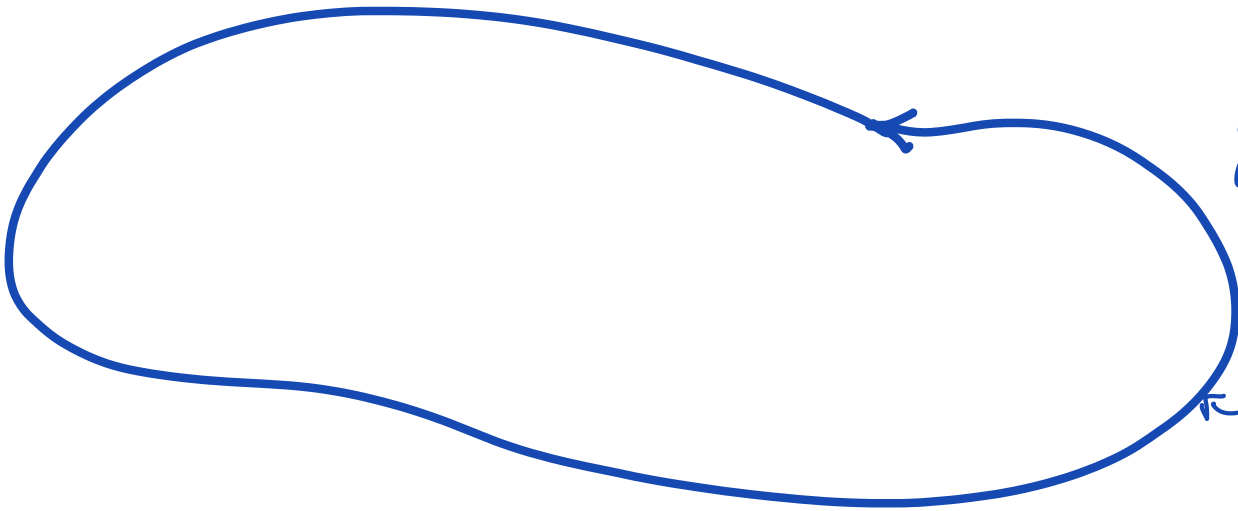
Stability of Periodic Orbits

autonomous system

$$\boxed{\frac{dX}{dt} = F(X), \qquad X(0) = x}$$

Periodic orbit $\gamma(t)$ There is a $T > 0$ s.t.

$$\boxed{\gamma(t+T) = \gamma(t) \qquad \forall t}$$



$\gamma(t) = \gamma(t+T)$

Γ – the whole curve

$$\Gamma = \bigcup_t \{\gamma(t)\}$$

Notions of Stability of γ (or Γ)

Stable: if for any neighborhood N of Γ, there is a neighborhood M of Γ such that

if $X(0) \in M$, then $X(t) \in N$, $\forall t > 0$.

Asymptotic Stable: there is a neighborhood N of Γ s.t.

if $X(0) \in N$, then $X(t) \longrightarrow \Gamma$ as $t \to \infty$

$\text{dist}(X(t), \Gamma) \longrightarrow 0$

Unstable: if Γ is not stable, i.e. there is a neighborhood N of Γ such that

for any $X(0) \notin \Gamma$, there is a t_i s.t. $X(t_i) \notin N$.

$t_i \to +\infty$

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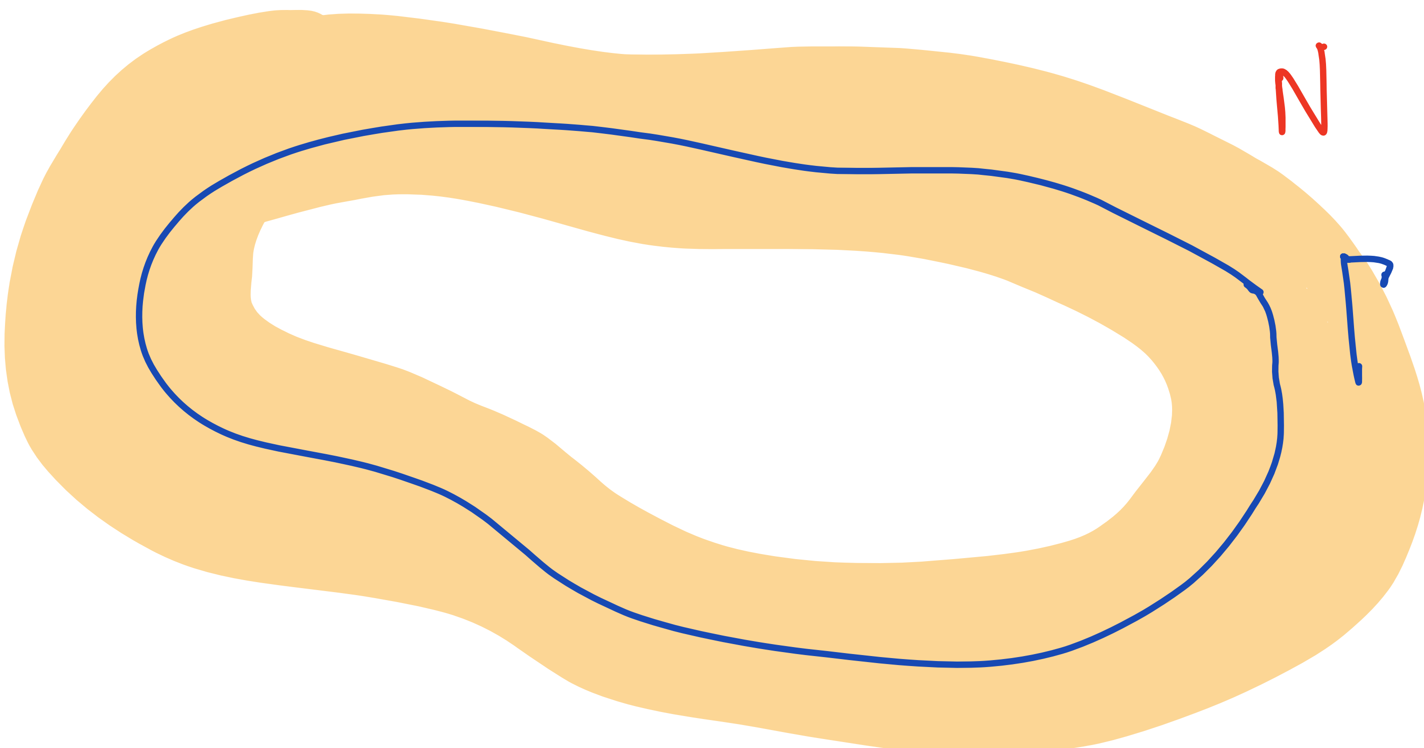

