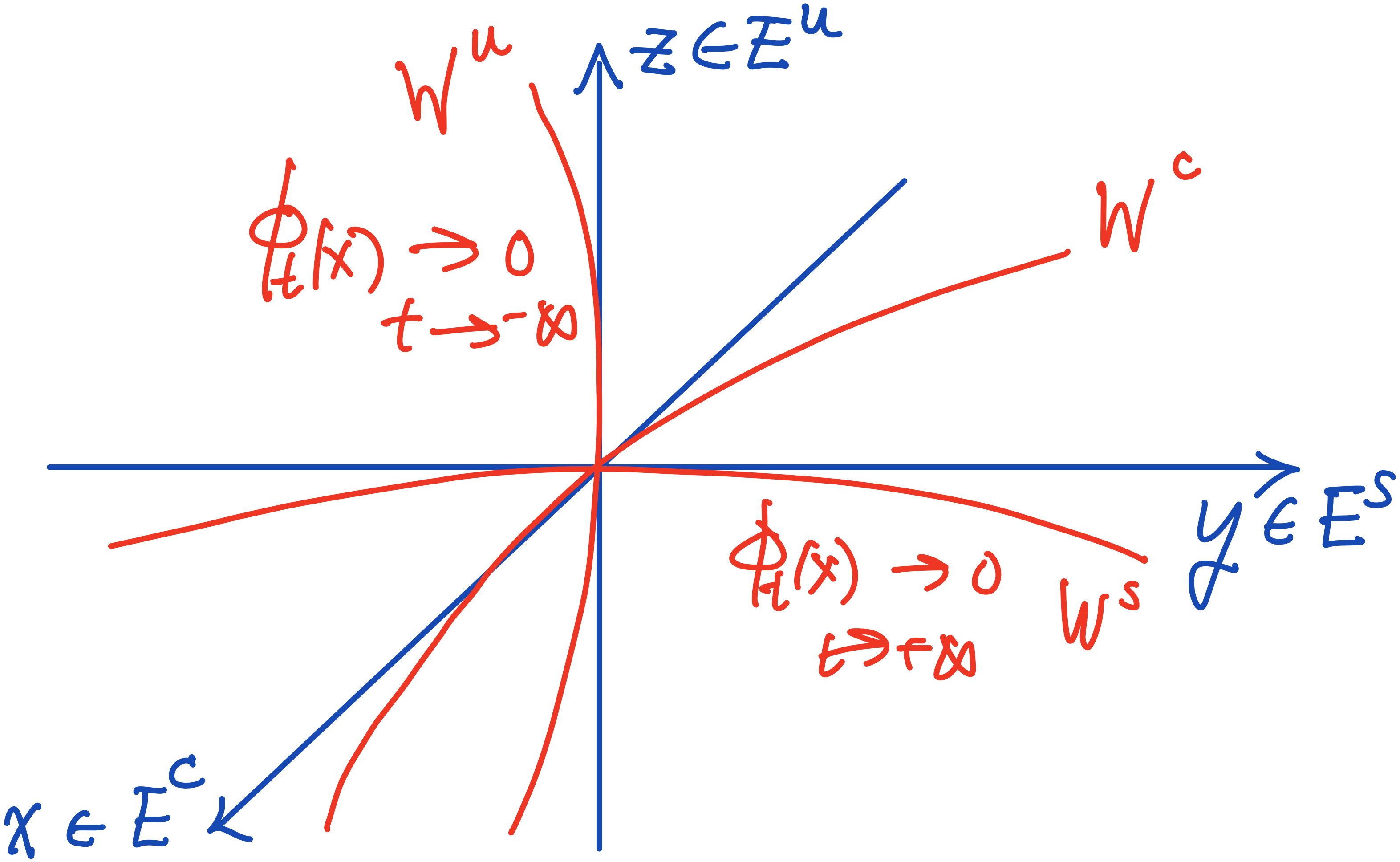
$$\boxed{\frac{dX}{dt} = AX + g(X), \quad |g(X)| \leq C|X|^2}$$



Non-hyperbolic Case: Center Manifold Thm. (M, Thm 5.21)

p. 179

$$\boxed{\frac{dX}{dt} = AX + g(X), \quad |g(X)| \leq C|X|^2}$$



Nonhyperbolic Hartman Grobman Thm. (M, Thm 5.23)

p. 182

$$\begin{cases} \dot{x} = Cx + P(x,y,z), & \mathrm{Re}(\lambda(C)) = 0 \\ \dot{y} = Sy + Q(x,y,z), & \mathrm{Re}(\lambda(S)) < 0 \\ \dot{z} = Uz + R(x,y,z), & \mathrm{Re}(\lambda(U)) > 0 \end{cases}$$

Topological conjugate to :

$$\begin{cases} \dot{x} = Cx + P(x, h(x), g(x)) \\ \dot{y} = Sy \\ \dot{z} = Uz \end{cases}$$

Nonhyperbolic Hartman Grobman Thm. (M, Thm 5.23)

p. 182

$$\begin{cases} \dot{x} = Cx + P(x, y, z), & \mathrm{Re}(\lambda(C)) = 0 \\ \dot{y} = Sy + Q(x, y, z), & \mathrm{Re}(\lambda(S)) < 0 \\ \dot{z} = Uz + R(x, y, z), & \mathrm{Re}(\lambda(U)) > 0 \end{cases}$$

Topological conjugate to:

$$\begin{cases} \dot{x} = Cx + P(x, \underbrace{h(x), g(x)}) \\ \dot{y} = Sy \\ \dot{z} = Uz \end{cases}$$

$W^c_{loc} = \{ (x, h(x), g(x)) : x \in E^c \}$