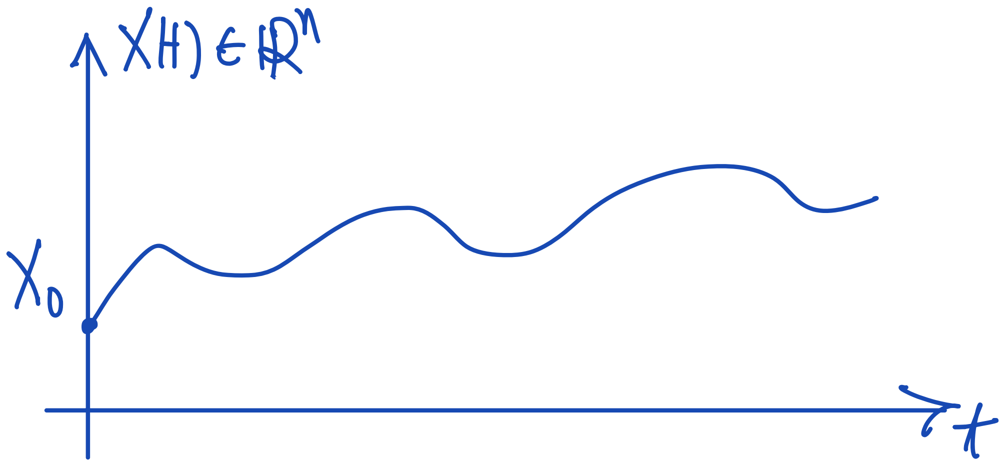
Existence and Uniqueness of Solutions (Lec 06)

$$\frac{d}{dt}X = F(X), \quad X(0) = X_0 \qquad \text{(autonomous)}$$

$$\left(\frac{dX}{dt} = F(X,t), \quad X(0) = X_0, \quad \text{non-autonomous} \right)$$

$$\boxed{X(t) = X(0) + \int_0^t F(X(s))\, ds}$$



constructive
(1) Iteration (Picard)
(2) Banach Fixed Point Thm (Contraction Map)
(3) Time Discretization (Stepping)

non-constructive
(4) Schauder Fixed Point Theorem

((1), (2) [M, Thm 3.19, p.82]; (3) [B, p.69])

(I) Iteration

Find $Y(t)$ s.t.

$$\boxed{Y(t) = X_0 + \int_0^t F(Y(s))\,ds} \quad (*)$$

$$Y_0(t) \equiv X_0$$

$$Y_1(t) = X_0 + \int_0^t F(Y_0(s))\,ds$$

$$Y_2(t) = X_0 + \int_0^t F(Y_1(s))\,ds$$

$$\vdots \qquad \vdots$$

$$Y_{n+1}(t) = X_0 + \int_0^t F(Y_n(s))\,ds$$

$$\vdots \qquad \vdots$$

Show that $Y_n(\cdot) \xrightarrow{n\to\infty} Y(\cdot)$ which indeed satisfies $(*)$

Linear Case $\quad \frac{dX}{dt} = AX + h(t), \quad X(0) = X_0$

Solution: $X(t) = e^{At} X_0 + \int_0^t e^{A(t-s)} h(s) ds$

Iteration: Look for $Y(t)$ s.t.

$$Y(t) = X_0 + \int_0^t A Y(s) ds + \int_0^t h(s) ds$$

$$Y_{n+1}(t) = X_0 + \int_0^t A Y_n(s) ds + \int_0^t h(s) ds$$

$$Y_0(t) \equiv X_0$$

$n = 0$, $\quad Y_1(t) = X_0 + \int_0^t A X_0 ds + \int_0^t h(s) ds$

$$= X_0 + tAX_0 + \int_0^t h(s) ds$$

$$Y_2(t) = X_0 + \int_0^t A Y_1(s) ds + \int_0^t h(s) ds$$

$$= X_0 + \int_0^t A\left[X_0 + sAX_0 + \int_0^s h(r) dr\right] ds + \int_0^t h(s) ds$$

$$= X_0 + tAX_0 + \frac{t^2}{2}A^2X_0 + \underbrace{\int_0^t A\left(\int_0^s h(r)\,dr\right)ds} + \underbrace{\int_0^t h(s)\,ds}$$

$$A\left[s\int_0^s h(r)\,dr \Big|_0^t - \int_0^t s\,h(s)\,ds \right]$$

$$= A\left[t\int_0^t h(r)\,dr - \int_0^t s\,h(s)\,ds \right]$$

$$= A\int_0^t (t-s)\,h(s)\,ds \quad +$$

$$= \int_0^t \left(A(t-s)\,h(s) + h(s)\right) ds$$

$$= \left(I + \frac{tA}{1!} + \frac{t^2A^2}{2!}\right)X_0 + \int_0^t \left(I + A(t-s)\right) h(s)\,ds$$

