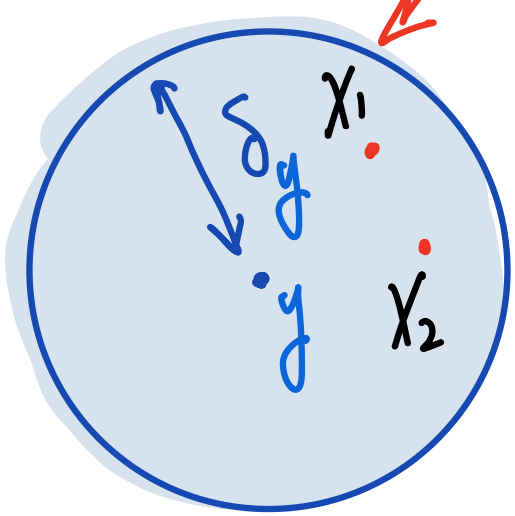
Local vs. Maximal Solution

Let $F: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be locally Lip.

i.e. $\forall y \in \mathbb{R}^n$, $\exists K_y, \delta_y > 0$ s.t.

$\forall$ $\boxed{x_1, x_2 \in B_{\delta_y}(y),}$ we have



$$\boxed{\|F(x_1) - F(x_2)\| \leq K_y \|x_1 - x_2\|}$$

Local vs. Maximal Solution

Let $F: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be locally Lip.

Then $\forall y \in \mathbb{R}^n$, $\exists$ $(-T(y), T(y)) \subseteq \mathbb{R}$ s.t.

the equation

$$\boxed{\frac{dX}{dt} = F(X), \quad X(0) = y}$$

local interval of existence

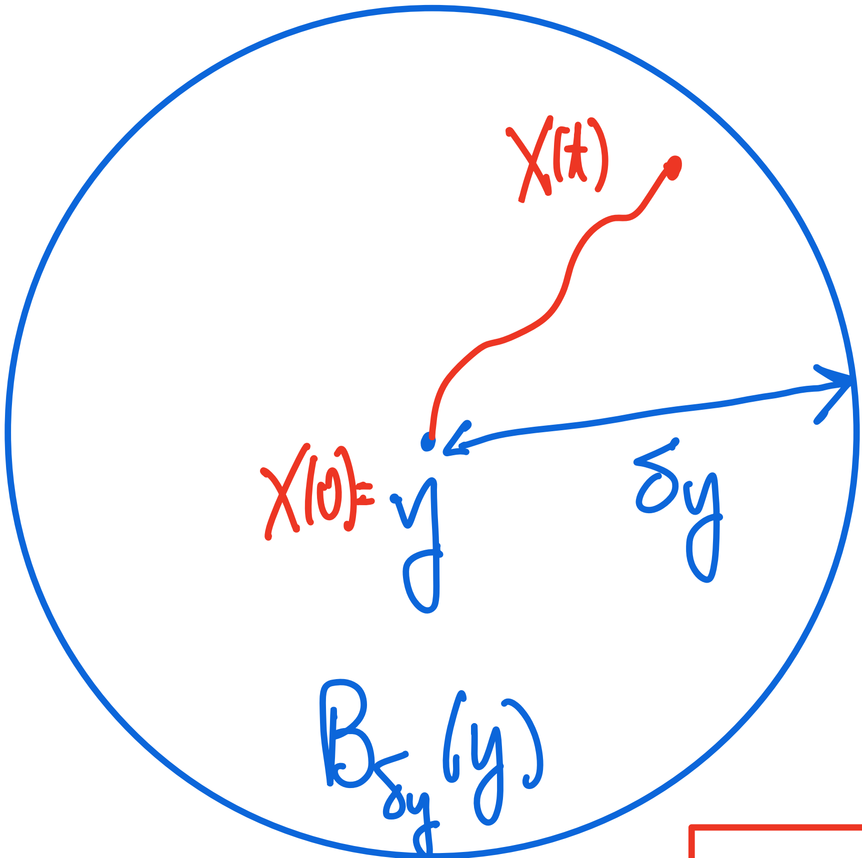
has a solution for $t \in (-T(y), T(y))$

[M, Thm 3.19, p. 82]

