Existence and Uniqueness of Solutions

(1) Does the solution exist for all $t > 0$?
(2) Is the solution unique?

Explicit Solution formula

(I) Linear Equation

$$\dot{x} = ax \quad - \; a \equiv \text{constant, lin. homog.}$$

$$\dot{x} = ax + h(t) \quad - \text{ linear inhomog.}$$

$$\boxed{\dot{x} = a(t)x + h(t), \quad x(0) = x_0}$$

integrating factor $I(t)$ $(=?)$

$I \times$

$$I(t)\dot{x} - I(t)a(t)x = I(t)h(t)$$

Let $I(t) = e^{-\int_0^t a(s)ds}$

Then $\dot{I}(t) = e^{-\int_0^t a(s)ds}(-a(t)) = -a(t)I$

$$I(t)\dot{x}(t) + \dot{I}(t)x(t) = I(t)h(t)$$

$$\frac{d}{dt}\left(I(t)x(t)\right) = I(t)h(t)$$

$$\int_0^t \frac{d}{ds}\left(I(s)x(s)\right)ds = \int_0^t \left(I(s)h(s)\right)ds$$

$$I(t)x(t) - \underbrace{\cancel{I(0)}}_{1}\,\underbrace{\cancel{x(0)}}_{x_0} = \int_0^t I(s)h(s)ds$$

$$I(t)x(t) = x_0 + \int_0^t I(s)h(s)ds$$

$$x(t) = I(t)^{-1}x_0 + I(t)^{-1}\int_0^t I(s)h(s)ds$$

$$= e^{\int_0^t a(r)dr}x_0$$

$$+ e^{\int_0^t a(r)dr}\int_0^t e^{\int_0^s -a(r)dr}h(s)ds$$

$$= e^{\int_0^t a(r)dr} x_0 + \int_0^t e^{\int_0^t a(r)dr} e^{\int_0^s -a(r)dr} h(s)$$

$$\boxed{x(t) = e^{\int_0^t a(r)dr} x_0 + \int_0^t e^{\int_s^t a(r)dr} h(s)ds}$$

(solution exists for as long as $\int_s^t a(r)dr$ is well-defined.)

(II) Nonlinear Example

$$\begin{cases} \dot{x} = x^p \\ x(0) = x_0 \ (>0) \end{cases}$$

$p > 1$ (eg. $p = 2$)

