Existence and Uniqueness of Solutions – II (Lec 07)

$$\frac{dX}{dt} = F(X), \quad X(0) = X_0 \iff X(t) = X_0 + \int_0^t F(X(s))\,ds$$

Uniqueness "$\iff$" Lipschitz condition for F

$$\|F(X) - F(Y)\| \leq L\|X - Y\|$$

$$\approx \quad \frac{\|F(X) - F(Y)\|}{\|X - Y\|} \leq L$$

i.e. $\quad \|DF(X)\| \leq L$

Let $X(\cdot)$ & $Y(\cdot)$ be two solutions:

$$X(t) = X_0 + \int_0^t F(X(s))\,ds$$

$$Y(t) = X_0 + \int_0^t F(Y(s))\,ds$$

$$\Rightarrow \underbrace{\|X(t) - Y(t)\|}_{0 \leq g(t)} \leq \int_0^t \|F(X(s)) - F(Y(s))\|\,ds$$

$$\leq \int_0^t L\underbrace{\|X(s) - Y(s)\|}_{g(s)}\,ds \Rightarrow ~~G~~.I$$

$$g(t) \equiv 0$$

Uniqueness is not true (in general) if F is not Lip.

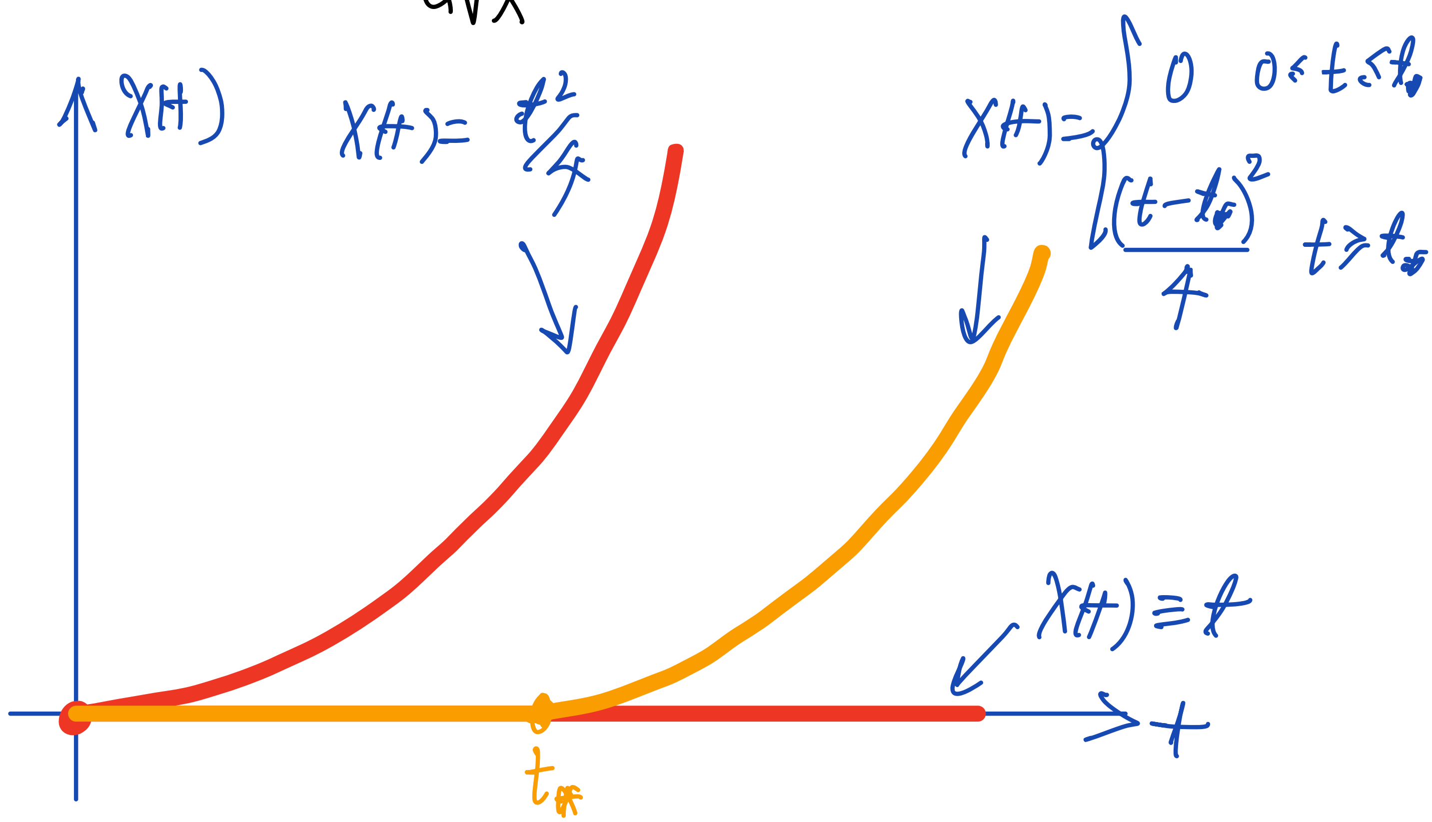
$$\dot{x} = \sqrt{x}, \quad x(0) = 0$$