Local vs. Maximal Solution

An estimation of $T(y)$:

$$M_y = \max_{x \in B_{\delta_y}(y)} \|F(x)\|$$



$$X(t) = y + \int_0^t F(X(s))\,ds$$

$$\|X(t) - y\| \leq \int_0^t M_y\,ds$$

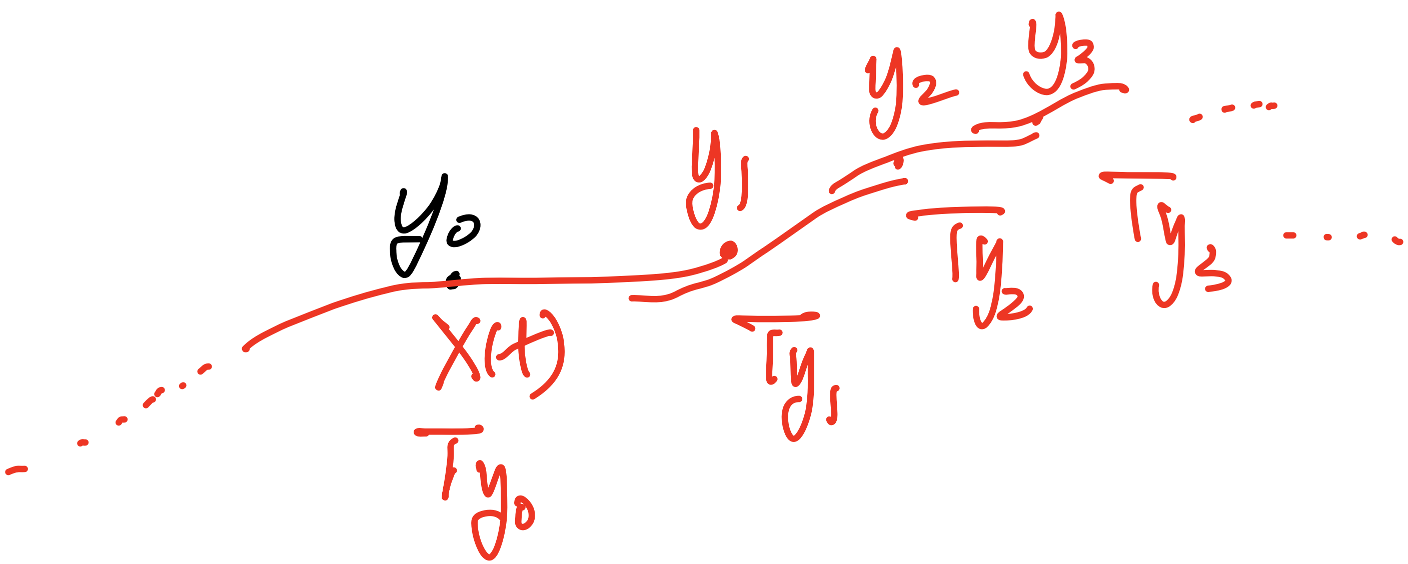
$$= M_y \times t \leq \delta_y$$

$$\boxed{t \leq \delta_y / M_y} \Rightarrow \boxed{T(y) \geq \frac{\delta_y}{M_y}}$$

Local vs. Maximal Solution

Let $F: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be <u>locally Lip.</u>