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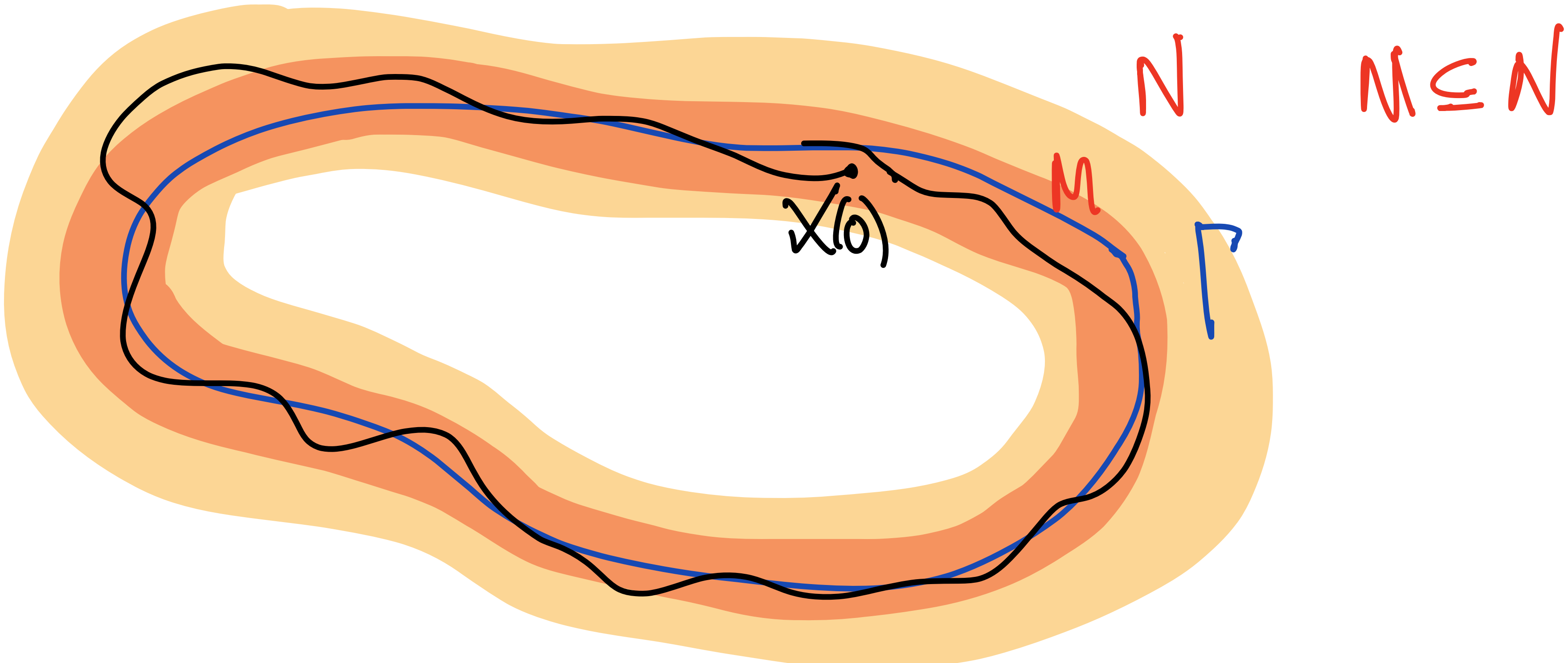

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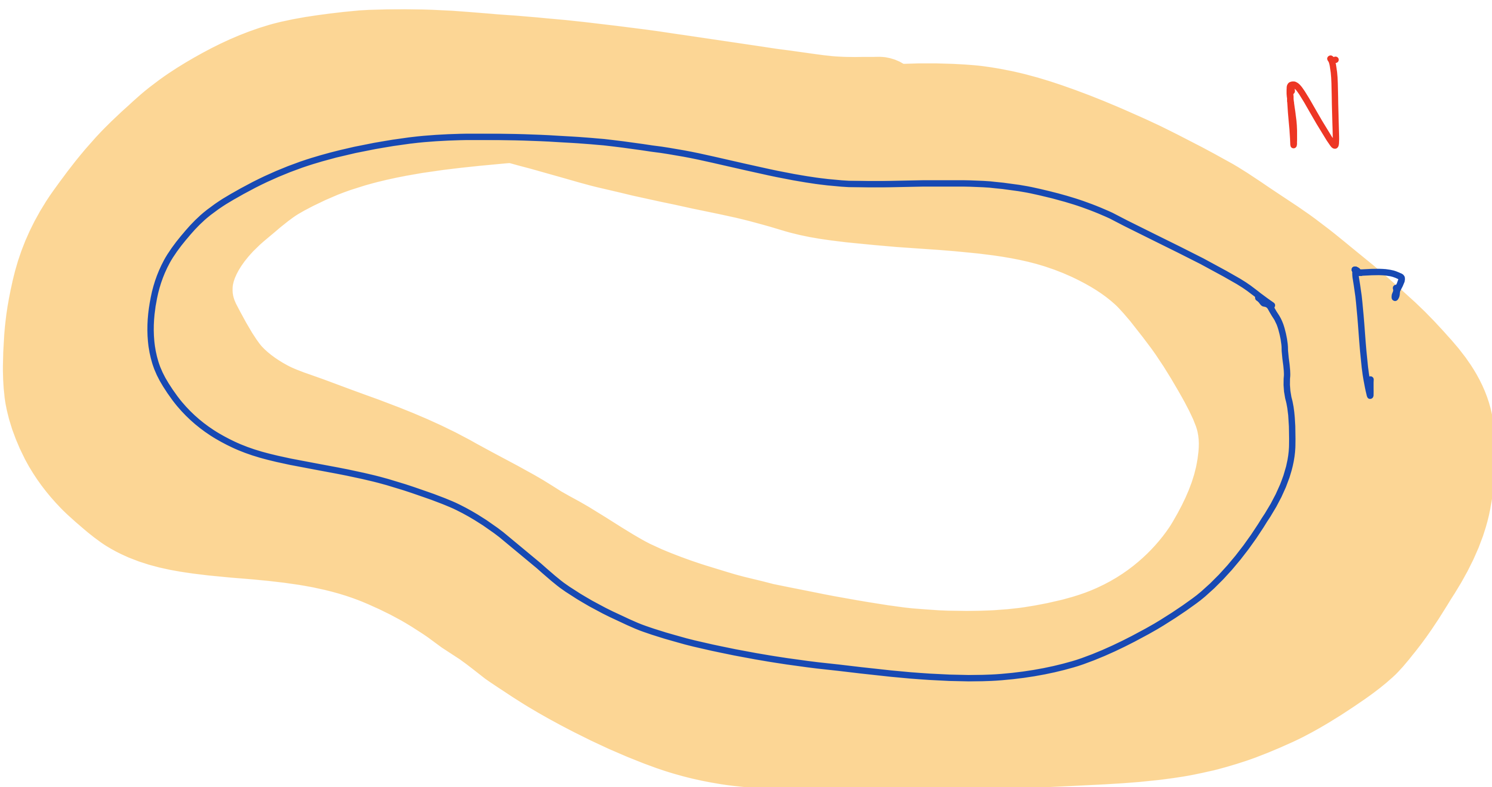

