Invariant Subspaces and Stability of ODEs (Lec 4)

$$\frac{dX}{dt} = AX + h(t), \quad X(0) = X_0 \qquad (A \equiv \text{a constant matrix})$$

Solution: $$X(t) = e^{At}X_0 + \int_0^t e^{A(t-s)} h(s)\, ds$$

Suppose A is diagonalizable, i.e.

A has n (linearly independent) eigenvectors, $\{v_i\}_{i=1}^n$

$\{ Av_i = \lambda_i v_i \}_{i=1}^n$

$\Rightarrow$ $\boxed{\text{Decompose the space } (\mathbb{R}^n) \text{ into } n \text{ invariant subspaces (directions)}}$

① $\{v_i\}_{i=1}^n$ forms a basis for $\mathbb{R}^n$

② Write $X(t) = \overset{?}{C_1(t)}\, v_1 + \overset{?}{C_2(t)}\, v_2 + \cdots + \overset{?}{C_n(t)}\, v_n$

$h(t) = b_1(t)\, v_1 + b_2(t)\, v_2 + \cdots + b_n(t)\, v_n$

substitute into equation

$$\frac{d}{dt}\left(\sum_i c_i(t) v_i\right) = A\left(\sum_i c_i(t) v_i\right) + \sum_i b_i(t) v_i$$

$$\sum_i \left(\frac{d}{dt} c_i(t)\right) v_i = \sum_i c_i(t) \underline{A v_i} + \sum_i b_i(t) v_i$$

$\lambda_i v_i$

$$\sum_i \left(\frac{d}{dt} c_i(t)\right) v_i = \sum_i \lambda_i c_i(t) v_i + \sum_i b_i(t) v_i$$

