Linear (In-) Stability — Perturbative Analysis

From properties of $\frac{dX}{dt} = AX$ to — homogeneous sys.

properties of $\frac{d}{dt}X = (A + B(t))X + h(t)$

inhomogeneous sys.

(I) Consider $\frac{dX}{dt} = AX$.

Suppose all solutions of the above go to zero as $t \to +\infty$.

($\Leftrightarrow$ $Re(\lambda_i) < 0$, for all λ_i of A)

Then for $h(t)$, s.t. $\|h(t)\| \leq Ce^{-\delta t}$.

The solution of

$$\frac{dX}{dt} = AX + h(t), \quad X(0) = X_0$$

also go to zero as $t \to +\infty$

(In fact, exponentially fast.)

<u>Pf</u> $\mathrm{Re}(\lambda_i) < 0$ for all λ_i $(i = 1, \cdots, n)$

$$\Rightarrow \| e^{At} x_0 \| \le C_0 e^{-K_0 t} \| x_0 \| , \text{ for } t > 0$$

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-s)} h(s) \, ds$$

$$\| x(t) \| \le \| e^{At} x_0 \| + \int_0^t \| e^{A(t-s)} h(s) \| ds$$

$$\le C_0 e^{-Kt} \| x_0 \| + \int_0^t C_0 e^{-K_0 (t-s)} \| h(s) \| ds$$

$$\le C_0 e^{-Kt} \| x_0 \| + \int_0^t C_0 e^{-K(t-s)} C e^{-\delta s} ds$$

$$= C_0 e^{-Kt} \| x_0 \| + C_0 C e^{-Kt} \int_0^t e^{(K-\delta)s} ds$$

$$= C_0 e^{-Kt} \| x_0 \| + C_0 C e^{-Kt} \left(\frac{e^{(K-\delta)t} - 1}{K - \delta} \right)$$

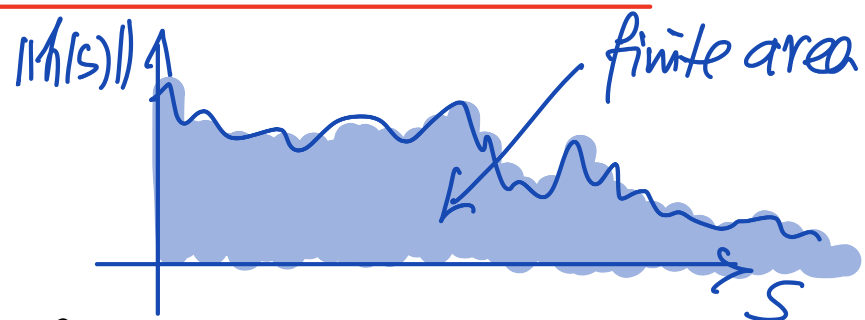
(what if $K = \delta$?)

$$= \underbrace{C_0 e^{-Kt}\|X_0\|}_{\downarrow \atop 0} + \underbrace{C_0 C \frac{\left(e^{-\delta t} - e^{-Kt}\right)}{K-\delta}}_{\downarrow \atop 0}$$

exp. fast.

(II) Under the same assumption as in (I) for A ($Re(\lambda_i) < 0$ for all λ_i's)