Extension of Solution in time



Local vs. Maximal Solution

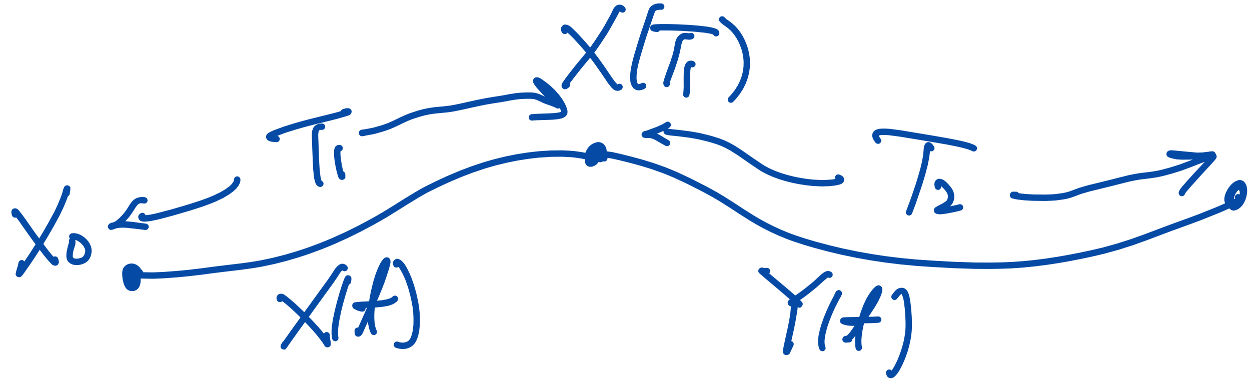
Concatenation of segments of solutions

Let $\{X(t)\}_{t\in[0,T_1]}$ satisfy $\dfrac{dX}{dt} = F(X)$

$X(0) = X_0$

$\{Y(t)\}_{t\in[0,T_2]}$ satisfy $\dfrac{dY}{dt} = F(Y)$

$Y(0) = X(T_1)$



Local vs. Maximal Solution

Concatenation of segments of solutions

Define

$$\boxed{Z(t) = \begin{cases} X(t) & t \in [0, T_1] \\ Y(t-T_1) & t \in [T_1, T_1+T_2] \end{cases}}$$

Then $Z(t)$ satisfies

$$\boxed{\frac{dZ}{dt} = F(Z), \quad Z(0) = X_0, \quad t \in [0, T_1+T_2]}$$

Local vs. Maximal Solution

Concatenation of segments of solutions

Define

$$\boxed{Z(t) = \begin{cases} X(t) & t \in [0, T_1] \\ Y(t-T_1) & t \in [T_1, T_1+T_2] \end{cases}}$$

Then $Z(t)$ satisfies

i.e.

$$\boxed{Z(t) = X_0 + \int_0^t F(Z(s))\,ds, \quad t \in [0, T_1+T_2]}$$

Local vs. Maximal Solution

Let $F: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be locally Lip.

Maximal Interval of Existence [M, Thm 3.33, p. 94]

There is a maximal (time) interval $(\alpha^*, \beta^*) \subseteq \mathbb{R}$
$(-\infty \le \alpha^* < \beta^* \le +\infty)$
of existence for the solution of

$$\frac{dX}{dt} = F(X); \quad X(0) = y$$

Local vs. Maximal Solution

Let $F: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be locally Lip.

Maximal Interval of Existence [M, Thm 3.33, p. 94]