$$Y_3(t) = X_0 + \int_0^t A\,Y_2(s)\,ds + \int_0^t h(s)\,ds$$

$$= X_0 + \int_0^t A\left[\left(I + \frac{sA}{1!} + \frac{s^2A^2}{2!}\right)X_0 + \int_0^s \left(I + A(s-r)\right)h(r)\,dr\right]ds + \int_0^t h(s)\,ds$$

$$X_0 + \left(tA + \frac{t^2A^2}{2!} + \frac{t^3}{3!}A^3\right)X_0$$

$$= \left(I + \frac{tA}{1!} + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!}\right)X_0$$

$$\int_0^t A \underbrace{\int_0^s (I + A(s-r))h(r)\,dr}\,ds + \int_0^t h(s)\,ds$$

$$H(s) = \int_0^s (I + A(s-r))h(r)$$

$$H'(s) = h(s) + \int_0^s A\,h(r)\,dr$$

$$= A\int_0^t H(s)\,ds + \int_0^t h(s)\,ds$$

$$= A\int_0^t H(s)\,\underline{d(s-t)} + \int_0^t h(s)\,ds$$

$$= A\left(\cancel{H(s)(s-t)}\Big|_0^t - \int_0^t (s-t)H'(s)\,ds\right) + \int_0^t h(s)\,ds$$

$$= A\int_0^t (t-s)\left(h(s)+\int_0^s Ah(r)\,dr\right)ds+\int_0^t h(s)\,ds$$

$$= \int_0^t h(s)\,ds+\int_0^t A(t-s)\,h(s)\,ds$$

$$+ A\int_0^t (t-s)\left(\int_0^s Ah(r)\,dr\right)ds$$

$$-A\int_0^t\left(\int_0^s Ah(r)\,dr\right)d\left(\frac{(s-t)^2}{2!}\right)$$

$$=-A\left[\frac{(s-t)^2}{2!}\int_0^s Ah(r)\,dr\Big|_0^t - \int_0^t \frac{(s-t)^2}{2!}Ah(s)\,ds\right]$$

$$= A\int_0^t \frac{(s-t)^2}{2!}Ah(s)\,ds$$

$$= \int_0^t h(s)\,ds+\int_0^t A(t-s)\,h(s)\,ds+\int_0^t \frac{(t-s)^2}{2!}A^2h(s)\,ds$$

$$Y_3(t) = \left(I + \frac{tA}{1!} + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!}\right)X_0 + \int_0^t \left(I + A(t-s) + \frac{A^2(t-s)^2}{2!}\right) h(s)\,ds$$

$$\Downarrow$$

$$Y_n(t) = \left(I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \cdots + \frac{A^n t^n}{2!}\right)X_0 + \int_0^t \left(I + \frac{A(t-s)}{1!} + \frac{A^2(t-s)^2}{2!} + \cdots + \frac{A^{n-1}(t-s)^{n-1}}{(n-1)!}\right) h(s)\,ds$$

$$\Downarrow$$

$$Y(t) = e^{At}X_0 + \int_0^t e^{A(t-s)} h(s)\,ds$$

<u>Uniqueness</u> Let $X(t)$ & $Y(t)$ be 2 solutions:

$$X(t) = X_0 + \int_0^t A X(s) ds + \int_0^t h(s) ds$$

$$Y(t) = X_0 + \int_0^t A Y(s) ds + \int_0^t h(s) ds$$

Subtract:

$$X(t) - Y(t) = \int_0^t (AX(s) - AY(s)) ds$$

$$= \int_0^t A(X(s) - Y(s)) ds$$

$$\underbrace{\| X(t) - Y(t) \|}_{g(t)} \leq \int_0^t \| A(X(s) - Y(s)) \| ds$$

$$\leq L \int_0^t \underbrace{\| X(s) - Y(s) \|}_{g(s)} ds$$

$$0 \leq g(t) \leq L \int_0^t g(s) ds$$

Gronwall $\Longrightarrow$ $g(t) \equiv 0$ for all t $\Rightarrow$ $X(t) = Y(t)$

Nonlinear Equation

(I) Picard Iteration

$$X(t) = X_0 + \int_0^t F(X(s))\,ds \qquad (*)$$

$$Y_0(t) \equiv X_0$$

$$Y_n(t) = X_0 + \int_0^t F(Y_{n-1}(s))\,ds$$

$$\Rightarrow \quad Y_0(\cdot),\ Y_1(\cdot),\ Y_2(\cdot),\ \cdots,\ Y_n(\cdot) \longrightarrow Y(\cdot)$$

which solves $(*)$?