Suppose $\displaystyle\int_0^\infty \|h(s)\|\,ds = C_2 < \infty$



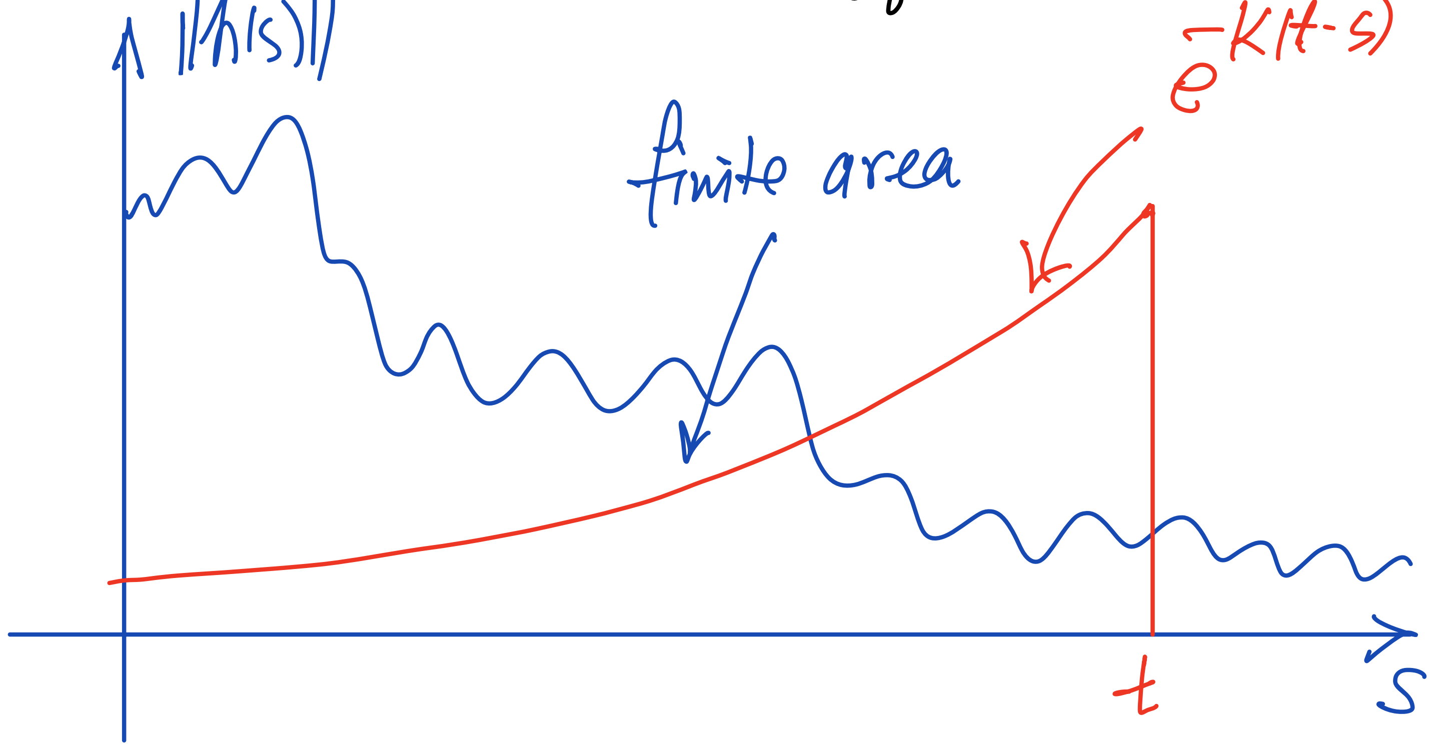
Then the solution of

$$\frac{dX}{dt} = AX + h(t), \quad X(0) = X_0$$

also go to zero as $t \to +\infty$ (but with no explicit rate)

Pf
$$X(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} h(s)\,ds$$

$$\|X(t)\| \le C_0 e^{-K_0 t}\|x_0\| + C_0\int_0^t e^{-K(t-s)}\|h(s)\|\,ds$$



Let ε be any small number, eg $\varepsilon = 10^{-6}$

Choose $N \gg 1$ s.t.

(1) $\int_N^\infty \|h(s)\|\,ds \le \varepsilon$

(2) $e^{-K_0 t} \le \varepsilon$ for $t \ge N$

Consider $\int_0^t e^{-K(t-s)} \|h(s)\| ds$, $\underline{t \geq 2N}$

$$= \int_0^N \underbrace{e^{-K(t-s)}}_{} \|h(s)\| ds + \underbrace{\int_N^t \underbrace{e^{-K(t-s)}}_{\leq 1} \|h(s)\| ds}_{}$$