(α^*, β^*) is maximal in the sense that

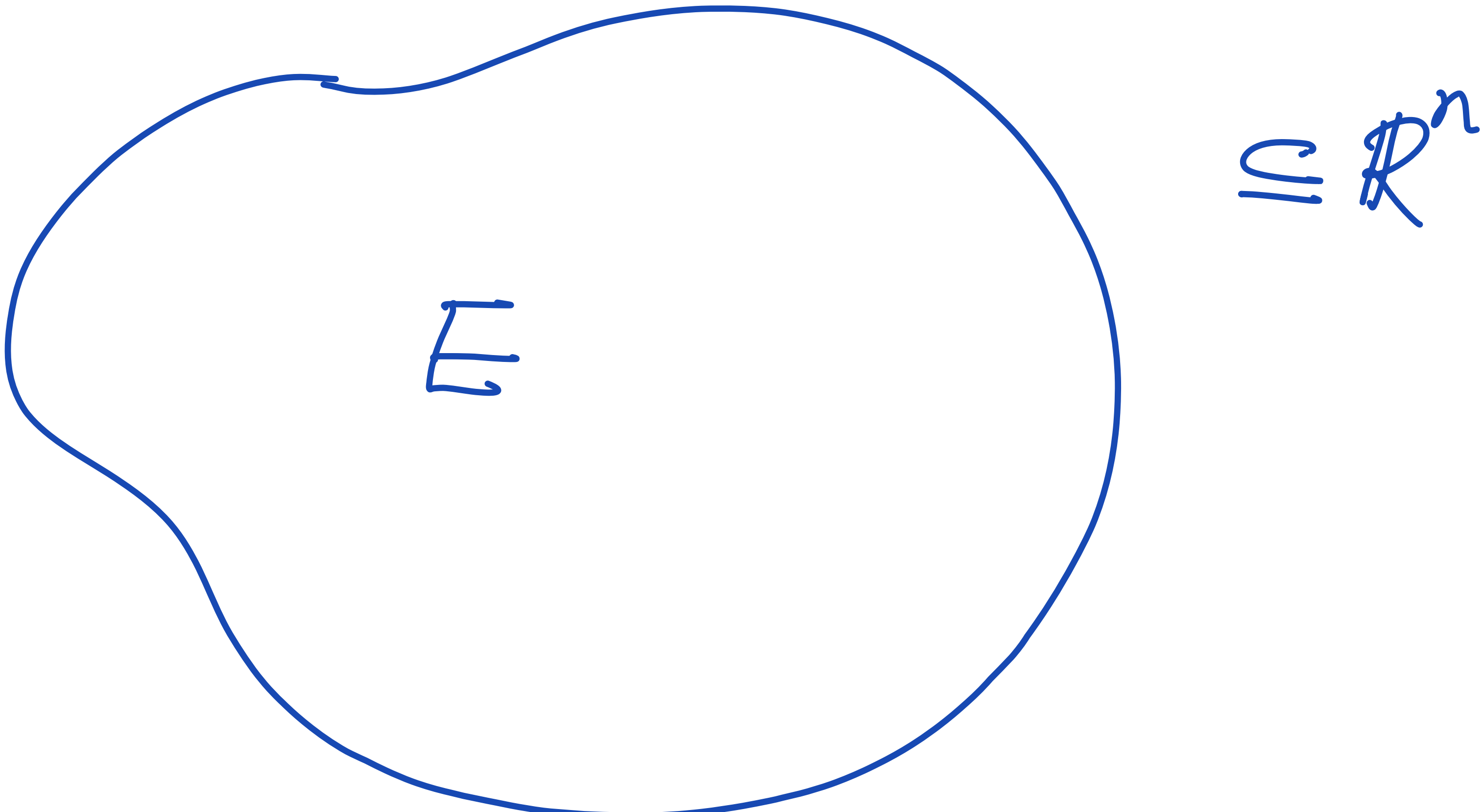
If $Z(t)$ satisfies $\frac{dZ}{dt} = F(Z)$, $Z(0) = y$

for $t \in (\alpha, \beta)$,

then $(\alpha, \beta) \subseteq (\alpha^*, \beta^*)$ i.e. $\alpha^* \leq \alpha < \beta \leq \beta^*$

Behavior of $X(t)$ as $t \to \beta^*$ (or α^*)