super-linear

$$\int \frac{dx}{x^p} = \int dt$$

$$\int_{x_0}^{x(t)} \frac{dx}{x^p} = \int_0^t dt = t$$

$$\left.\frac{x^{-p+1}}{-p+1}\right|_{x_0}^{x(t)} = t$$

$$\frac{x(t)^{-p+1} - x_0^{-p+1}}{-p+1} = t$$

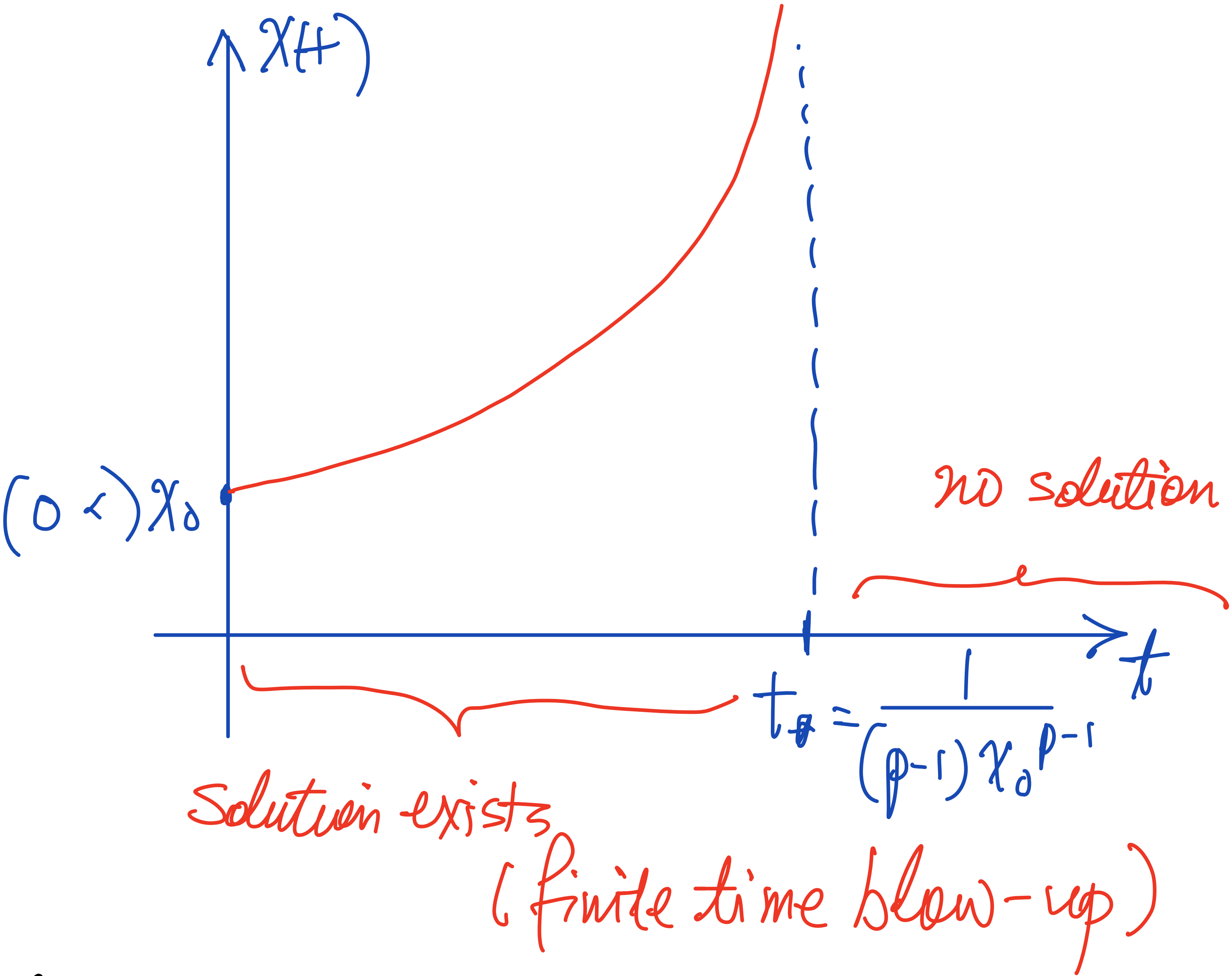
$$x(t)^{-p+1} = x_0^{-p+1} - (p-1)t$$

$$\boxed{x(t) = \left(\frac{1}{x_0^{-p+1} - (p-1)t}\right)^{\frac{1}{p-1}}}$$

Solution exists until

$$x_0^{-p+1} = (p-1)t$$

$$t = t_* = \left(\frac{1}{p-1}\right)\frac{1}{x_0^{p-1}}$$



(Note: The larger the x_0 is, the smaller is the existence interval.)

$$\begin{cases} \dot{x} = x^{p} & 0 < p < 1, \quad (\text{eg. } p = \frac{1}{2}) \\ x(0) = x_0 = 0 \end{cases}$$

$$\begin{cases} \dot{x} = x^{\frac{1}{2}} \\ x(0) = 0 \end{cases}$$

(1) Trivial Solution: $x(t) \equiv 0$

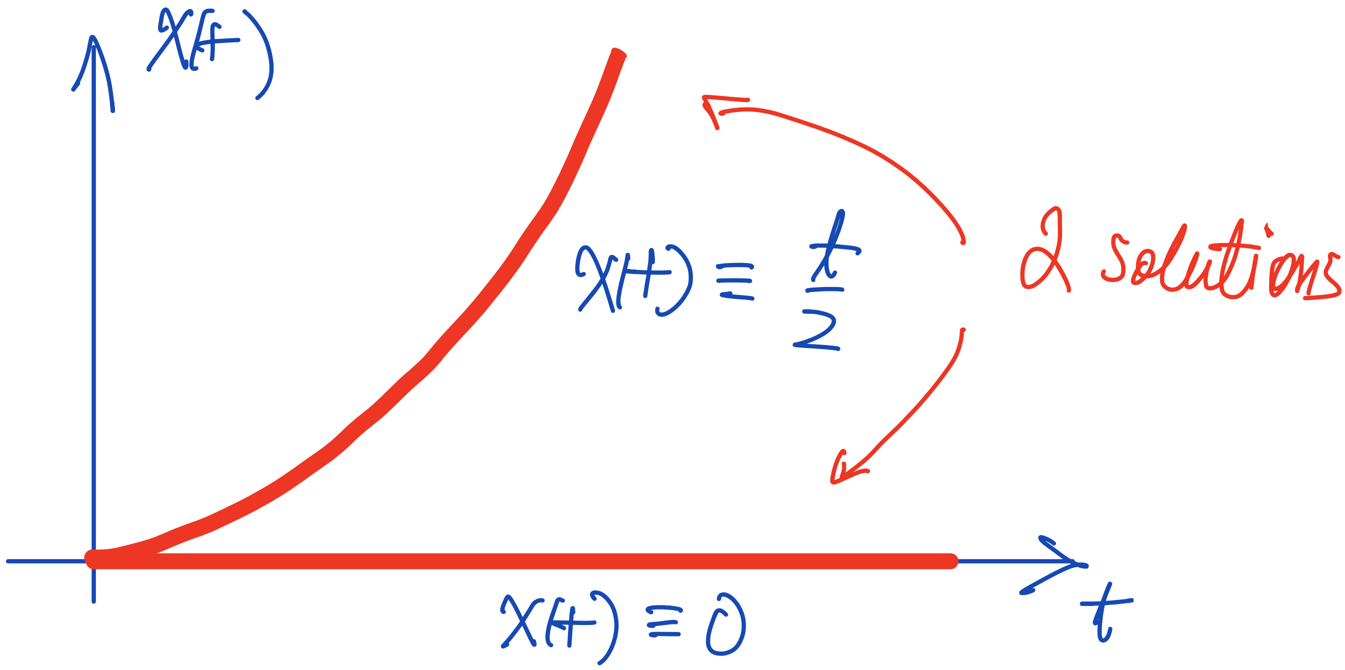
(2) Non-trivial Solution:

$$\int_0^{x(t)} \frac{dx}{x^{\frac{1}{2}}} = \int_0^t dt$$

$$2\left(x^{\frac{1}{2}}(t) - \cancel{x_0^{\frac{1}{2}}}\right) = t$$

$$\boxed{x(t) = \left(\frac{t}{2}\right)^2} \quad (x(0) = 0)$$

$$\left(\dot{x} = 2\left(\frac{t}{2}\right)\frac{1}{2} = \frac{t}{2} = x^{\frac{1}{2}} = \frac{t}{2}\right)$$



In fact, there are infinitely many soln's

$$x(t) = \begin{cases} 0 & 0 \le t \le t_* \\ \left(\frac{t-t_*}{2}\right)^2 & t \ge t_*, \end{cases}$$

$\boxed{t_* = \text{Waiting Period}}$