compare the coefficients for each i

$$\frac{d}{dt} c_i(t) = \lambda_i c_i(t) + b_i(t) \qquad i = 1, 2, \cdots, n$$

Variation of parameter formula

Solve for each i, independently of each other

$$c_i(t) = e^{\lambda_i t} c_i(0) + \int_0^t e^{\lambda_i (t-s)} b_i(s)\, ds$$

$t=0$

$$X(t) = \sum_i c_i(t) v_i \qquad \left(X(0) = \sum_i c_i(0) v_i \right)$$

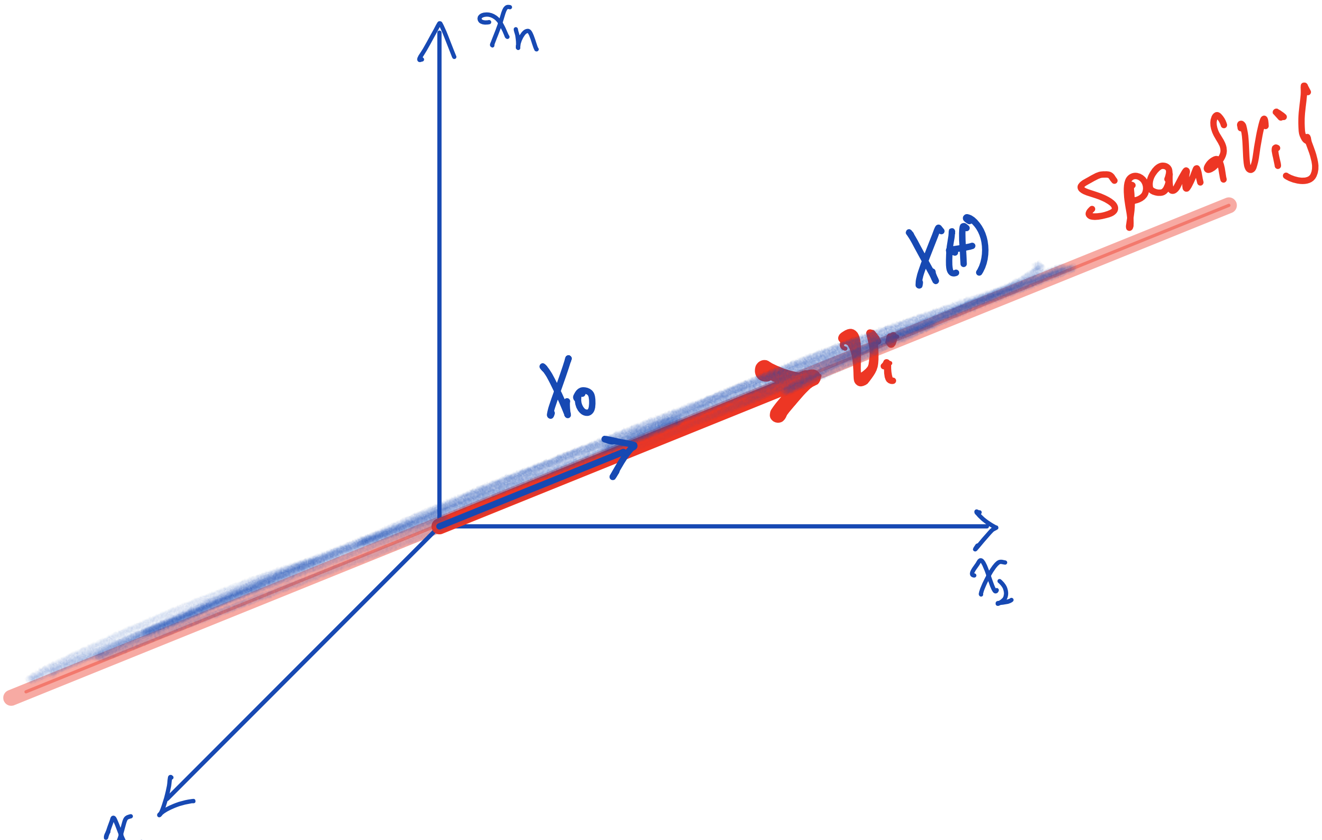
Invariant Subspaces

Consider $\frac{d}{dt}X = AX, \quad X(0) = X_0$

just one term

Suppose initially $(t=0)$ $\underline{X_0 = c_i(0)V_i \in \text{Span}\{V_i\}}$

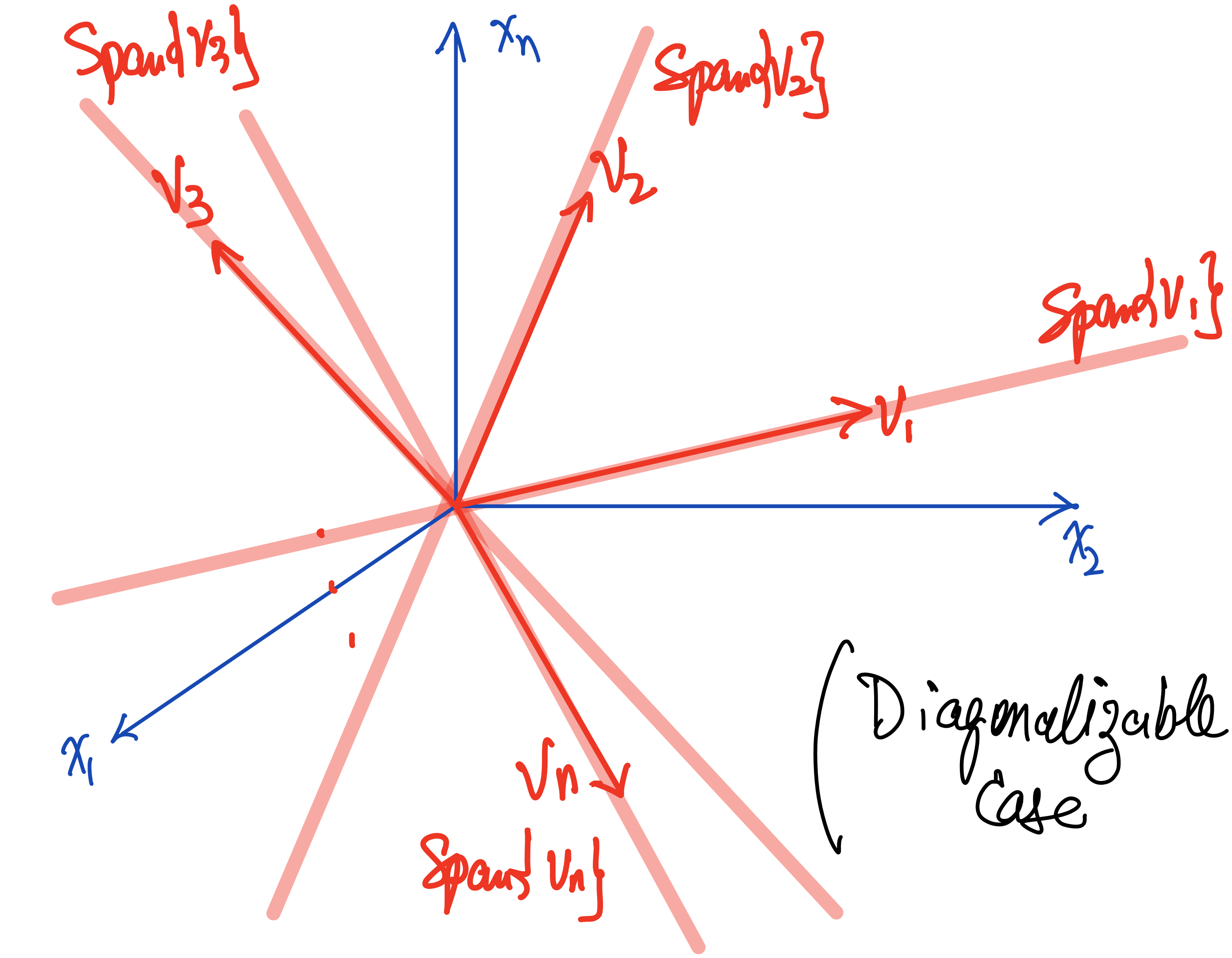
Then $\underline{X(t) \in \text{Span}\{V_i\}}$ for all t