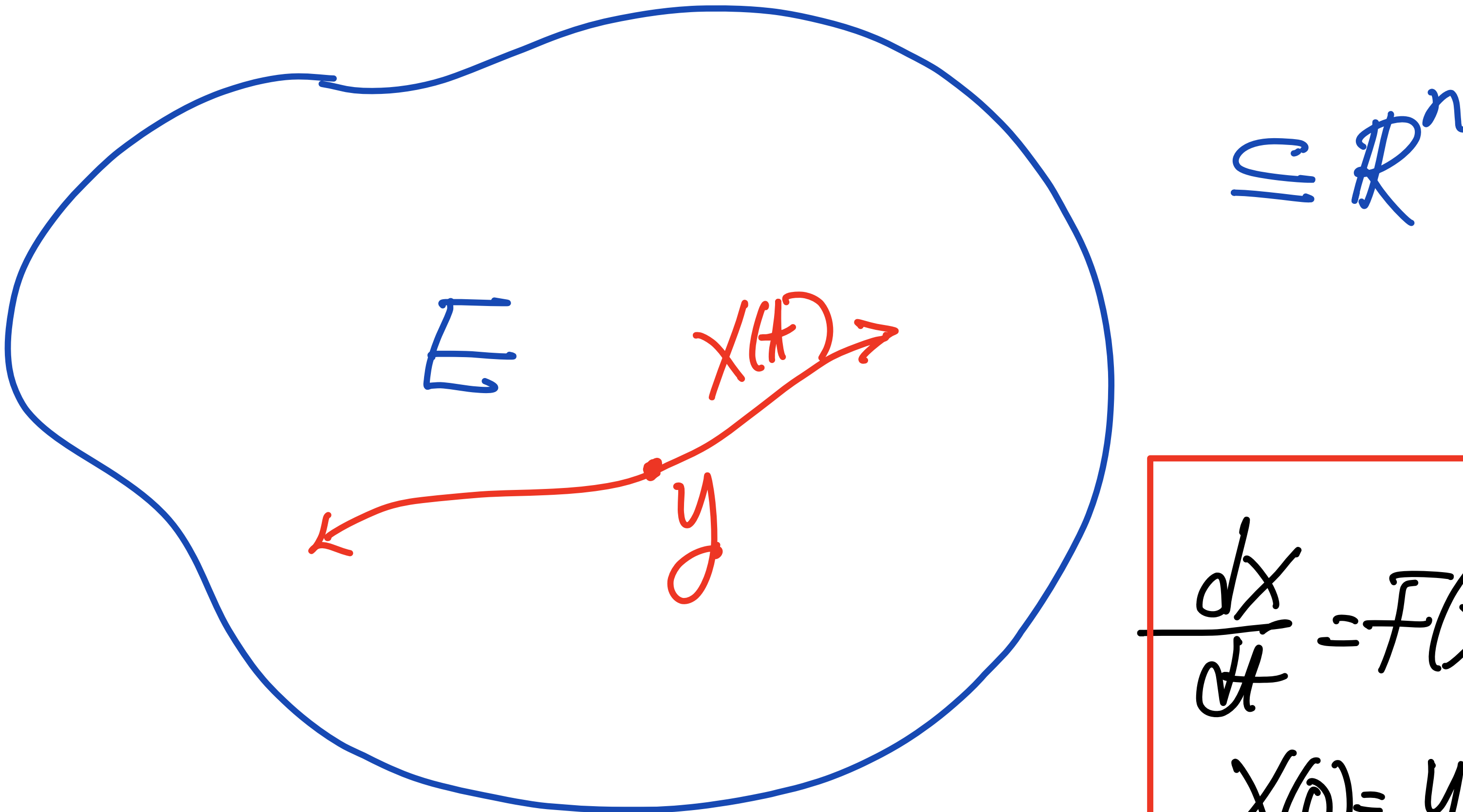
Let $F: E \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be locally Lip in E



[M, Thm 3.35, Cor 3.36, p. 96]

Behavior of $X(t)$ as $t \to \beta^*$ (or α^*)

Let $F : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be locally Lip in E



$$\boxed{\begin{aligned} \frac{dX}{dt} &= F(X) \\ X(0) &= y \end{aligned}}$$

Behavior of $X(t)$ as $t \to \beta^*$ (or α^*)

Let $F: E \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be locally Lip in E

If $\beta^* < \infty$, then as $t \longrightarrow \beta^*$,

$X(t)$ must approach the boundary of E (∂E)