Assumptions on F (for now)

(1) $\|F(X)\| < M$ for all X

(boundedness of F)

(2) $\|F(X) - F(Y)\| \leq L\|X-Y\|$ for all X, Y

(Lipschitz condition of F)

$\boxed{\text{Consider the difference between } Y_{n-1} \text{ \& } Y_n}$

Let $\delta_n(t) = Y_n(t) - Y_{n-1}(t)$

$$\delta_1(t) = Y_1(t) - Y_0(t) = \int_0^t F(Y_0(s))ds$$

$$\|\delta_1(t)\| \leq \int^t \|F(X_0)\| ds \leq Mt$$

$$\delta_2(t) = Y_2(t) - Y_1(t) = \int_0^t F(Y_1(s))ds - \int_0^t F(Y_0(s))ds$$

$$\|\delta_2(t)\| \leq \int_0^t \|F(Y_1(s)) - F(Y_0(s))\| ds$$

$$\leq \int_0^t L \|Y_1(s) - Y_0(s)\| ds$$

$$\leq \int_0^t L\, \delta_1(s)\, ds$$

$$\leq L \int_0^t Ms\, ds = \frac{LMt^2}{2}$$

Similarly $\|\delta_3(t)\| \leq L \int_0^t \delta_2(s)\, ds$

$$\leq L \int_0^t \frac{LMs^2}{2} ds = M\frac{L^2 t^3}{3!}$$

$\vdots \quad \vdots$ (By induction)

$$\|\delta_{n+1}(t)\| \leq L\int_0^t \|\delta_n(s)\| ds$$

$$\leq L\int_0^t M \frac{L^{n-1} s^n}{n!} ds$$

$$= M\frac{L^n t^{n+1}}{(n+1)!}$$

$$\boxed{\|\delta_n(t)\| \leq \frac{M L^{n-1} t^n}{n!} \qquad n = 1, 2, 3 \cdots}$$

$$\|\delta_1(t)\| + \|\delta_2(t)\| + \|\delta_3(t)\| + \cdots\cdots$$

$$\leq \left(\frac{M}{L}\right)\left(1 + \frac{Lt}{1!} + \frac{(Lt)^2}{2!} + \frac{(Lt)^3}{3!} + \cdots\right)$$

$$= \frac{M}{L} e^{Lt} < \infty \quad \text{for all } t$$

$\Longrightarrow Y_n(\cdot) \xrightarrow{n \to \infty} Y(\cdot)$ which is well-defined.

Cauchy sequence, Completeness of function space

$$Y_n(t) = X_0 + \int_0^t F(Y_{n-1}(s))\,ds$$

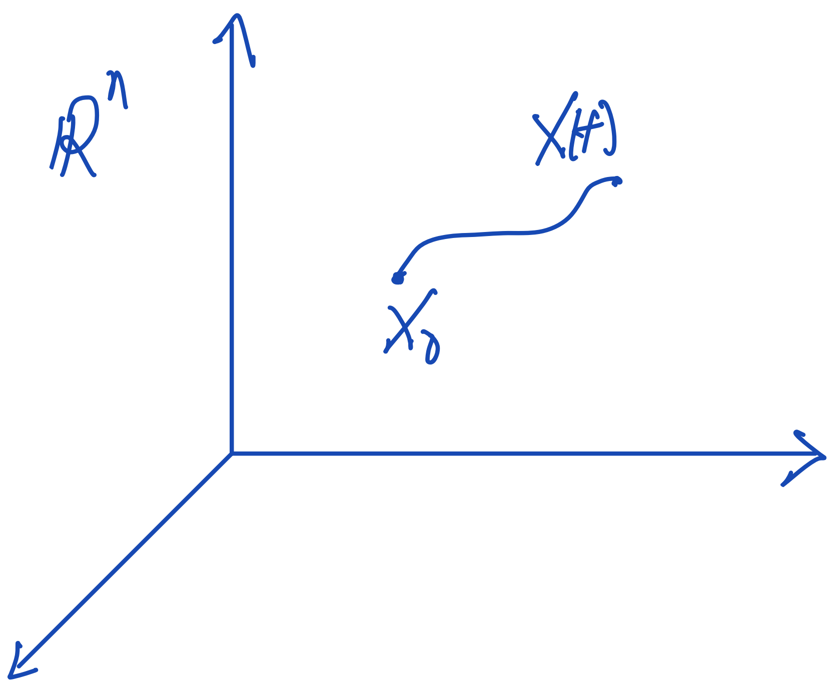
$$\downarrow \qquad\qquad \downarrow$$

$$Y(t) = X_0 + \int_0^t F(Y(s))\,ds$$

(F is Lip, in particular, F is continuous.)