$$\boxed{\text{Span}\{V_i\} \text{ is invariant under the dynamics}}$$

Eigenvector: $AV_i = \lambda V_i \iff (A - \lambda_i I)V_i = 0$

$E_{\lambda_i} = \text{Span}\{V_i\} = \{cV_i : c \in \mathbb{R}\} = \text{Null}(A - \lambda_i I)$



(Diagonalizable case)

Pf (M1): from solution formula:

$$X_0 = c_i(0) v_i \Rightarrow X(t) = e^{\lambda_i t} c_i(0) v_i)$$

M2:

$$X(t) = e^{At} X_0$$

$$= \left(I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots \right) X_0$$

$$= \left(I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \cdots \cdots \right) (c_i(0) v_i)$$

$$= c_i(0)\left(v_i + \frac{tAv_i}{1!} + \frac{t^2A^2v_i}{2!} + \cdots\right)$$

$$= c_i(0)\left(v_i + \frac{t\lambda_i v_i}{1!} + \frac{t^2\lambda_i^2 v_i}{2!} + \cdots\right)$$

$$= c_i(0)\left(1 + \frac{t\lambda_i}{1!} + \frac{(t\lambda_i)^2}{2!} + \cdots\right)v_i$$

$$= e^{\lambda_i t} c_i(0) v_i \in \text{Span}\{v_i\}$$

For $\frac{dX}{dt} = AX + h(t)$, $X(0) = \overset{= c_i(0)v_i}{X_0} \in \text{Span}\{v_i\}$

If $\overset{= b_i(t)v_i}{h(t)} \in \text{Span}\{v_i\}$ for all t,

then $X(t) \in \text{Span}\{v_i\}$ for all t.

Pf (M1: from solution formula:

$X(t) = c_i(t)\, v_i \leftarrow$ only one term)

M2:

$$X(t) = e^{At}\underbrace{X_0}_{c_i(0)v_i} + \int_0^t e^{A(t-s)}\underbrace{h(s)}_{b_i(s)v_i}\,ds$$

What if A is not diagonalizable?
(eg when eigenvalues of A repeats)

① $A \rightarrow$ characteristic polynomial

② $p(\lambda) = \det(A - \lambda I)$
$$= (-1)^n (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k}$$
$$m_1 + m_2 + \cdots + m_k = n$$