Two solutions: (1) $x(t) \equiv 0$

(2) $x(t) = \frac{t^2}{4}$

$$F(x) = \sqrt{x}$$

$$F'(x) = \frac{1}{2\sqrt{x}} \to +\infty \text{ when } x \to 0^+$$



(two, in fact infinitely many solutions)

Generalize $\|F(x)\| \le M$ to

$$\boxed{\|F(x)\| \le A\|x\| + B \leftarrow \text{linear growth}}$$

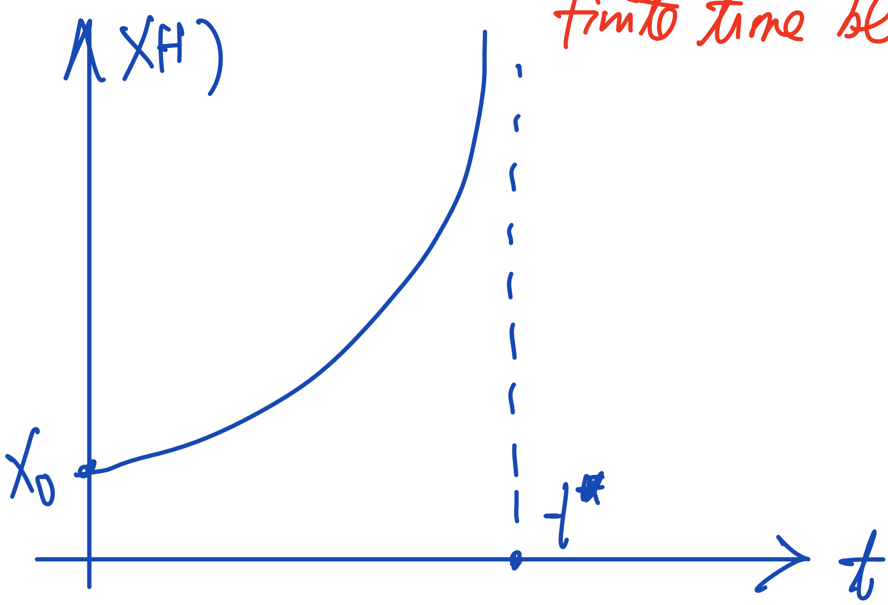
$\Longrightarrow$ solution exists for all t (global soln)

($\dot{x} = x^p$, $x(0) = x_0 > 0$, $p > 1$, superlinear growth

$$\int \frac{dx}{x^p} = \int dt \Rightarrow \frac{x^{-p+1} - x_0^{-p+1}}{-p+1} = t$$

$$\Rightarrow x(t) = \frac{1}{\left(\frac{1}{x_0^{p-1}} - (p-1)t\right)^{\frac{1}{p-1}}}$$

$$\underbrace{x(t) \longrightarrow +\infty \text{ as } t \rightarrow t_* = \frac{1}{(p-1)x_0^{p-1}}}_{\text{finite time blow-up.}}$$