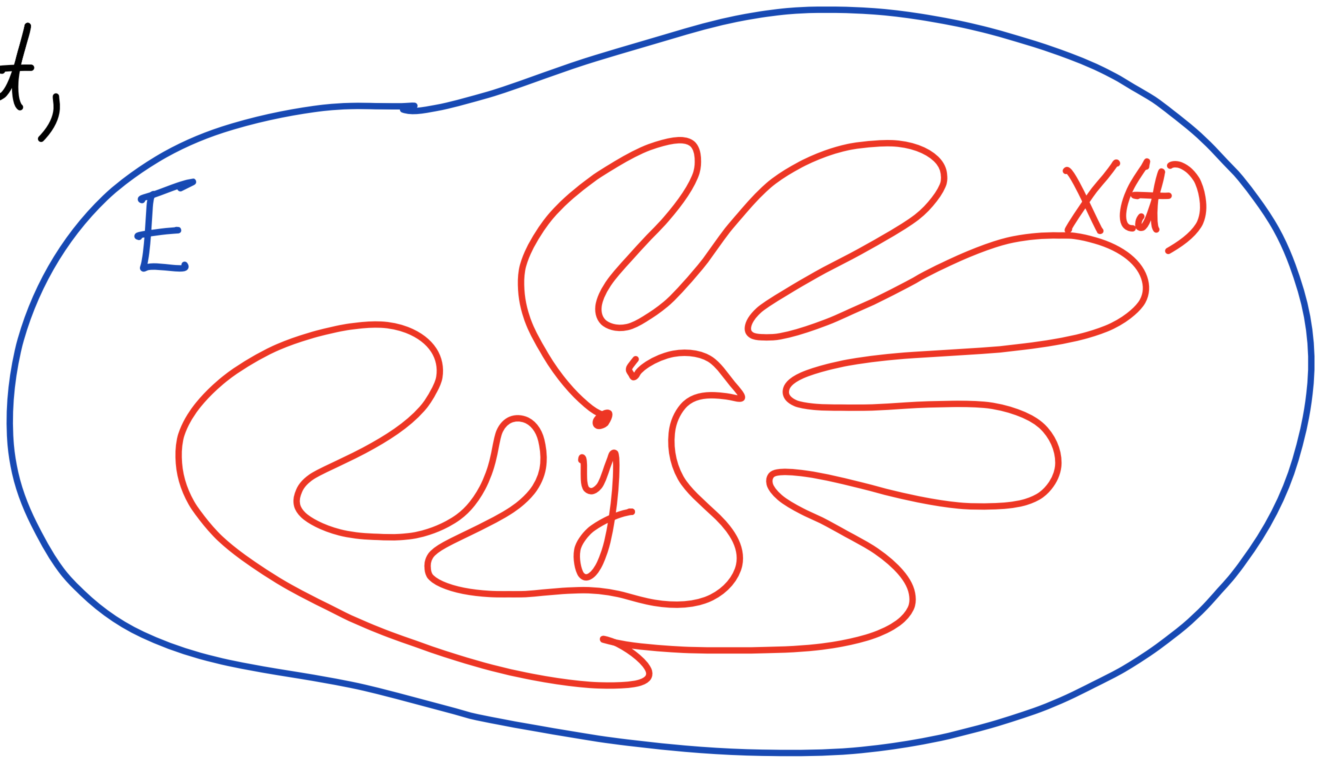
(or $|X(t)| \longrightarrow \infty$ if $E = \mathbb{R}^n$)

Behavior of $X(t)$ as $t \to \beta^*$ (or α^*)

Let $F: E \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be locally Lip in E