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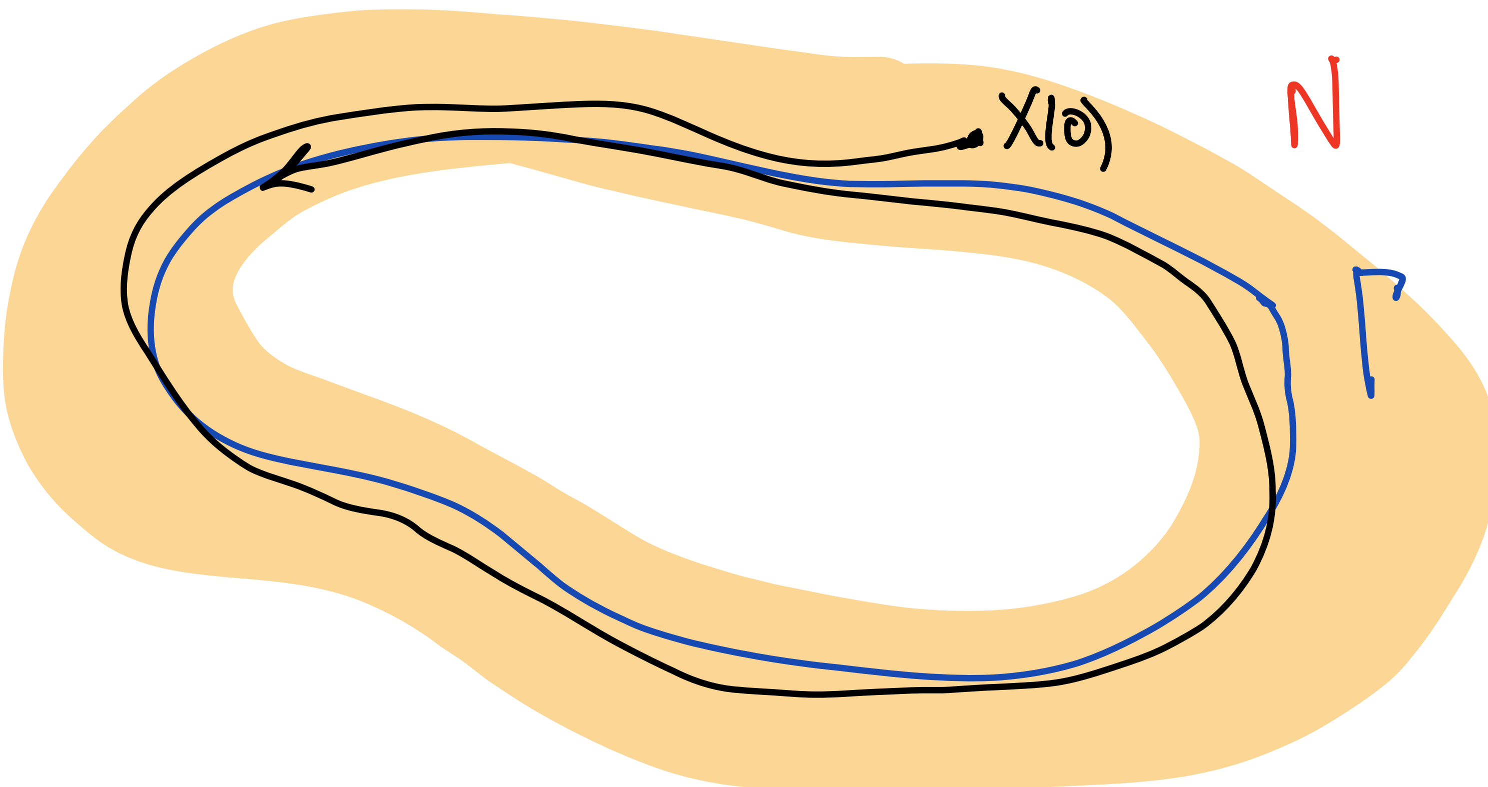

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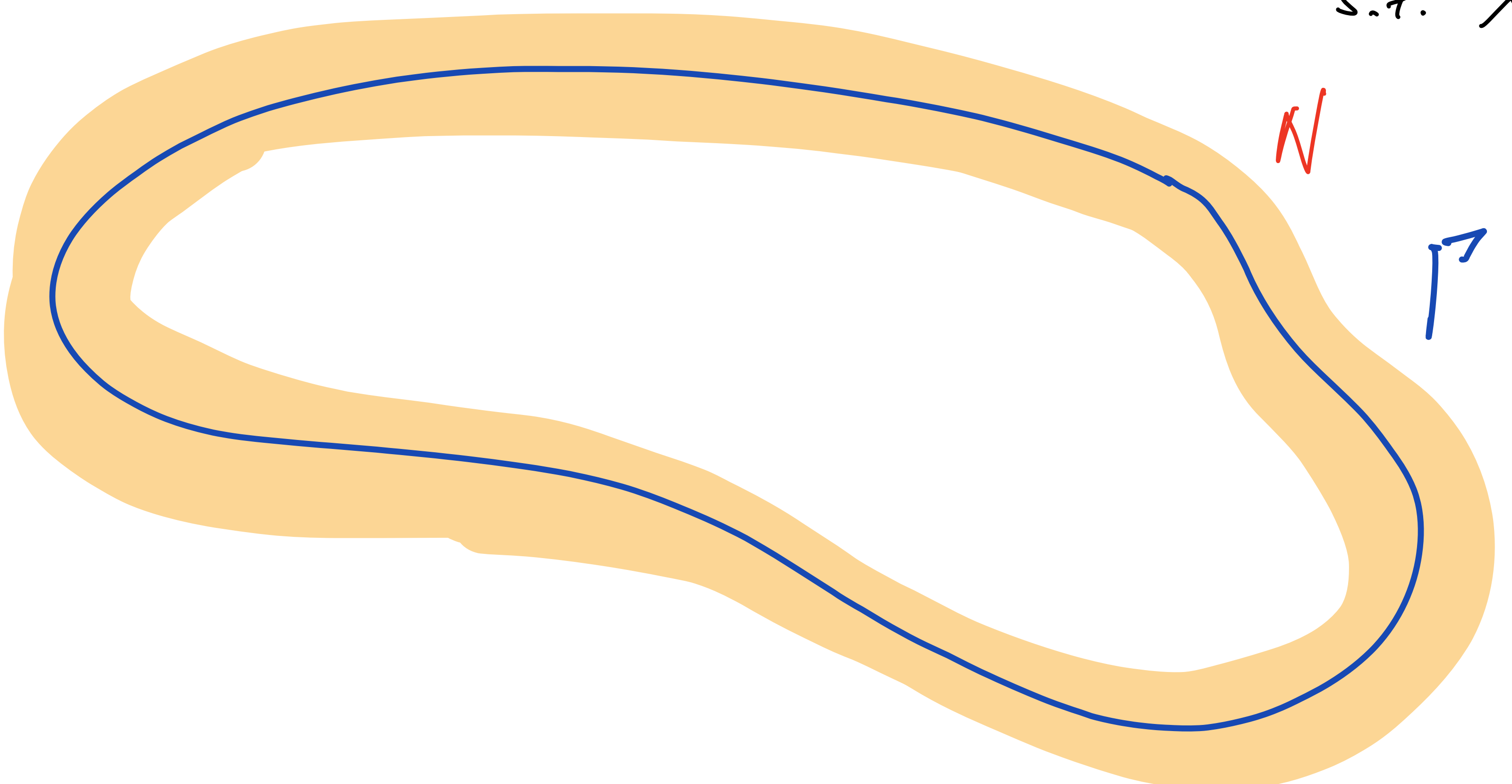

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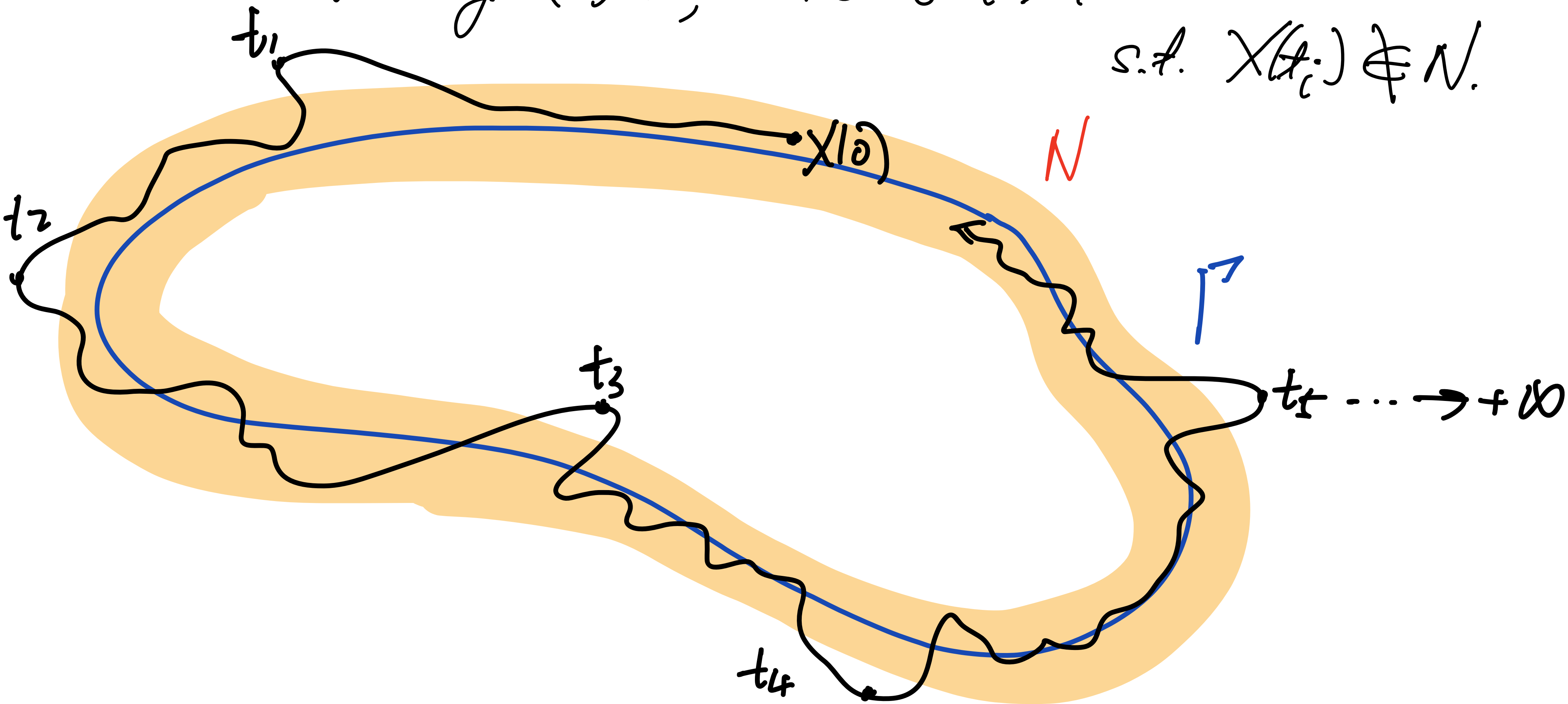