(II) Banach Fixed Point Thm

$$\mathcal{X} = \{ X(t) : \underline{0 \leq t \leq T,} \ X(0) = X_0 \}$$

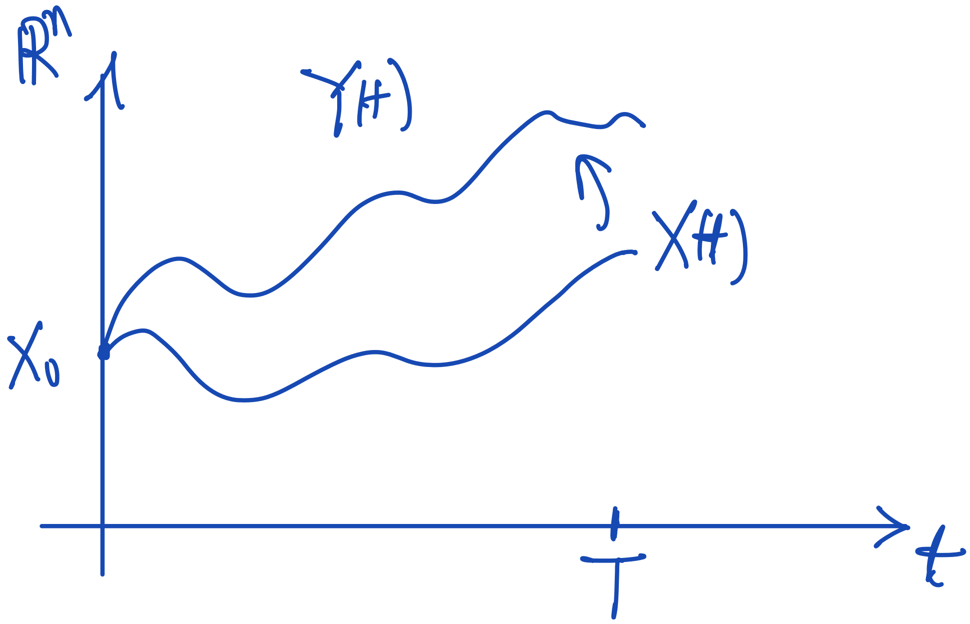


$$\boxed{\begin{gathered} X(\cdot) \xrightarrow{\quad \mathcal{T} \quad} Y(\cdot) = (\mathcal{T}X)(\cdot) \\[2ex] Y(t) = X_0 + \int_0^t F(X(s))\,ds \end{gathered}}$$

$$Y(t) - X_0 = \int_0^t F(X(s))\,ds$$

$$\|Y(t) - X_0\| \le \int_0^t \|F(X(s))\|\,ds \le Mt \le MT$$

$$(0 \le t \le T)$$



$$\mathcal{J}(X_0, MT) = \{ X(\cdot) : \|X(t) - X_0\| \le MT,\ 0 \le t \le T \}$$