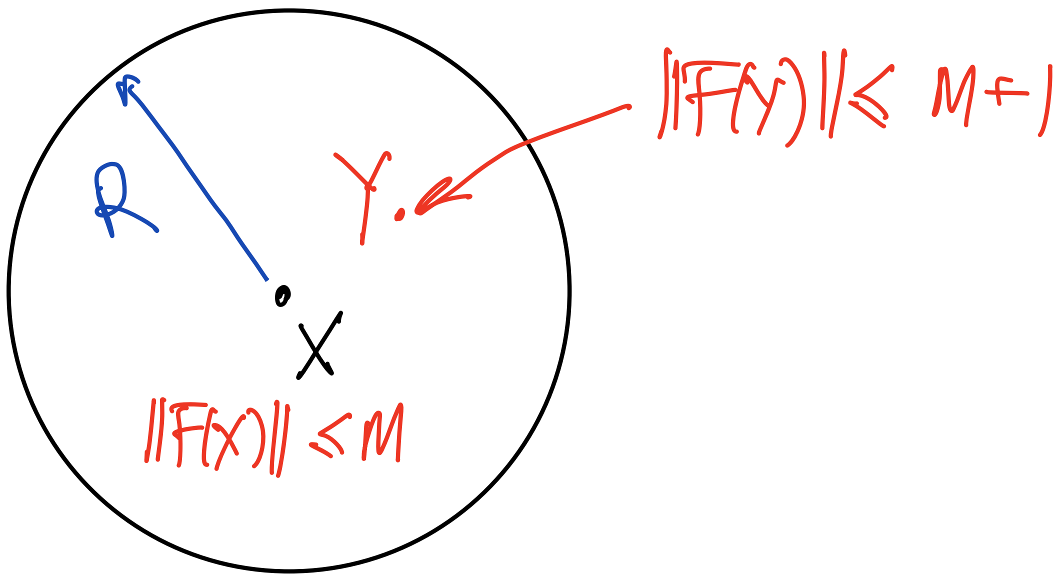
)

$$\|F(X)\| \leq A\|X\| + B$$

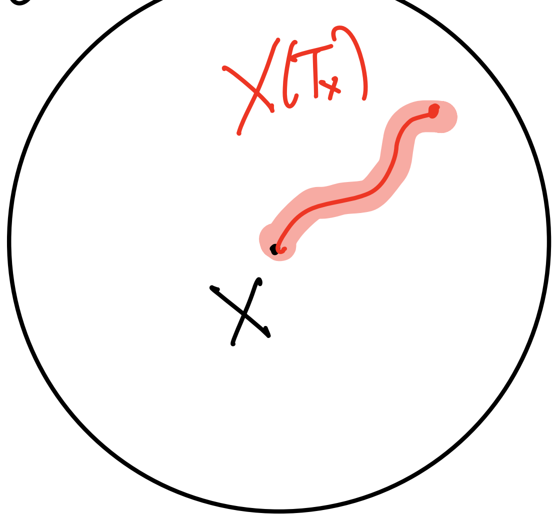
① Suppose $\|F(X)\| \leq M$.

There exist $R < \infty$ s.t.

$\|F(Y)\| \leq M+1$ for $\|Y - X\| \leq R$

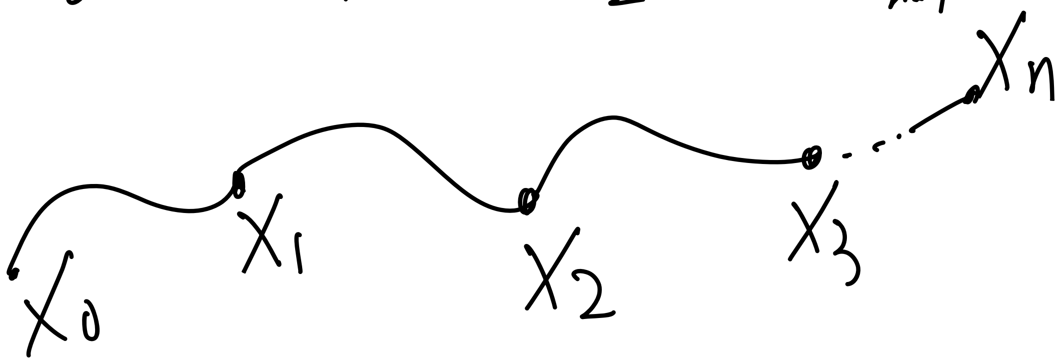


② Then starting from X, can solve the diff. equation up to time T_X



③ Paste the pieces together:

$$X_0 \xrightarrow[T_{X_0}]{} X_1 \xrightarrow[T_{X_1}]{} X_2 \xrightarrow[T_{X_2}]{} \cdots \xrightarrow[T_{X_{n-1}}]{} X_n$$