③ $\mathbb{R}^n = E_1 \oplus E_2 \oplus \cdots \oplus E_k$

④ $\dim(E_i) = m_i$, with basis vectors $\{v_1^i, v_2^i, \cdots v_{m_i}^i\}$

(Generalized) eigenvectors which satisfy:

how to find them?
Jordan Canonical Form

$$(A - \lambda_i I)^{m_i} v_j^i = 0, \quad j = 1, 2, \cdots, m_i$$

or $v_j^i \in \mathrm{Null}\left((A - \lambda_i I)^{m_i}\right)$

⑤ Generalized eigenspace

$$E_i \; (\text{or } E_{\lambda_i}) = \{ v : (A - \lambda_i I)^{m_i} v = 0 \}$$
$$= \mathrm{Null}\left((A - \lambda_i I)^{m}\right)$$

Each E_i is invariant under the dynamics

$$\frac{dX}{dt} = AX$$

If $X(0) \in E_i$, then $X(t) \in E_i$ for all t

Pf $\quad X(t) = e^{At} X_0$

$$= \left(I + \frac{At}{1!} + \frac{A^2 t^2}{2!} + \cdots \right) X_0$$

$$= \left(X_0 + \frac{t A X_0}{1!} + \frac{t^2 A^2 X_0}{2!} + \cdots \right)$$

If $X_0 \in E_i$ i.e. $(A - \lambda_i I)^{m_i} X_0 = 0$

then $A X_0 \in E_i$: $\quad (A - \lambda_i I)^{m_i} A X_0$

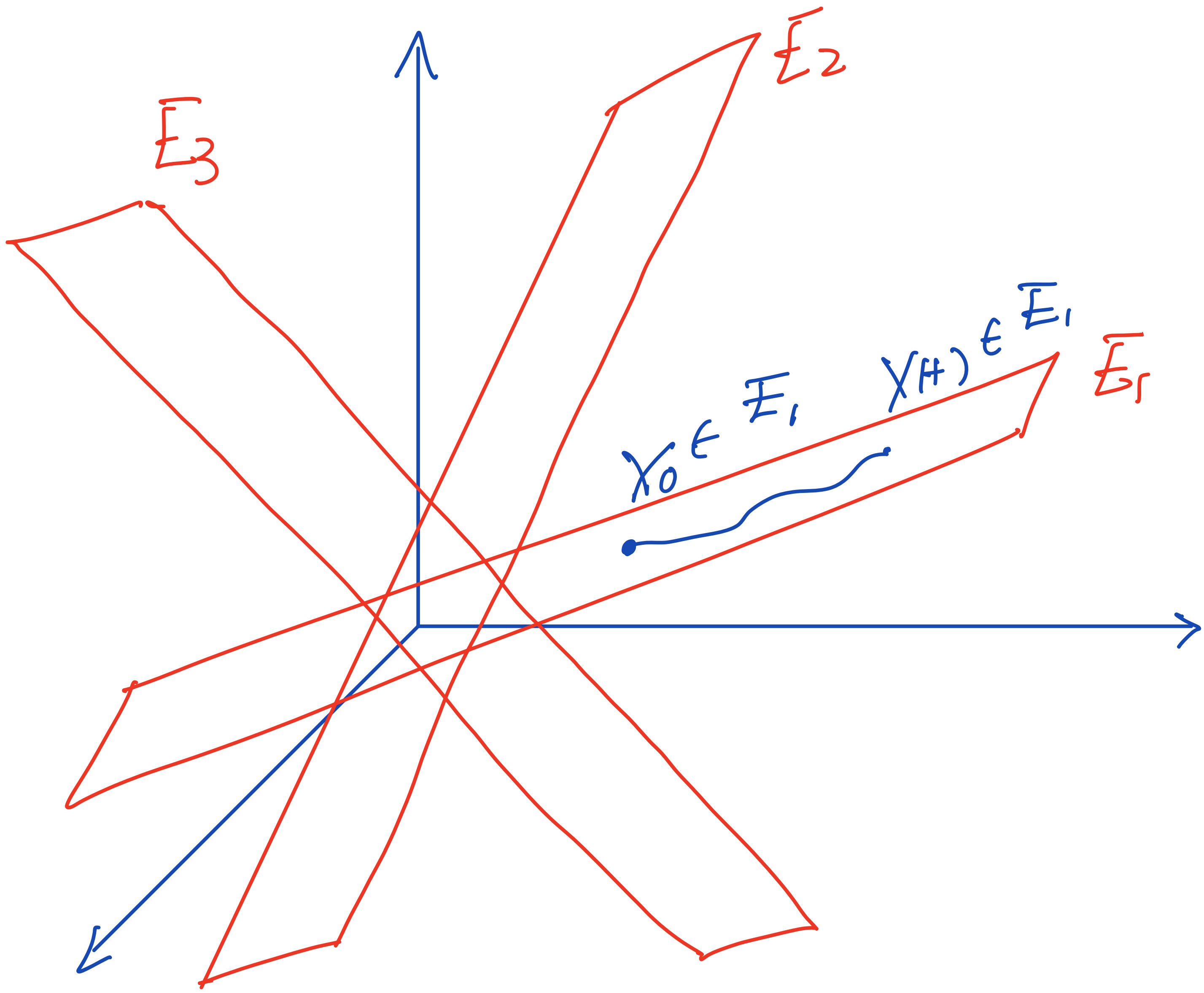
$$= A (A - \lambda_i I)^{m_i} X_0 = 0$$

$A^2 X_0 \in E_i$: $\quad (A - \lambda_i I)^{m_i} A^2 X_0$

$$= A^2 (A - \lambda_i I)^{m_i} X_0 = 0$$

$\vdots \qquad \vdots$

$A^l X_0 \in E_i \quad \cdots\cdots$



Similarly, for $\dfrac{dX}{dt} = AX + h(t)$,

$X(0) \in E_i$, $h(t) \in E_i$ for all t

then $X(t) \in E_i$ for all t.

Pf: $X(t) = e^{At} X_0 + \int_0^t e^{A(t-s)} h(s)\, ds$

Stable, Unstable and Center Subspaces

(E_s) (E_u) (E_c)

for $\frac{dX}{dt} = AX$, $X(0) = X_0$ $\quad AV_i = \lambda V_i$

Consider an eigenspace $E_{\lambda_i} = \mathrm{Span}\{V_i\}$, λ_i