$$\mathcal{F}(X_0, MT) \xrightarrow{\ T\ } \mathcal{F}(X_0, MT)$$

$$X \longrightarrow Y = TX$$

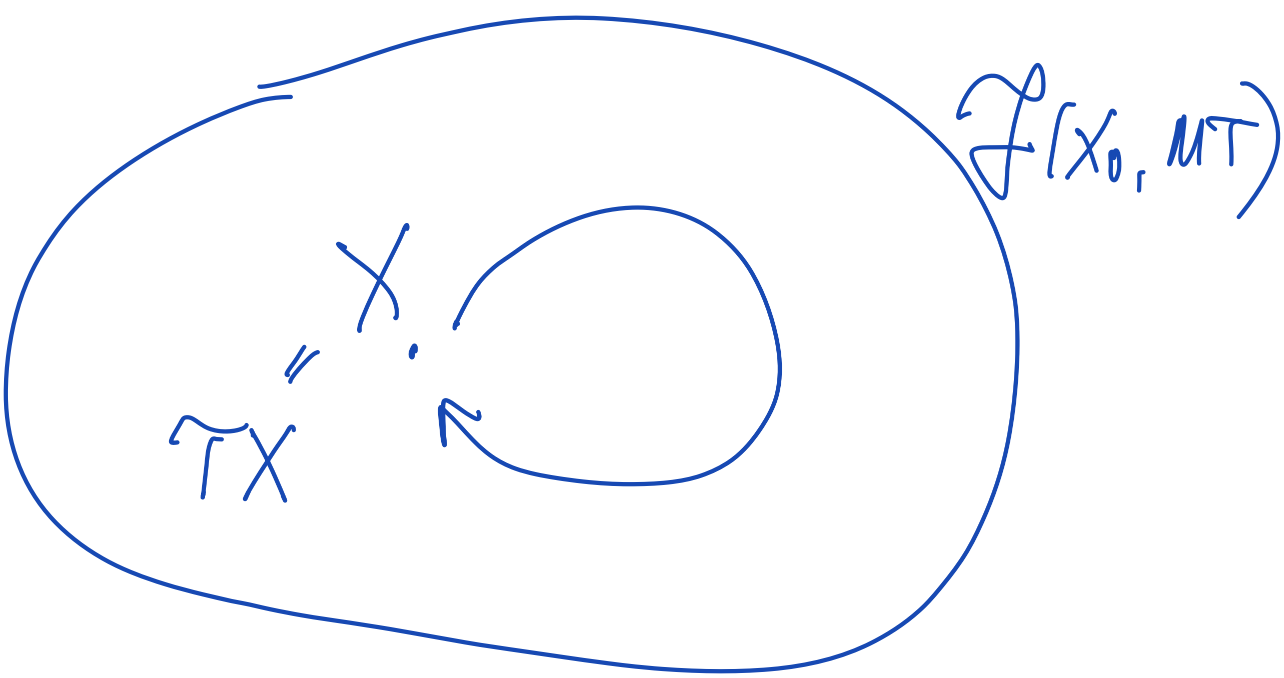
X is a solution, i.e. it satisfies

$$X(t) = X_0 + \int_0^t F(X(s))\,ds$$

if and only if

$$\underline{X = TX}$$

i.e. <u>X is a fixed point of T</u>



Take $X_1(\cdot), X_2(\cdot) \in \mathcal{F}(X_0, MT)$

$$Y_1 = TX_1, \quad Y_2 = TX_2 \in \mathcal{F}(X_0, MT)$$

Consider

$$Y_1(t) - Y_2(t) = X_0 + \int_0^t F(X_1(s))ds - \left(X_0 + \int_0^t F(X_2(s))ds\right)$$

$$= \int_0^t \left(F(X_1(s)) - F(X_2(s))\right)ds$$

$$\|Y_1(t) - Y_2(t)\| \le \int_0^t \|F(X_1(s)) - F(X_2(s))\| ds$$

$$\le L \int_0^t \|X_1(s) - X_2(s)\| ds$$

Let $\|Y_1 - Y_2\| = \max\limits_{t \in [0,T]} \text{(or sup)} \|Y_1(t) - Y_2(t)\|$