If $\beta^* < \infty$, then as $t \longrightarrow \beta^*$,

(If not,

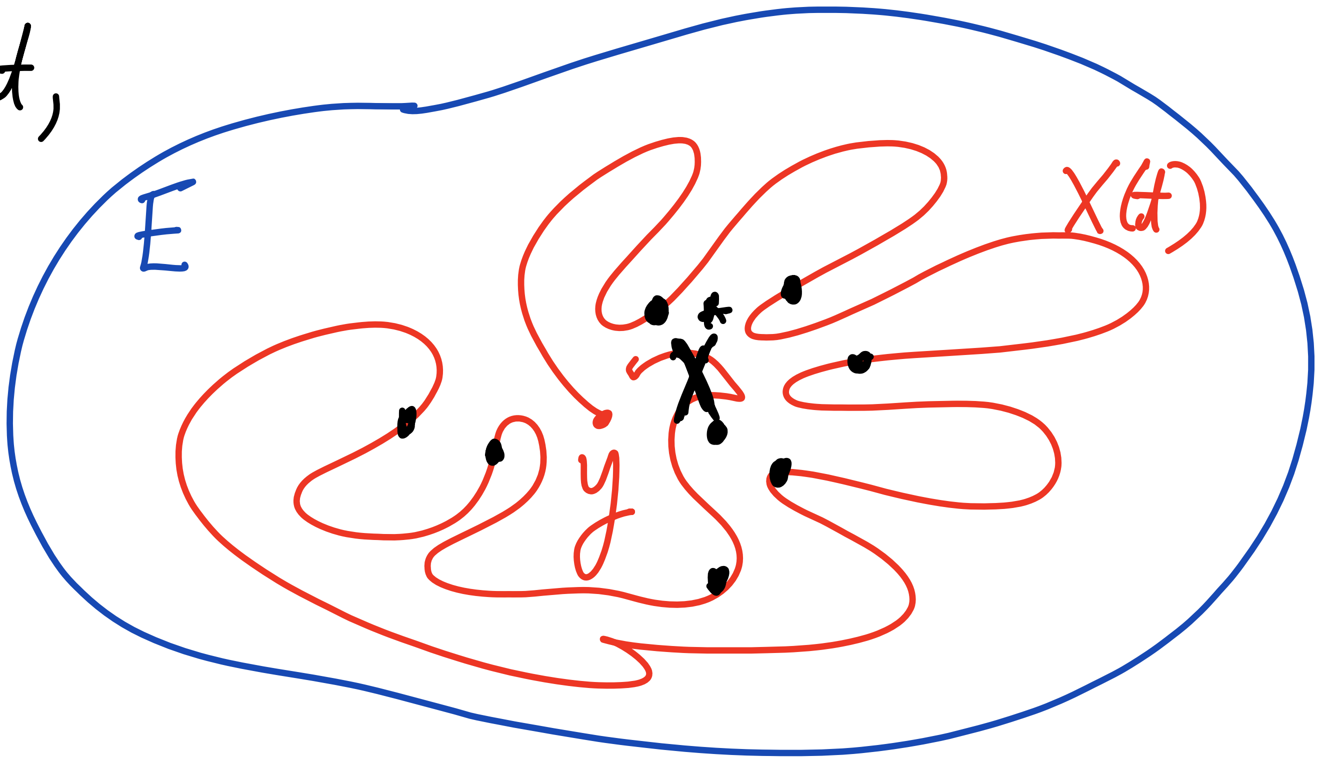


Behavior of $X(t)$ as $t \to \beta^*$ (or α^*)

Let $F : E \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be locally Lip in E

If $\beta^* < \infty$, then as $t \longrightarrow \beta^*$,

(If not,



$\exists\, t_i \to \beta^*$

s.t.