① If $\lambda_i < 0$, $X(0) = c_i(0) V_i$,

then $X(t) = \underbrace{\left(e^{\lambda_i t} c_i(0)\right)}_{\substack{\to 0 \text{ as } t \to +\infty \\ \to (\pm)\infty \text{ as } t \to -\infty}} V_i$

$\|X(t)\| \leq K \|X(0)\| e^{\lambda_i t}$ as $t \to +\infty$

② If $\lambda_i > 0$, $X(t) = \underbrace{\left(e^{\lambda_i t} c_i(0)\right)}_{\substack{\to (\pm)\infty \text{ as } t \to +\infty \\ \to 0 \text{ as } t \to -\infty}} V_i$

$\|X(t)\| \leq K \|X(0)\| e^{\lambda_i t}$ as $t \to -\infty$

③ <u>If $\lambda_i = 0$,</u> $\quad x(t) = \underbrace{\left(e^{\lambda_i t} c_i(0)\right) v_i}_{\text{remains bounded as } t \to \pm\infty}$

Often times λ_i can be complex

$\lambda_i = a_i + b_i \qquad a_i = Re(\lambda_i),\ b_i = Im(\lambda_i)$

$a_i = Re(\lambda_i) < 0 \Rightarrow$ Same as ①

$a_i = Re(\lambda_i) > 0 \Rightarrow$ Same as ②

$a_i = Re(\lambda_i) = 0 \Rightarrow$ Same as ③

$$e^{\lambda_i t} = e^{(a_i + i b_i)t}$$

$$= e^{a_i t} e^{i b_i t}$$

$$= e^{a_i t} \Big(\underbrace{\cos(b_i t) + i \sin(b_i t)}_{\text{bounded}} \Big)$$

$\to \pm\infty$ as $t \to \pm\infty$

Stable Subspaces

$$E_s = \bigoplus_{\mathrm{Re}\,\lambda_i < 0} E_{\lambda_i}$$

Unstable Subspaces

$$E_u = \bigoplus_{\mathrm{Re}\,\lambda_i > 0} E_{\lambda_i}$$

Center Subspaces

$$E_c = \bigoplus_{\mathrm{Re}\,\lambda_i = 0} E_{\lambda_i}$$

(can be generalized eigenspace)

Properties of E_s, E_u, E_c

① Each of E_s, E_u, E_c is invariant under e^{At}

i.e. if $x_0 \in E_s$ (E_u, E_c),

then $e^{At}x_0 \in E_s$ (E_u, E_c)