Time Shift and Orbital Stability

For an obvious reason, it is not possible that if $X(0)$ is close to $\gamma(0)$, then $X(t) - \gamma(t) \xrightarrow{t\to\infty} 0$

Reason: Due to time shift of solution:

if $X(0) = \gamma(\delta)$, then $X(t) = \gamma(t+\delta)$, that will never converge to $\gamma(t)$.

Orbital Stability:

if $|X(0) - \gamma(0)|$ is small, then there is a δ s.t.

$|X(t) - \gamma(t+\delta)|$ is small or $\longrightarrow 0$ as $t \to +\infty$.

Linearization Near γ

perturbation around γ

Consider the solution as $\gamma(t) + X(t)$.

Then
$$\frac{d}{dt}\left(\cancel{\gamma(t)} + X(t)\right) = F(\gamma(t) + X(t))$$
$$= \cancel{F(\gamma(t))} + DF(\gamma(t))X(t) + O(|X(t)|^2)$$

and
$$\frac{dX(t)}{dt} = DF(\gamma(t))X(t) + O(|X(t)|^2)$$

Linearization:
$$\boxed{\frac{dX}{dt} = A(t)X}$$

$A(t+T) = A(t)$

$A(t) = DF(\gamma(t))$ is T-periodic

Floquet Theory (M, Sec 2.8)

Consider $\dfrac{dX}{dt} = A(t)X$, $A(t+T) = A(t)$.

The fundamental solution $\Phi(t)$ of the above:

$$\frac{d\Phi(t)}{dt} = A(t)\Phi(t), \quad \Phi(0) = I$$

can be written as

$$\boxed{\Phi(t) = Q(t)e^{tB}}$$

for some T-periodic matrix $Q(t)$ $(Q(t+T) = Q(t))$
and constant matrix B.

Floquet Theory (M, Sec 2.8)

- All the stability property is captured by B:

$$X(t) = \Phi(t)\, X_0 = \underbrace{Q(t)}_{T\text{-periodic}}\, e^{tB} X_0$$

- Let/introduce $Y(t)$:

$$X(t) = Q(t)\,Y(t) \quad \text{or} \quad Y(t) = Q(t)^{-1} X(t).$$

Then

$$\dot{X}(t) - A(t)\,X(t) = Q(t)\left(\dot{Y}(t) - B\,Y(t)\right)$$

Hence

$$\dot{X} = A(t)\,X \iff \dot{Y} = BY$$

Stability Using Floquet Theory

$$\frac{dX}{dt} = A(t)\,X, \qquad X(0) = x$$

$$X(t) = \Phi(t)\,X_0 = Q(t)\,e^{tB}X_0$$

$$\begin{aligned} X(mT) &= Q(mT)\,e^{mTB}X_0 \\ &= Q(0)\,e^{mTB}X_0 \\ &= e^{mTB}X_0 \\ &= \left(e^{TB}\right)^m X_0 \\ &= M^m X_0 \end{aligned}$$

Def: Monodromy matrix

$$\boxed{M = e^{TB}}$$

Stability Using Floquet Theory

If the eigenvalues μ of M satisfy

$$|\mu| < 1$$

then
$$\|M^n x_0\| \le \underbrace{C\,|\mu|^n \|x_0\|}_{\substack{\downarrow \; n \to +\infty \\ 0}}$$

Hence
$$X(nT) = M^n X_0 \xrightarrow{n \to +\infty} 0$$

Stability Using Floquet Theory

- Consider: $\dot{Y} = BY$, $Y(0) = Y_0$;

  Solution: $Y(t) = e^{tB} Y_0$

  If $Re(\lambda(B)) < 0$, then $\exists\, K, \alpha > 0$ s.t.

  $$|Y(t)| \leq K e^{-\alpha t} |Y_0|, \quad t > 0.$$

- Consider: $Y(m) = M\, Y(m-1)$, $Y(0) = Y_0$;

  Solution: $Y(m) = M^m Y_0$

  If $|\lambda(M)| < 1$, then $\exists\, K > 0$, $0 < \alpha < 1$, s.t.

  $$|Y(m)| \leq K \alpha^m |Y_0|, \quad m > 0.$$

Stability Using Floquet Theory

$$M = e^{TB}$$

- Eigenvalues of M (μ_i) are called characteristic multipliers of M
- Eigenvalues of B (λ_i) are called characteristic exponents of B
- $\mu_i = e^{T\lambda_i}$

