Linear System

Lec 3

$$\frac{d\vec{X}}{dt} = A\vec{X} + \vec{h}(t), \quad X(0) = X_0$$

$A \equiv$ a constant matrix (autonomous)

$$\frac{dX}{dt} = A(t)\,X + h(t)$$

A depends on time (non-autonomous)

$$\left(X,\ h \in \mathbb{R}^n;\quad A \in \mathbb{R}^{n\times n} \right)$$

Constant coefficient matrix A

$$e^{At} := I + \frac{tA}{1!} + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} + \cdots$$

$$\left(= \sum_{i=0}^{\infty} \frac{(tA)^i}{i!} \right)$$

Properties of e^{At} (Matrix Exponential)

(1) $e^{At}\Big|_{t=0} = I$

(2) $\frac{d}{dt}(e^{At}) = Ae^{At} = e^{At}A$

(3) $e^{At}e^{As} = e^{A(t+s)}$

(4) $(e^{At})^{-1} = e^{-At}$, i.e. $(e^{At})(e^{-At}) = I$

i.e. e^{At} is invertible

(5) If $AB = BA$, then

$$e^{At}e^{Bt} = e^{Bt}e^{At} = e^{(A+B)t}$$

(In fact, $e^{At}e^{Bt} = e^{(A+B)t}$ for all t if and only if $AB = BA$ [M, p.64 #6])

(Use $I(t)=e^{-At}$ as integrating factor.)

$$\frac{dX}{dt} = AX + h(t)$$

$$\frac{dX}{dt} - AX = h(t)$$

$$e^{-At}\frac{d}{dt}X \underbrace{- \; A e^{-At}}_{= \, \frac{d}{dt} e^{-At}} X = e^{-At} h(t)$$

$$e^{-At}\frac{d}{dt}X + \left(\frac{d}{dt}e^{-At}\right)X = e^{-At}h(t)$$

$$\frac{d}{dt}\left(e^{-At}X\right) = e^{-At}h(t)$$

$$\int_0^t \frac{d}{ds}\left(e^{-As}X(s)\right)ds = \int_0^t e^{-As}h(s)\,ds$$

$$e^{-At}X(t) - X(0) = \int_0^t e^{-As}h(s)\,ds$$

$$e^{-At}X(t) = X(0) + \int_0^t e^{-As}h(s)\,ds$$

$$\boxed{\begin{aligned} X(t) &= e^{At}X(0) + e^{At}\int_0^t e^{-As}h(s)\,ds \\ &= e^{At}X(0) + \int_0^t e^{A(t-s)}h(s)\,ds \end{aligned}}$$

(In particular, if $h(s) \equiv 0$, i.e. $\dot{X} = AX$, then, $X(t) = e^{At}X(0)$.)

Non-autonomous Case

$$\boxed{\frac{dX}{dt} = A(t)X + h(t), \quad X(0) = X_0}$$

Variation of Parameters

① Let $\{\Phi(t)\}_{t\geq 0}$ be a matrix $\overset{(n\times n)}{\text{function}}$ that solve:

$$\boxed{\frac{d\Phi(t)}{dt} = A(t)\Phi(t), \qquad \Phi(0) = I}$$

(Identity matrix)

$$[\Phi(t)]^{n\times n} = \begin{bmatrix} | & | & & | \\ \varphi_1(t) & \varphi_2(t) & \cdots & \varphi_n(t) \\ | & | & & | \end{bmatrix} \qquad \varphi_i \in \mathbb{R}^n$$

Then $\varphi_i(t)$ solves

$$\boxed{\frac{d}{dt}\varphi_i(t) = A(t)\varphi_i(t), \quad \varphi_i(0) = e_i}$$

② Let $X(t) = \Phi(t)\vec{u}(t)$ $\qquad \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix}$

$$\left(= u_1(t)\vec{\varphi}_1(t) + \cdots + u_n(t)\vec{\varphi}_n(t) \right)$$

Try to find $\vec{u}(t)$ (=?)

$$\frac{dX}{dt} = A(t)X + h(t), \quad X(0) = X_0$$

$$\frac{d}{dt}\left(\underline{\Phi(t)u(t)}\right) = A(t)\left(\underline{\Phi(t)u(t)}\right) + h(t)$$

$$\cancel{\left(\frac{d}{dt}\Phi(t)\right)}u(t) + \Phi(t)\frac{du(t)}{dt} = \cancel{A(t)\Phi(t)}u(t) + h(t)$$

$$\Phi(t)\frac{du(t)}{dt} = h(t)$$

$\Phi^{-1}(t)$ (assume exists)

$$\frac{du(t)}{dt} = \Phi^{-1}(t)h(t)$$

$$\int_0^t \left(\qquad \right) dt$$

$$u(t) - u(0) = \int_0^t \Phi^{-1}(s)h(s)\,ds$$

$$u(t) = u(0) + \int_0^t \Phi^{-1}(s)h(s)\,ds$$

$$X(t) = \Phi(t)\, u(t)$$

$$= \Phi(t)\, u_0 + \Phi(t) \int_0^t \Phi(s)^{-1} h(s)\, ds$$

$u_0 = ?$

$$X_0 = \Phi(0) u_0 + \cancel{\Phi(0) \int_0^0 \cdots ds}$$

$$= I\, u_0$$

$$\Longrightarrow u_0 = X_0$$

Hence

$$\boxed{\begin{aligned} X(t) &= \Phi(t) X_0 + \Phi(t) \int_0^t \Phi(s)^{-1} h(s)\, ds \\ &= \Phi(t) X_0 + \int_0^t \Phi(t)\Phi(s)^{-1} h(s)\, ds \end{aligned}}$$

Let $\Phi(t,s) = \Phi(t)\Phi(s)^{-1} \quad (0 \leq s \leq t)$

$$\Phi(t,0) = \Phi(t)$$

Then

$$\boxed{X(t) = \Phi(t,0) X_0 + \int_0^t \Phi(t,s)\, h(s)\, ds}$$

(In particular, if $h(s) \equiv 0$, then $X(t) = \Phi(t,0) X_0$)

$\Phi(t)$ is called the fundamental matrix:

$$\boxed{\frac{d\Phi(t)}{dt} = A(t)\Phi(t), \qquad \Phi(0) = I}$$

$\Phi(t,s)$ is called the propagation operator:

$$\boxed{\frac{d}{dt}\Phi(t,s) = A(t)\Phi(t,s), \qquad \Phi(s,s) = I}$$

Existence of $\Phi(t)$

(Method 1) Let $\Psi(t)$ be the solution of

$$\dot{\Psi}(t) = -\Psi(t)A(t), \qquad \Psi(0) = I$$

Consider $\frac{d}{dt}\left(\Psi(t)\Phi(t)\right) = ?$

(Method 2) Use Abel's Thm [M, Thm 2.34]

$$\frac{d\Phi(t)}{dt} = A(t)\Phi(t), \quad \Phi(0) = I$$

Then $\frac{d}{dt}\left(\det\Phi(t)\right) = \left(\operatorname{tr}A(t)\right)\left(\det\Phi(t)\right)$

$$\rightarrow \quad \det\Phi(t) = e^{\int_0^t (\operatorname{tr}A(s))\,ds}$$

......