② If $x_0 \in E_s$, then there $C, K > 0$ s.t.

$$\| e^{At} x_0 \| \leq C e^{-Kt} \| x_0 \| \text{ for all } t > 0.$$

($\to 0$ exponentially fast as $t \to +\infty$)

If $X_0 \in E_u$, then there $C, K > 0$ s.t.

$$\| e^{At} X_0 \| \leq C e^{-Kt} \|X_0\| \text{ for all } t < 0.$$

($\to 0$ exponentially fast as $t \to -\infty$)

In general, it is not true that

if $X_0 \in E_c$, then $e^{At} X_0$ remains bounded.

eg. $$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}^{A}\begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 + y_0 t \\ y_0 \end{pmatrix} \to \infty \text{ if } y_0 \neq 0$$

$$e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

③ The convergent rate constant K in general is not equal to the eigenvalues (but almost ...)

eg $\quad \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

$$\frac{dy}{dt} = -y \implies y(t) = e^{-t} y_0$$

$$\frac{dx}{dt} = -x + y = -x + \underbrace{e^{-t} y_0}_{h(t)}$$

$$x(t) = e^{-t} x_0 + \int_0^t e^{-(t-s)} \underbrace{\left(e^{-s} y_0\right)}_{h(s)} ds$$

$$= e^{-t} x_0 + \int_0^t e^{-t} \cancel{e^{s}} \cancel{e^{-s}} y_0 \, ds$$

$$= e^{-t} x_0 + e^{-t} t y_0$$

$$= e^{-t}(x_0 + t y_0)$$

$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$

$e^{At} = \begin{pmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{pmatrix}$

$$|x(t)| = e^{-t}|x_0 + t y_0| \leq \underline{C e^{-Kt} \left\| \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\|, \quad 0 < K < 1}$$

Examples of e^{At}, when A is not diagonalizable

① $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (From solution formula, $e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$)

$$e^{At} = I + At + \underbrace{\frac{A^2t^2}{2!} + \frac{A^3t^3}{2!} + \cdots}$$

$A^2 = A^3 = \cdots = 0$, A is nilpotent

$$= I + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

② $A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ (From solution formula, $e^{At} = \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix}$)

$$A = \underbrace{\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}}_{D} + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{N} = D + N$$

diagonal

Nilpotent

$DN = ND$, $N^2 = 0$

$$e^{At} = e^{(D+N)t} = e^{Dt}e^{Nt}$$

$$= e^{Dt}\left(I + \frac{Nt}{1!} + \cancel{\frac{N^2t^2}{2!} + \cdots}\right)$$

$$= \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

③ Jordan Canonical Form:

$$A = P J P^{-1} \qquad J = \begin{bmatrix} \boxed{J_1} & & & \\ & \boxed{J_2} & & \\ & & \ddots & \\ & & & \boxed{J_k} \end{bmatrix}$$

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix} = \underbrace{\lambda_i \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}}_{D_i = \lambda_i I} + \underbrace{\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}}_{N_i}$$

$D_i N_i = N_i D_i$, $\quad N_i$ – nilpotent, $N^{\square} = 0$

$$e^{At} = P e^{Jt} P^{-1}$$

$$e^{Jt} = \begin{bmatrix} \boxed{e^{J_1 t}} & & & \\ & \boxed{e^{J_2 t}} & & \\ & & \ddots & \\ & & & \boxed{e^{J_k t}} \end{bmatrix}$$

$$e^{J_i t} = e^{(D_i + N_i)t}$$

$$= e^{D_i t} e^{N_i t}$$

$$= \underbrace{\begin{bmatrix} e^{\lambda_i t} & & & \\ & e^{\lambda_i t} & & \\ & & \ddots & \\ & & & e^{\lambda_i t} \end{bmatrix}} \underbrace{\left(I + \frac{N_i t}{1!} + \frac{N_i^2 t^2}{2!} + \cdots \right)}_{\text{at most finitely many terms,}}$$

$$= e^{\lambda_i t} I \, e^{N_i t}$$

$$= \underbrace{\left(e^{\lambda_i t}\right)}_{\text{exponential}} \underbrace{e^{N_i t}}_{\text{polynomial}}$$