$t \geq 2N$

$s \leq N$

$\Rightarrow t-s \geq N$

$\Rightarrow e^{-K_0(t-s)} \leq \varepsilon$

$\Rightarrow \int_0^N e^{-K_0(t-s)} \|h(s)\| ds$

$\leq \varepsilon \int_0^N \|h(s)\| ds$

$\leq C_2 \varepsilon$

$\leq \int_N^t \|h(s)\| ds$

$\leq \int_N^\infty \|h(s)\| ds < \varepsilon$

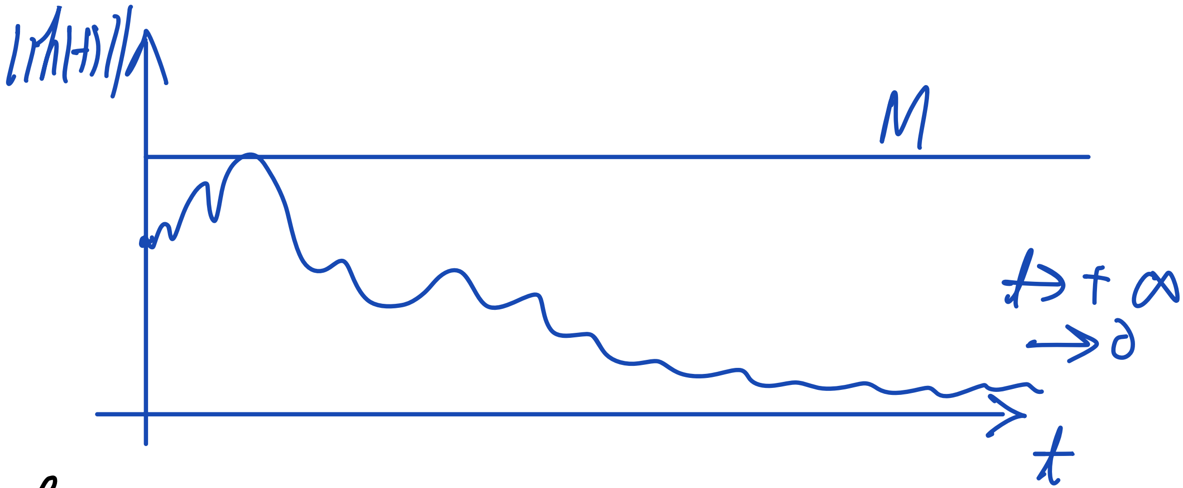
$\lesssim \varepsilon$

$$\|X(t)\| \leq \underbrace{C_0 e^{-K_0 t} \|X_0\| + C_2 \varepsilon + \varepsilon}_{\lesssim C\varepsilon}$$

(III) Under the same assumption as in (I) for A ($Re(\lambda_i) < 0$ for all λ_i's)

Suppose $\lim_{t \to +\infty} \|h(t)\| = 0$

and $\|h(t)\| \leq M$ for all t.



Then $\lim_{t \to \infty} \|X(t)\| = 0$

(The proof is very similar.)

For any $\varepsilon > 0$ (e.g. $\varepsilon = 10^{-6}$)

Choose N s.t. for all $t \geq N$

(1) $\|h(t)\| \leq \varepsilon$ and (2) $e^{-K_0 t} < \varepsilon$

Consider $\int_0^t e^{-K(t-s)} \|h(s)\| ds$, $\underline{t \geq 2N}$

$$= \int_0^N e^{-K(t-s)} \underbrace{\|h(s)\|}_{\leq M} ds + \int_N^t e^{-K(t-s)} \underbrace{\|h(s)\|}_{\leq \varepsilon} ds$$

$t \geq 2N$

$s \leq N$

$t - s \geq N$

$$\leq M \int_0^N e^{-K_0(t-s)} ds$$

$$= M e^{-K_0 t} \int_0^N e^{K_0 s} ds$$

$$= M e^{-K_0 t} \frac{e^{K_0 N} - 1}{K_0}$$

$$= \frac{M}{K_0} \left(e^{-K_0(t-N)} - e^{-K_0 t} \right) \leq \frac{M}{K_0} \varepsilon$$

$$\leq \varepsilon \int_N^t e^{-K_0(t-s)} ds$$

$$\leq \frac{\varepsilon}{K_0}$$

(III) Gronwall Inequality [M, p.90, Lem. 3.28]

Suppose $k(t) \geq 0$, and $g(t)$ satisfies

$$g(t) \leq c + \int_0^t k(s)\, g(s)\, ds$$

Then

$$g(t) \leq c\, e^{\int_0^t k(s)\, ds}$$

Pf Let $G(t) = \underbrace{c + \int_0^t k(s)\, g(s)\, ds}$

Then $G(0) = c$

$$\dot{G}(t) = k(t)\, g(t) \leq k(t)\, G(t)$$

$$\dot{G}(t) - k(t)\, G(t) \leq 0$$

Integrating factor: $I(t) = e^{-\int_0^t k(s)\, ds} > 0$

$$\left(\dot{I}(t) = -I(t)\, k(t) \right)$$

$$I(t)\dot{G} - k(t) I(t) G \leq 0$$

$$I(t)\dot{G} + \dot{I}(t) G \leq 0$$

$$\int_0^t \quad \frac{d}{dt}\left(I(t) G(t)\right) \leq 0$$

$$I(t)G(t) - I(0)G(0) \leq 0$$

$$\underbrace{I(t)}_{e^{-\int_0^t k(s)ds}} G(t) \leq \underbrace{I(0)}_{I} \underbrace{G(0)}_{C}$$

$$G(t) \leq C e^{\int_0^t k(s)\,ds}$$

as $\quad g(t) \leq G(t)$

Then $\quad g(t) \leq C e^{\int_0^t k(s)ds}$

In particular,

if $C = 0$, then $g(t) \leq 0$

if furthermore, $g(t) \geq 0$, then $g(t) = 0$

(IV) (Perturbation of A) [Bellman, p.34, Thm 1]

Suppose all solutions of $\frac{dX}{dt} = AX$ are bounded as $t \to \infty$, i.e.