$$\underline{|\mu_i| < 1} \iff \underline{Re(\lambda_i) < 0}$$

M $\nearrow$ B $\nearrow$

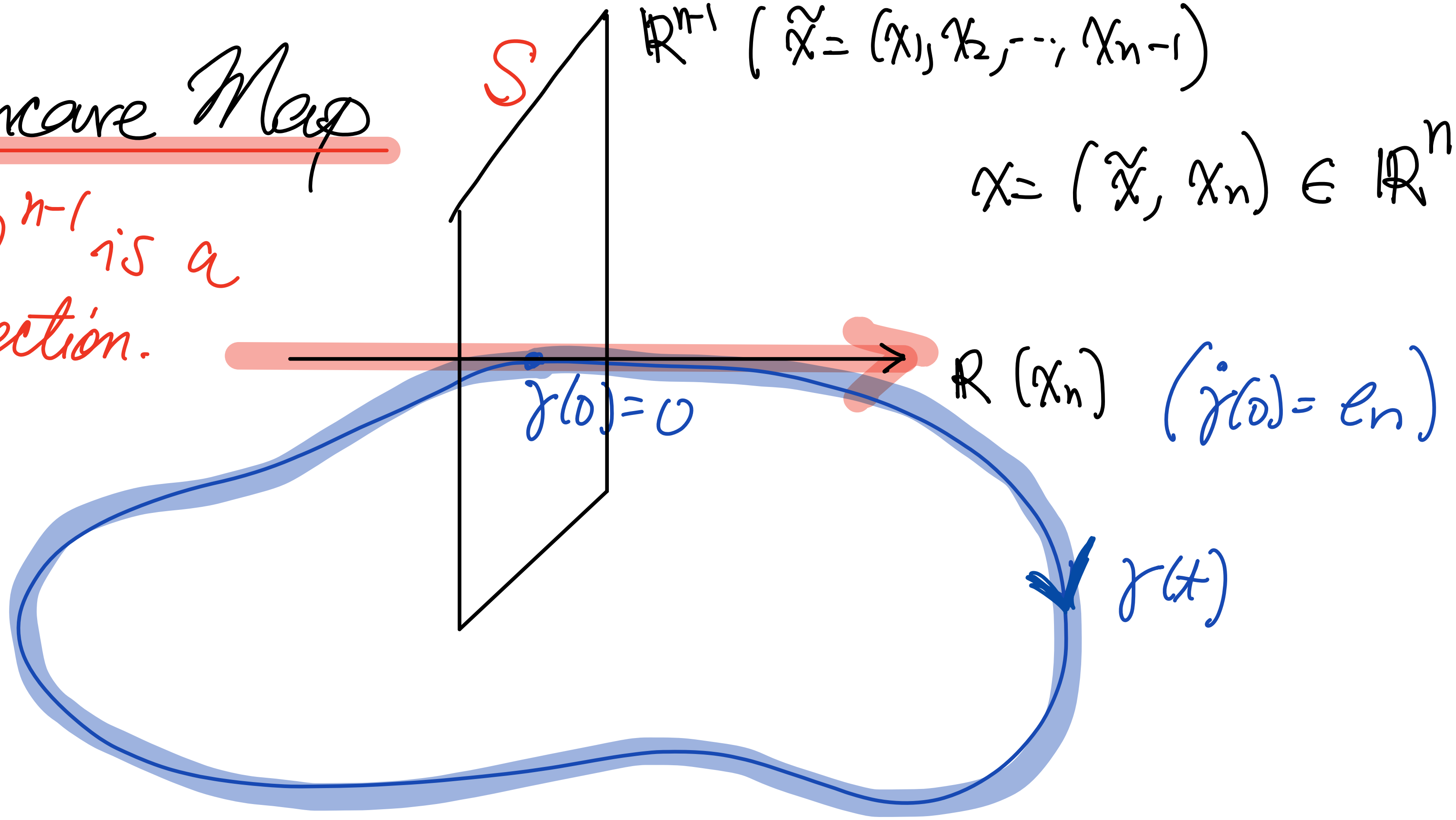
Poincare Map

$S \cong \mathbb{R}^{n-1}$ is a cross-section.

$\perp e_n$

$\mathbb{R}^{n-1}$ $(\tilde{x} = (x_1, x_2, \cdots, x_{n-1}))$

$x = (\tilde{x}, x_n) \in \mathbb{R}^n$



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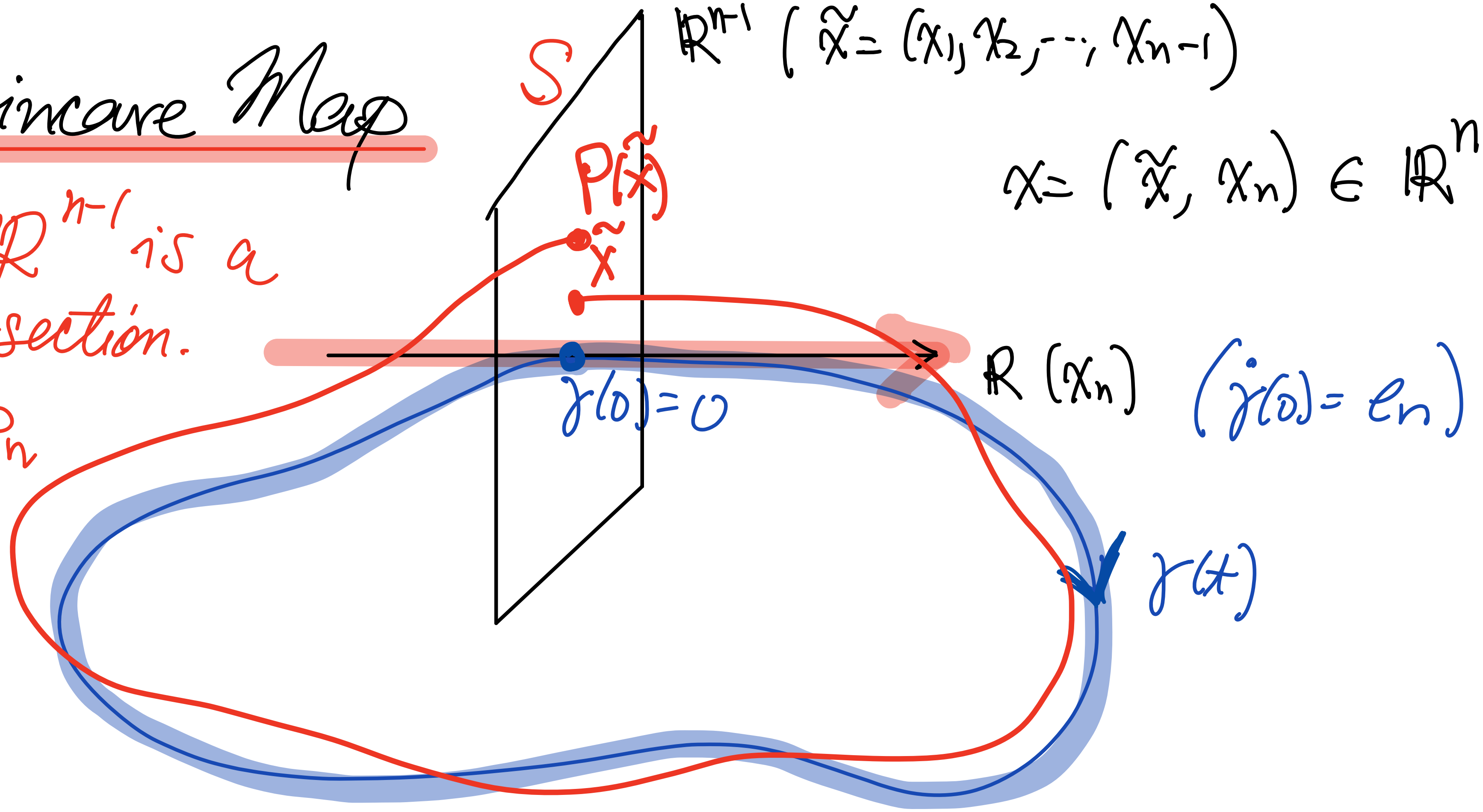