To obtain a solution $X(t)$ from

$t=0$ to $t=T\ (=T_{X_0}+T_{X_1}+\cdots+T_{X_n})$

④ Claim: as long as $T<\infty$, then $\|X(T)\|<\infty$

i.e. no finite time blow-up.

$$X(t) = X_0 + \int_0^t F(X(s))\,ds$$

$$\underbrace{\|X(t)\|}_{g(t)} \leq \|X_0\| + \int_0^t \|F(X(s))\|\,ds$$

$$\leq \|X_0\| + \int_0^t \left(A\|X(s)\| + B\right) ds$$

$$\leq \|X_0\| + Bt + \int_0^t A\|X(s)\|ds$$

$$\leq \underbrace{(\|X_0\| + BT)}_{c} + \int_0^t A\underbrace{\|X(s)\|}_{g(s)}ds$$

$$\|X(t)\| \leq (\|X_0\| + BT)e^{At}$$

$$\|X(T)\| \leq (\|X_0\| + BT)e^{AT} < \infty$$

(An alternative, "better" estimate:

$$\|X(t)\| \leq \|X_0\| + \int_0^t (A\|X(s)\| + B)ds$$

$$\leq \|X_0\| + \int_0^t A\left(\|X(s)\| + \frac{B}{A}\right)ds$$

$$\underbrace{\left(\|X(t)\| + \frac{B}{A}\right)}_{g(t)} \leq \underbrace{\left(\|X_0\| + \frac{B}{A}\right)}_{c} + \int_0^t A\underbrace{\left(\|X(s)\| + \frac{B}{A}\right)}_{g(s)}ds$$

$$\|X(t)\| \leq \|X(t)\| + \frac{B}{A} \leq \left(\|X_0\| + \frac{B}{A}\right)e^{At}$$

Continuous Dependent on initial data

$$X(t) = X_0 + \int_0^t F(X(s))\,ds$$

$$Y(t) = \underbrace{Y_0}_{\text{2 different initial data}} + \int_0^t F(Y(s))\,ds$$

$$X(t) - Y(t) = X_0 - Y_0 + \int_0^t \underbrace{\big(F(X(s)) - F(Y(s))\big)}_{\text{Lip, } L.}\,ds$$

$$\underbrace{\|X(t) - Y(t)\|}_{g(t)} \leq \|X_0 - Y_0\| + \int_0^t L\,\underbrace{\|X(s) - Y(s)\|}_{g(s)}\,ds$$

$$\Downarrow \text{ G.I}$$

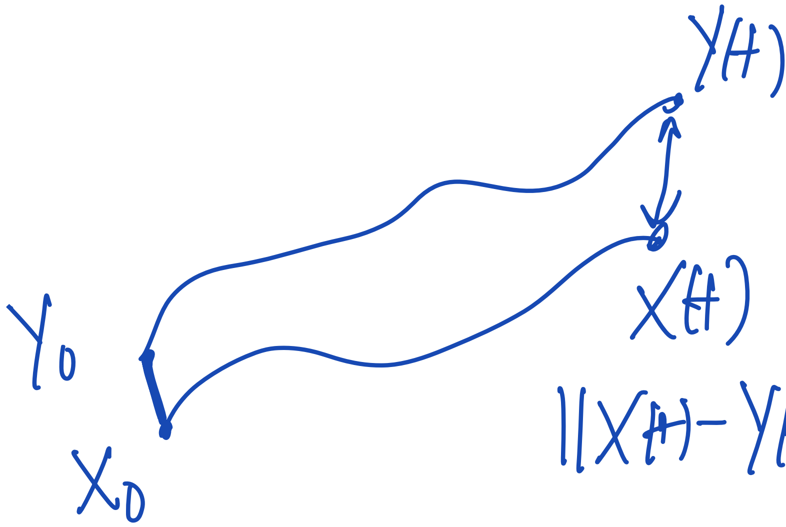
$$\boxed{\|X(t) - Y(t)\| \leq \|X_0 - Y_0\|\,e^{Lt}}$$

① Uniqueness. $\left(X_0 = Y_0 \Rightarrow X(t) = Y(t) \right)$

② The solution depends continuously on the initial data

If $X_0 \longrightarrow Y_0$, then $X(t) \longrightarrow Y(t)$

③ An estimate on $\|X(t) - Y(t)\|$ on the difference on initial data



$$\|X(t) - Y(t)\| \leq \|X_0 - Y_0\| e^{Lt}$$

$\|X_0 - Y_0\|$