$$\|X(t)\| = \|e^{At}X_0\| \leq C\|X_0\| \text{ for all } t$$

Then the same holds for the solution of

$$\frac{dX}{dt} = (A + B(t))X, \quad X(0) = X_0$$

Provided $\int_0^\infty \|B(t)\|dt < \infty$

Pf

$$\frac{dX}{dt} = AX + \underbrace{B(t)X}_{h(t)}$$

$$X(t) = e^{At}X_0 + \int_0^t e^{A(t-s)}B(s)X(s)ds$$

$$\|X(t)\| \leq \|e^{At}X_0\| + \int_0^t \|e^{A(t-s)}B(s)X(s)\|ds$$

$$\|X(t)\| \leq \underbrace{C_0\|X_0\|}_{C} + \int_0^t \underbrace{C_0\|B(s)\|}_{k(s)}\,\|Y(s)\|\,ds$$

Apply Gronwall $\Rightarrow$

$$\|X(t)\| \leq C_0\|X_0\|\, e^{\int_0^t C_0\|B(s)\|ds}$$

$$\leq C_0\|X_0\|\, e^{\int_0^\infty C_0\|B(s)\|ds}$$

(V) (Perturbation of A) [Bellman, p.36, Thm 2]

Suppose all solutions of $\frac{dX}{dt} = AX$ go to zero as $t \to +\infty$

(i.e. $Re(\lambda_i) < 0$ for all λ_i

or $\|e^{At}X_0\| \leq C_0 e^{-K_0 t}\|X_0\|$)

Then the same holds for the solution of

$$\underline{\frac{dX}{dt} = (A + B(t))X, \quad X(0) = X_0}$$

provided $\|B(t)\| \leqslant \underline{\varepsilon(A)}$ for $t \geq t_0$

(i.e. $\|B(t)\|$ is small enough for t large.

the smallness depends on A.)

Pf

$$X(t) = e^{At}X_0 + \int_0^t e^{A(t-s)}B(s)X(s)\,ds$$

$$\|X(t)\| \leqslant \|e^{At}X_0\| + \int_0^t \|e^{A(t-s)}B(s)X(s)\|\,ds$$

$$\leqslant C_0 \underline{e^{-K_0 t}} \|X_0\| + \int_0^t C_0 \underline{e^{-K_0(t-s)}} \|B(s)\| \|X(s)\|\,ds$$

$$\underbrace{e^{K_0 t}\|X(t)\|}_{Z(t)} \leqslant \underbrace{C_0\|X_0\|}_{C} + \int_0^t \underbrace{C_0\|B(s)\|}_{k(s)} \underbrace{e^{K_0 s}\|X(s)\|}_{Z(s)}\,ds$$

Gronwall $\Longrightarrow$

$$Z(t) \leqslant C e^{\int_0^t k(s)\,ds}$$

i.e. $$e^{K_0 t}\|X(t)\| \leqslant C_0\|X_0\| e^{\int_0^t C_0\|B(s)\|\,ds}$$

i.e. $$\|X(t)\| \leqslant C_0\|X_0\| \underbrace{e^{-K_0 t + \int_0^t C_0\|B(s)\|\,ds}}_{\text{for } t \geqslant t_0}$$

$$= C_0\|X_0\| e^{-K_0(t - t_0 + t_0) + \int_0^{t_0} C_0\|B(s)\|\,ds + \int_{t_0}^{t} \underbrace{C_0\|B(s)\|}_{\leqslant C_0 \varepsilon(A)}\,ds}$$

$$\leqslant C_0\|X_0\| e^{-K_0 t_0 + \int_0^{t_0} C_0\|B(s)\|\,ds}\, e^{-K_0(t-t_0) + C_0\varepsilon(A)(t-t_0)}$$

$$= \underbrace{C_0\|X_0\| e^{-K_0 t_0 + \int_0^{t_0} C_0\|B(s)\|\,ds}}_{\text{some constant.}}\, e^{-\underbrace{(K_0 - C_0\varepsilon)}(t-t_0)}$$

Need $K_0 - C_0\varepsilon > 0$

$$\underbrace{\text{i.e. } \varepsilon < \frac{K_0}{C_0}}_{\text{then } \to 0 \text{ as } t \to +\infty}$$

The conclusion of the above also holds if

(1) $\int_0^\infty \|B(t)\| ds < \infty$

or

(2) $\lim_{t \to +\infty} \|B(t)\| = 0$

(VI) "Counter-Example" for $A = A(t)$.

[Bellman, p. 42. Thm 5] There are $A(t)$, $B(t)$ s.t.

(1) all solutions of $\frac{d}{dt} X = A(t) X$ go to zero

(2) $\int_0^\infty \|B(s)\| ds < \infty$

and yet any solution of

$$\frac{dX}{dt} = (A(t) + B(t)) X$$

will go to infinity as $t \to \infty$

$(\|X(t)\| \longrightarrow +\infty)$