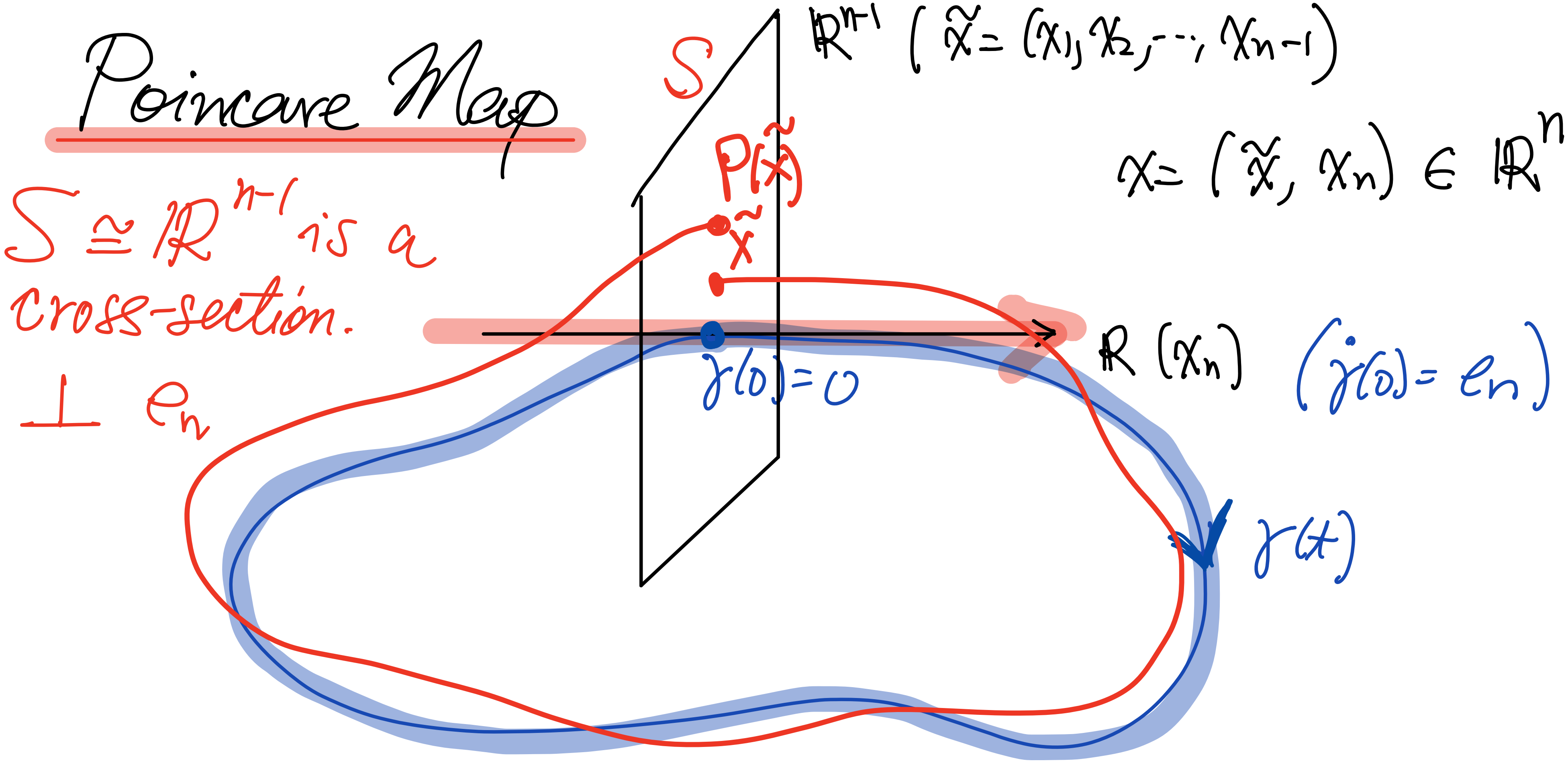
- For any $X(0) = \tilde{x} \in S$, close to $\gamma(0)$, there is a $\tau(x)$ s.t.

$$\underline{\phi_{\tau(\tilde{x})}(\tilde{x}) \in S}$$

- Poincare Map: $P(\tilde{x}) = \phi_{\tau(\tilde{x})}(\tilde{x})$

$P : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$
$(S \to S)$

Poincare Map



- The solution $X(t)$ starting at $\tilde{x} \in S$ is a periodic orbit if and only if x is a fixed point of P, i.e. $P(\tilde{x}) = \tilde{x}$

e.g. $P(0) = 0$

Poincare Map

Discrete Time Map

Stability of the periodic orbit γ is equivalent to the stability of 0 $(P(0)=0)$ under iterates of P, $\{P^m(x)\}_m$

- 0 is stable: $\forall \varepsilon, \exists \delta$ s.t. $\forall x \in S \cap B_\delta(0)$, then $P^m(x) \in B_\varepsilon(0)$ for any $m > 0$.

- 0 is asymptotically stable: if there is an $\varepsilon > 0$ s.t. for any $x \in S \cap B_\varepsilon(0)$, $P^m(x) \xrightarrow{m \to +\infty} 0$

- 0 is unstable: it is not stable, i.e. there is an $\varepsilon > 0$, s.t. $\forall x \in B_\varepsilon(0)$, $\exists m > 0$ s.t. $P^m(x) \notin B_\varepsilon(0)$

Relationship between Floquet Theory & Poincare Map.

$$M = e^{TB}$$

- Thm (M, 4.55)

collection of eigenvalues

$$\underbrace{\mathrm{Spec}(M)}_{n\times n \text{ matrix}} = \underbrace{\mathrm{Spec}(DP(0))}_{(n-1)\times(n-1) \text{ matrix}} \cup \underbrace{\{1\}}_{\text{Comes from time shift.}}$$

- Thm (M, 4.56)

If $\underline{\mathrm{Spec}(DP(0)) \subseteq \{\mu : |\mu| < 1\}}$,

then γ is asymptotically stable.

Relationship between Floquet Theory & Poincare Map.

Generally, the computation of the stability of a periodic orbit requires that we consider the linearization of the flow in the neighborhood of the periodic orbit. One must typically resort to numerical methods to solve for the Floquet multipliers, even if the periodic orbit is known analytically. It is often convenient numerically to compute the Poincaré map (4.53) and study stability of an orbit by this method. One advantage is that the Poincaré map acts on the section S that has dimension $n-1$, one less than the flow. Moreover, the removed dimension corresponds to the motion along the periodic orbit and thus to the neutral Floquet multiplier $\mu_1 = 1$. Consequently, stability computed using the Poincaré map is the same as that from the Floquet spectrum:

Theorem 4.55. *Let γ be a periodic orbit of a C^2 flow φ, S be a local section through a point $x_o \in \gamma$, and $x_o \subset S_o \subset S$ so that $\varphi_\tau(x)(x) \in S$ for each $x \in S_o$. Then there is a Poincaré return map $P : S_o \to S$. If the monodromy matrix of γ is M, then*

$$\mathrm{spec}(M) = \mathrm{spec}(DP(x_o)) \cup \{1\}.$$

Theorem 4.56. *If γ is a periodic orbit of a C^2 flow that is linearly asymptotically stable (the spectrum of its Poincaré map is inside the unit circle), then it is asymptotically stable.*

Relationship between Floquet Theory & Poincare Map.

$$
\Big[\; M \;\Big]^{n\times n} = \left[\begin{array}{c|c} DP(0)^{(n-1)\times(n-1)} & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \times\times\times\times\times\times\times\times & 1 \end{array}\right]
$$

$\mu = 1$

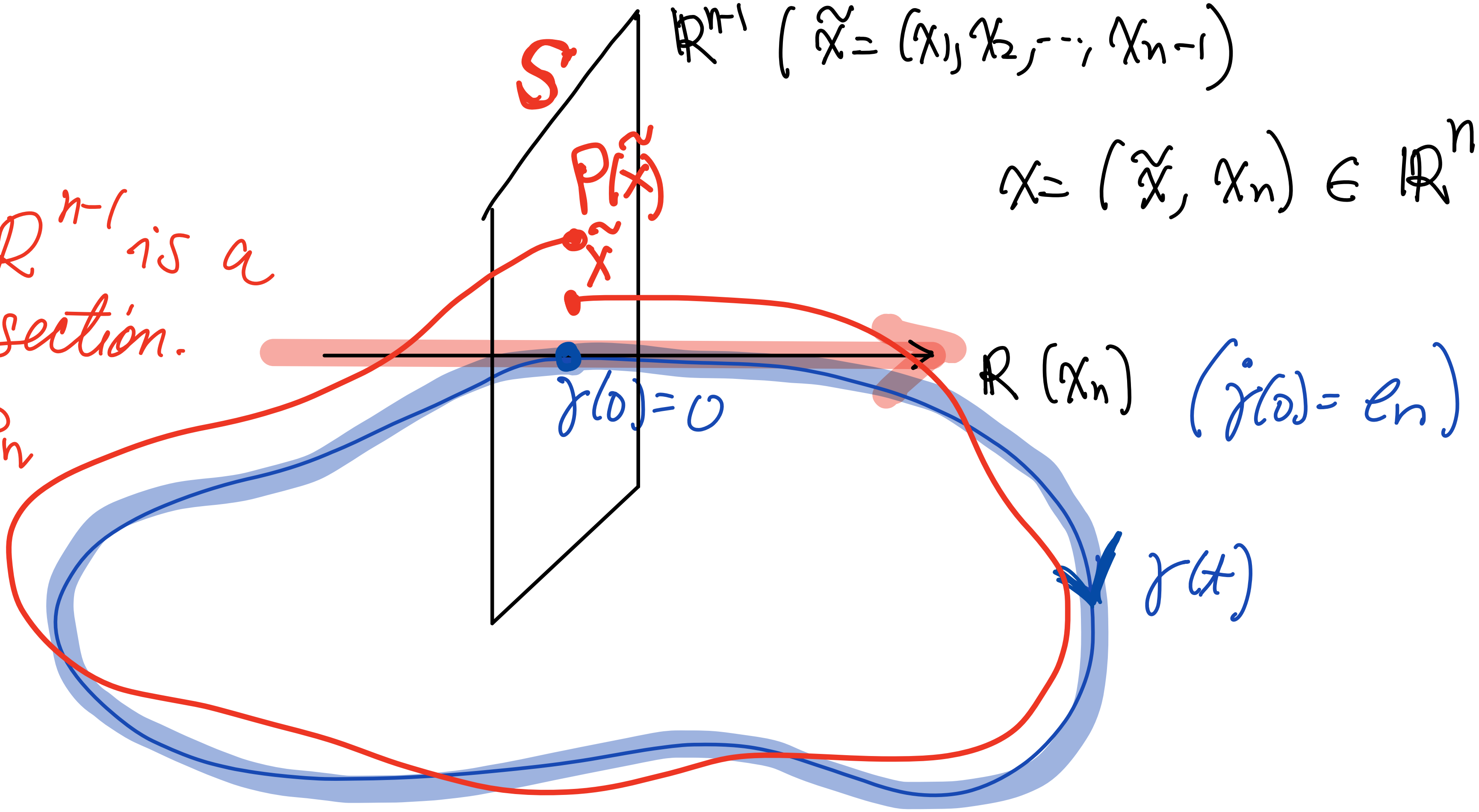
(time shift)