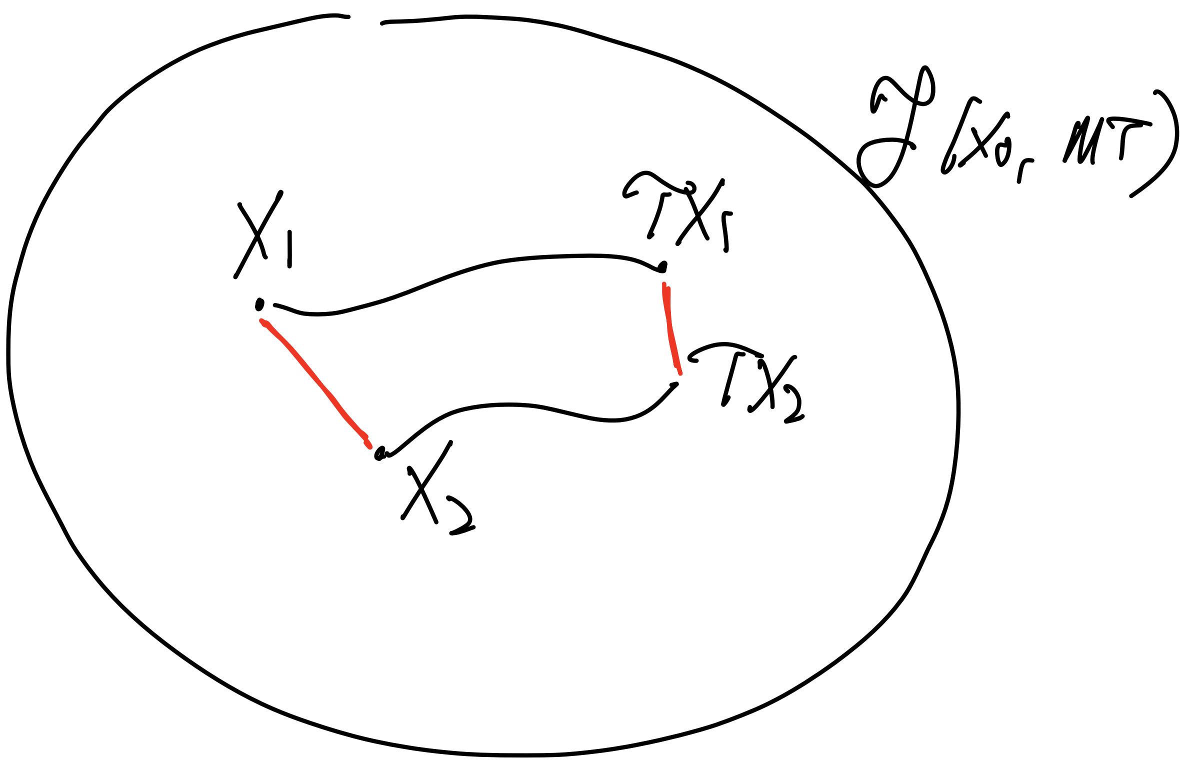
Then

$$\|Y_1 - Y_2\| \leq L \|X_1 - X_2\| \int_0^T ds = \underbrace{LT}_{<1 \text{ if } T < \frac{1}{L}} \|X_1 - X_2\|$$

ie. $\|TX_1 - TX_2\| \leq C \|X_1 - X_2\|$

$C = LT < 1$

ie. <u>T is a contraction map.</u>



Then use iteration to find a fixed pt.

$$X_0 = x_0 \quad (X_0(t) = x_0,\ 0 \leq t \leq T)$$

$$X_1 = \mathcal{T} X_0$$

$$X_2 = \mathcal{T} X_1$$

$$\vdots$$

$$X_{n+1} = \mathcal{T} X_n$$

$$\Longrightarrow \{X_0, X_1, X_2, \cdots, X_n, \cdots\}$$

$$X_n \longrightarrow X \text{ a fixed pt of } \mathcal{T}\,?$$

Again, look at the difference between iterates.

$$\|X_1 - X_0\| \leq \max_{t\in[0,T]} \int_0^t F(X_0(s))\,ds$$

$$= \underline{MT}$$

$$\|X_2 - X_1\| \leq C\,\|X_1 - X_0\| = C\,\underline{MT}$$

$$\|X_3 - X_2\| \leq C\,\|X_2 - X_1\| \leq C^2 MT$$

$$\|X_4 - X_2\| \le C\,\|X_3 - X_2\| \le C^3 MT$$

$$\vdots \quad \vdots$$

$$\|X_n - X_{n-1}\| \le C\,\|X_{n-1} - X_{n-2}\| \le C^{n-1} MT$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\sum_{i=1}^{\infty} \|X_i - X_{i-1}\| = MT\Big(\underbrace{1 + C + C^2 + C^3 + \cdots}_{<\infty \text{ as } C<1}\Big)$$

$$= \frac{MT}{1-C} < \infty$$

Again, making use of the concept of Cauchy sequence, Completeness of function space

$$\Longrightarrow X_n(\cdot) \xrightarrow{n\to\infty} X(\cdot)$$

$$\left(\max_{t\in[0,T]} \|X_n(t) - X(t)\| \xrightarrow{n\to\infty} 0 \right)$$

Claim: <u>X is indeed a fixed pt of T</u>:

$$X_{n+1} = TX_n \qquad \left(\|TX_n - TX\| \le C\|X_n - X\| \to 0\right)$$

$$\downarrow \qquad \downarrow$$

$$X = TX$$

Uniqueness

(1) Let X_1, X_2 be two fixed pts,

i.e. $X_1 = TX_1, \quad X_2 = TX_2$

$$\Longrightarrow \quad \|X_1 - X_2\| = \|TX_1 - TX_2\|$$

$$\leq C\|X_1 - X_2\|$$

$$\Longrightarrow \quad \|X_1 - X_2\| = 0 \quad \text{i.e. } X_1 = X_2$$

or

(2)
$$X_1(t) = X_0 + \int_0^t F(X_1(s))\,ds$$

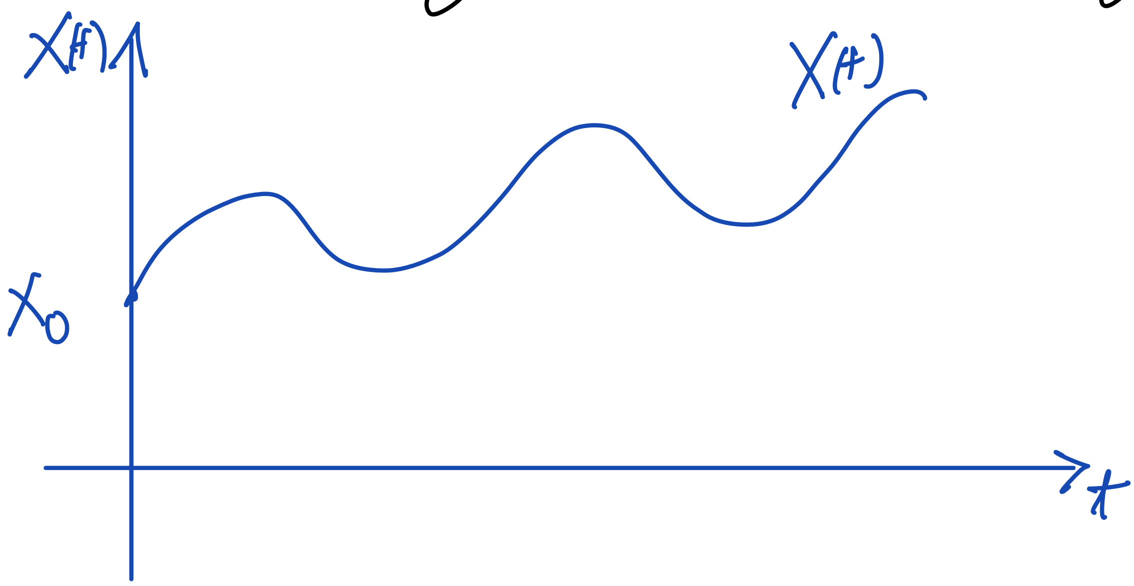
$$X_2(t) = X_0 + \int_0^t F(X_2(s))\,ds$$

$$X_1(t) - X_2(t) = \int_0^t \big(F(X_1(s)) - F(X_2(s))\big)\,ds$$

$$\underbrace{\|X_1(t) - X_2(t)\|}_{g(t)} \leq \int_0^t L\,\underbrace{\|X_1(s) - X_2(s)\|}_{g(s)}\,ds$$

$$0 \leq g(t) \leq \int_0^t L g(s)\,ds \Rightarrow (\text{G.I})\ g(t) \equiv 0$$

(III) Time discretization (Time Stepping)



$0 < \Delta t, \quad X_{\Delta t}(0) = X_0$

$$X_{\Delta t}(\Delta t) = X_{\Delta t}(0) + F(X_{\Delta t}(0))\Delta t$$

$$X_{\Delta t}(2\Delta t) = X_{\Delta t}(\Delta t) + F(X_{\Delta t}(\Delta t))\Delta t$$