- $M: \quad x_n\text{-axis} \longrightarrow x_n\text{-axis}$
- $M\big|_S = [DP]$;

Relationship between Floquet Theory + Poincare Map.

$S \cong \mathbb{R}^{n-1}$ is a cross-section.

$\perp e_n$



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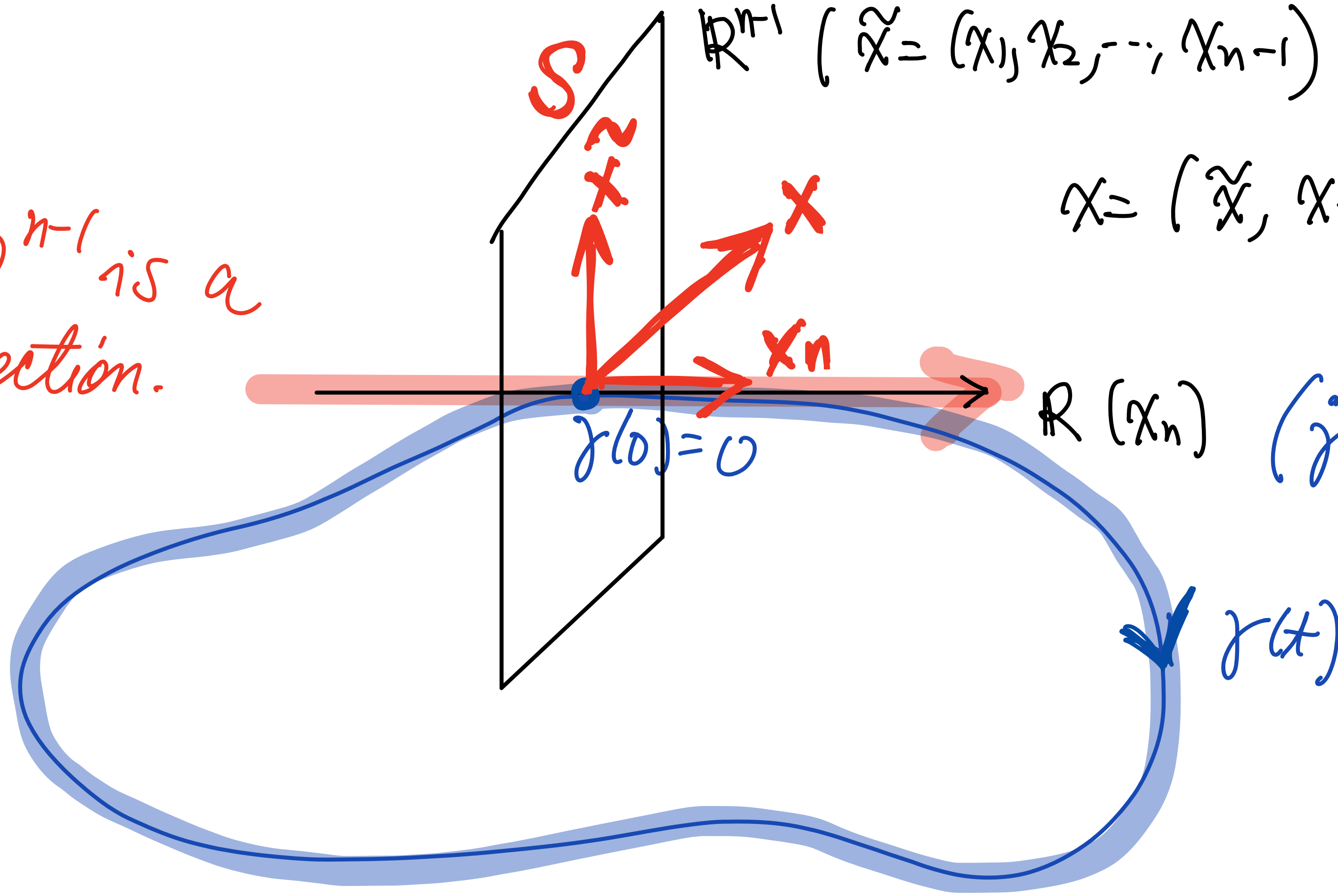


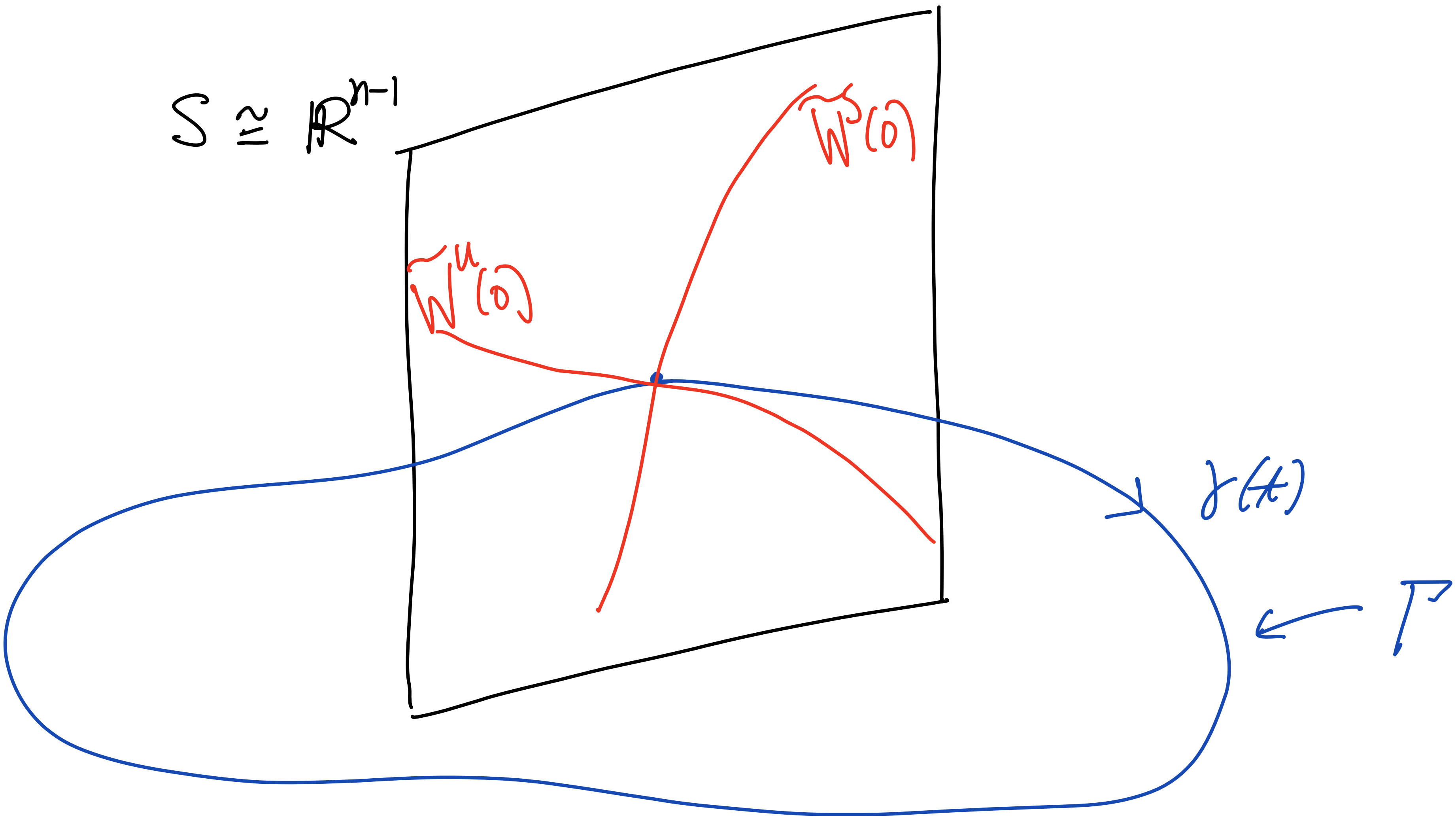
$x = (\tilde{x}, x_n) \in \mathbb{R}^n$

$(\dot{\gamma}(0) = e_n)$

Stable and Unstable Manifolds of γ

Suppose $\#\{\lambda_i : Re(\lambda_i(DP(0))) < 0\} = k$
$\#\{\lambda_i : Re(\lambda_i(DP(0))) > 0\} = n-k-1$ $\Bigg\}$ total $= n-1$

$S \cong \mathbb{R}^{n-1}$



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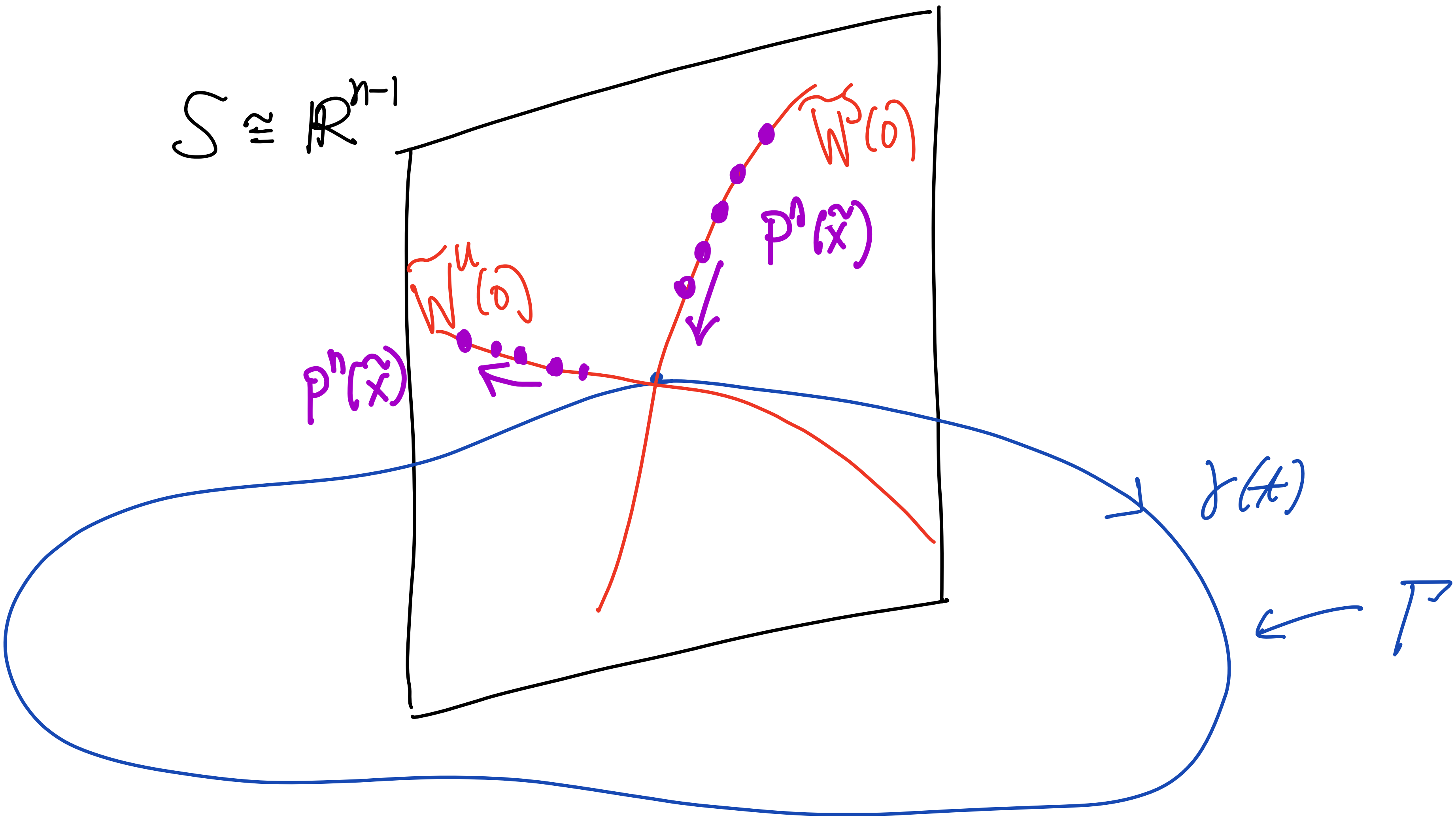

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Suppose
$$\left.\begin{array}{l} \#\{\lambda_i : Re(\lambda_i(DP(0))) < 0\} = k \\ \#\{\lambda_i : Re(\lambda_i(DP(0))) > 0\} = n-k-1 \end{array}\right\} \begin{array}{c} \text{total} \\ \shortparallel \\ n-1 \end{array}$$

- $\tilde{W}^S(0) \subseteq S \ (\cong \mathbb{R}^{n-1})$: stable manifold
- $\dim(\tilde{W}^S(0)) = k$
- $\forall x \in \tilde{W}^s(0)$, $P(x) \in \tilde{W}^s(0)$, i.e. $\tilde{W}^S(0)$ is inv. under P.

$$P^m(x) \longrightarrow 0 \quad \text{as} \quad m \longrightarrow +\infty$$

$$\phi_t(x) \longrightarrow \Gamma \quad \text{as} \quad t \longrightarrow +\infty$$

Orbital asymp. stable: $\quad \exists \delta : |\phi_t(x) - \gamma(t+\delta)| \leq K e^{-\alpha t}, \ t > 0$

Stable and Unstable Manifolds of γ

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- $\tilde{W}^u(0) \subseteq S \ (\cong \mathbb{R}^{n-1})$: unstable manifold
- $\dim(\tilde{W}^u(0)) = n-k-1$
- $\forall x \in \tilde{W}^u(0)$, $P^{-1}(x) \in \tilde{W}(0)$, i.e. $\tilde{W}(0)$ is inv. under P^{-1}.

$$P^m(x) \longrightarrow 0 \quad \text{as} \quad m \longrightarrow -\infty$$

$$\phi_t(x) \longrightarrow \Gamma \quad \text{as} \quad t \longrightarrow -\infty$$

Orbital asymp. "un" stable: $\exists \delta : |\phi_t(x) - \gamma(t+\delta)| \leq K e^{-\alpha t}, \ t<0$