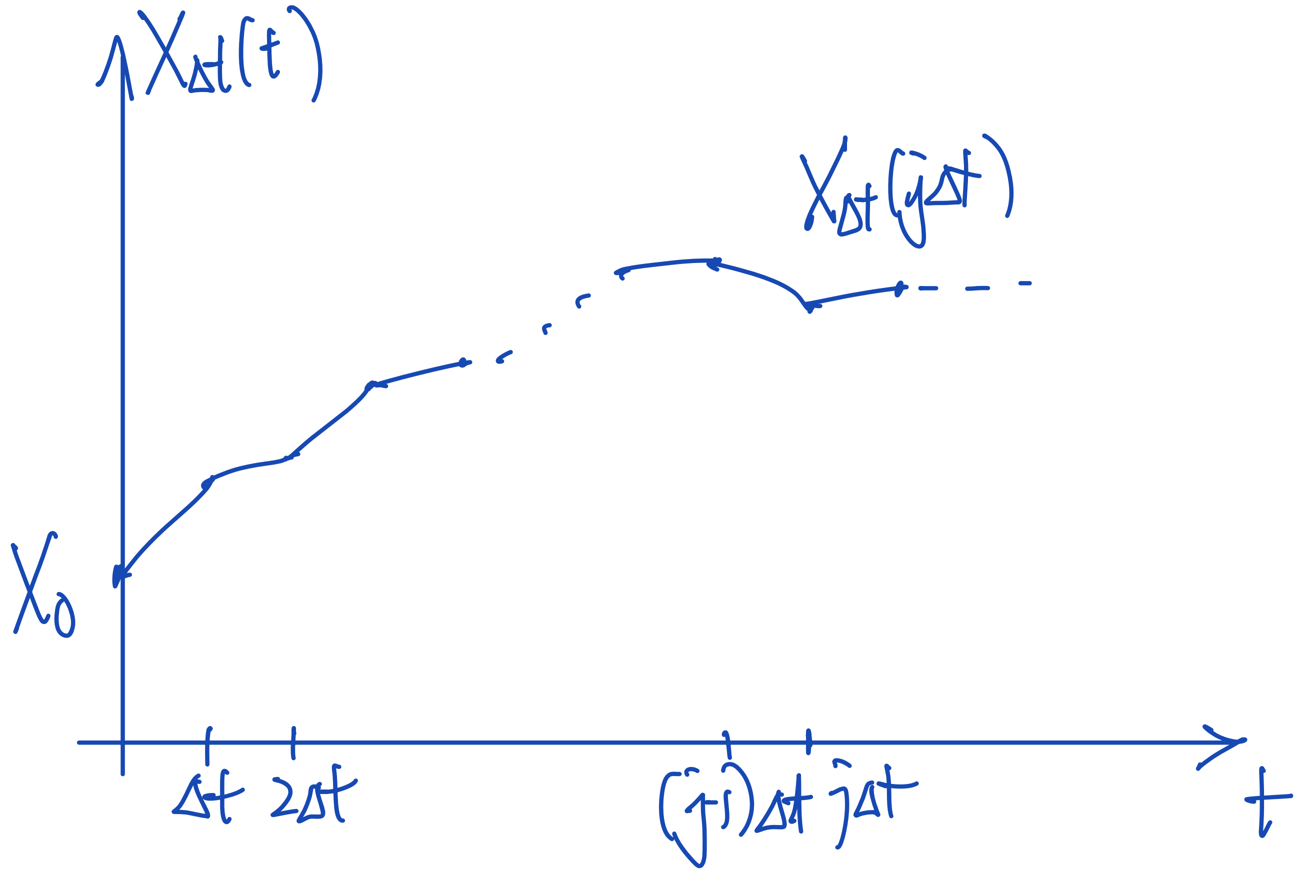
$$\vdots \qquad \vdots \qquad \vdots$$

$$X_{\Delta t}(j\Delta t) = X_{\Delta t}((j-1)\Delta t) + F(X_{\Delta t}((j-1)\Delta t))\Delta t$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\left(\frac{dX(t)}{dt} = F(X(t)), \qquad \frac{X(t+\Delta t) - X(t)}{\Delta t} \cong F(X(t)) \right.$$

$$\left. X(t+\Delta t) = X(t) + F(X(t))\Delta t \right)$$



$$\{ X_{\Delta t}(j\Delta t) : 0 \le j\Delta t \le T \}_{\Delta t > 0}$$

(1) Take a sequence of $\Delta t \longrightarrow 0$

(2) Show that $\{X_{\Delta t}(\cdot)\}_{\Delta t \to 0}$ is compact (equi-continuous)

and hence $X_{\Delta t}(\cdot) \xrightarrow{\Delta t \to 0} X(\cdot)$

(3) $X(\cdot)$ satisfies $X(t) = X_0 + \int_0^t F(X(s))\,ds$

$$X_{\Delta t}(0) = X_0$$

$$X_{\Delta t}(\Delta t) = X_0 + F(X_{\Delta t}(0))\Delta t$$

$$X_{\Delta t}(2\Delta t) = X_{\Delta t}(\Delta t) + F(X_{\Delta t}(\Delta t))\Delta t$$

$$= X_0 + F(X_{\Delta t}(0))\Delta t + F(X_{\Delta t}(\Delta t))\Delta t$$

$$X_{\Delta t}(3\Delta t) = X_{\Delta t}(2\Delta t) + F(X_{\Delta t}(2\Delta t))\Delta t$$

$$= X_0 + F(X_{\Delta t}(0))\Delta t + F(X_{\Delta t}(\Delta t))\Delta t + F(X_{\Delta t}(2\Delta t))\Delta t$$

$$\vdots \quad \vdots \quad \vdots$$

$$X_{\Delta t}(j\Delta t) = X_0 + \sum_{i=0}^{j-1} F(X_{\Delta t}(i\Delta t))\Delta t$$

$$\downarrow$$

$$\boxed{X(t) = X_0 + \int_0^t F(X(s))\,ds}$$