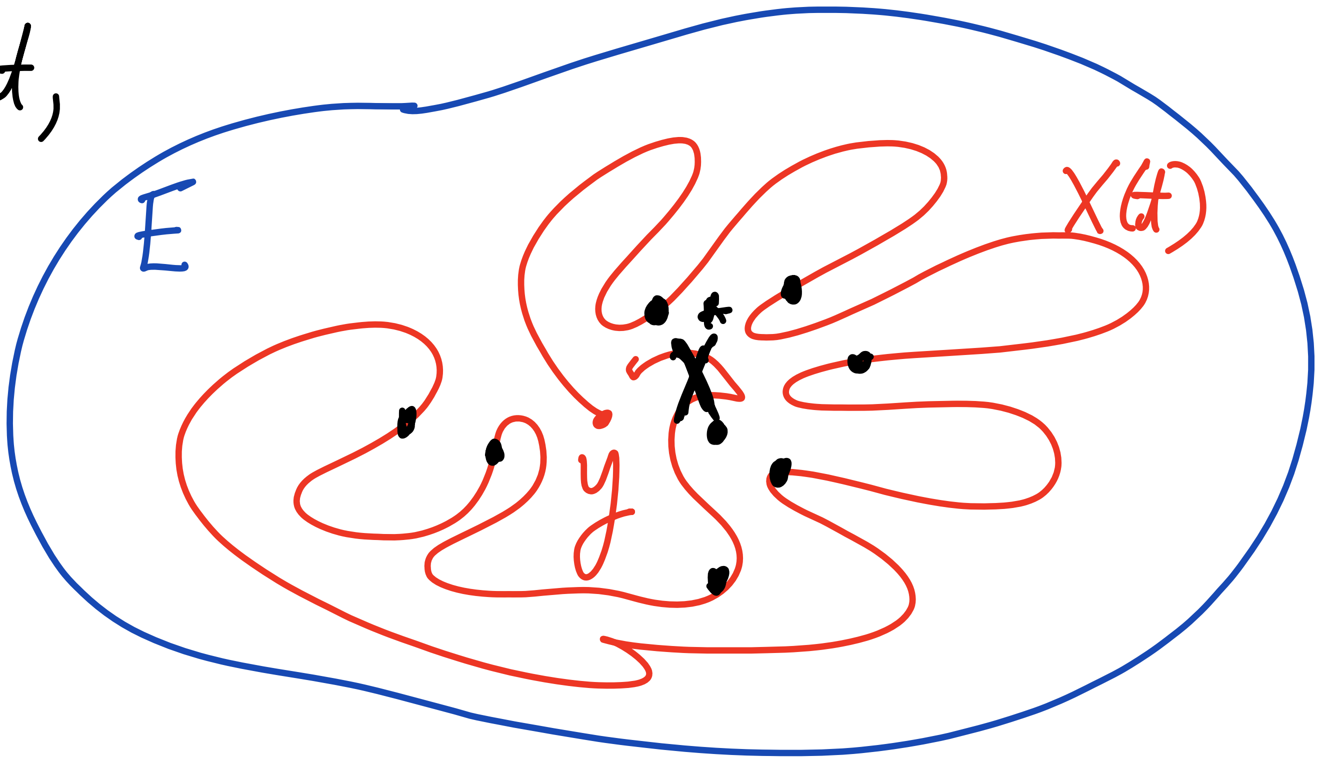
$X(t_i) \to X^*$
$\cap$
E

Behavior of $X(t)$ as $t \to \beta^*$ (or α^*)

Let $F: E \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be locally Lip in E

If $\beta^* < \infty$, then as $t \longrightarrow \beta^*$,

(If not,

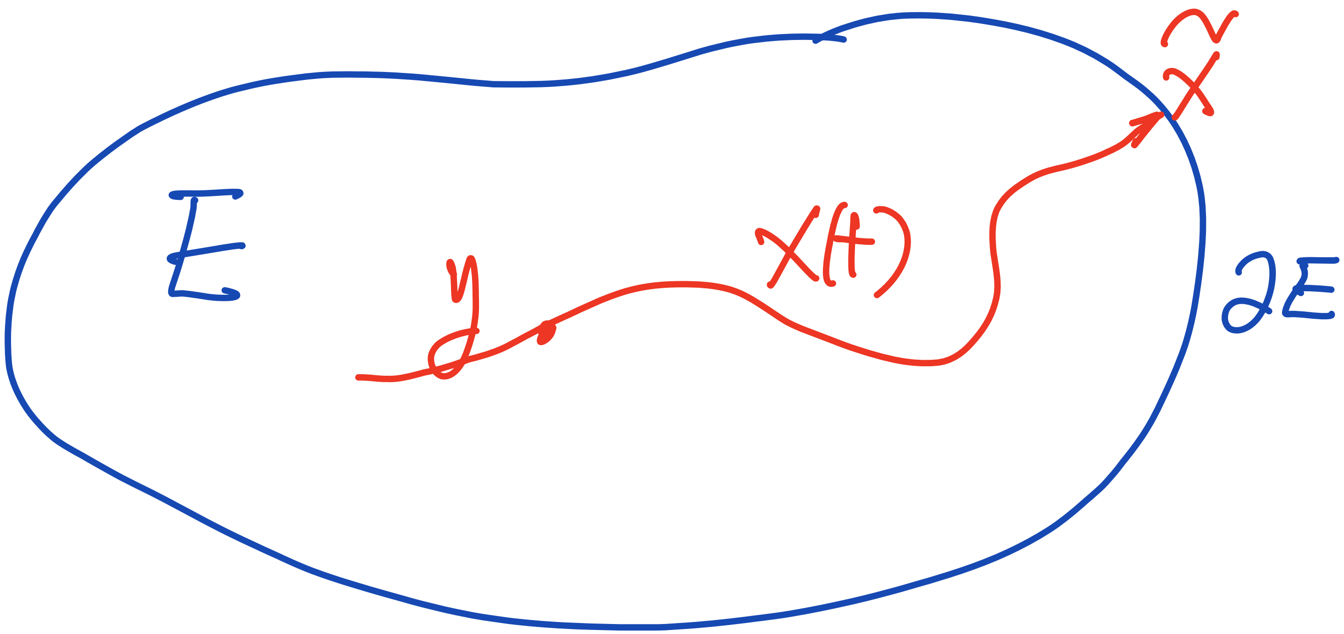


By constructing the solution starting at X^*, can increase β^*!!)

Behavior of $X(t)$ as $t \to \beta^*$ (or α^*)

Let $F: \underline{E} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be locally Lip in E

If $\beta^* < \infty$, and as $t \to \beta^*$, $X(t)$ converges to $\tilde{X}$, then $\tilde{X} \in \partial E$:



Behavior of $X(t)$ as $t \to \beta^*$ (or α^*)

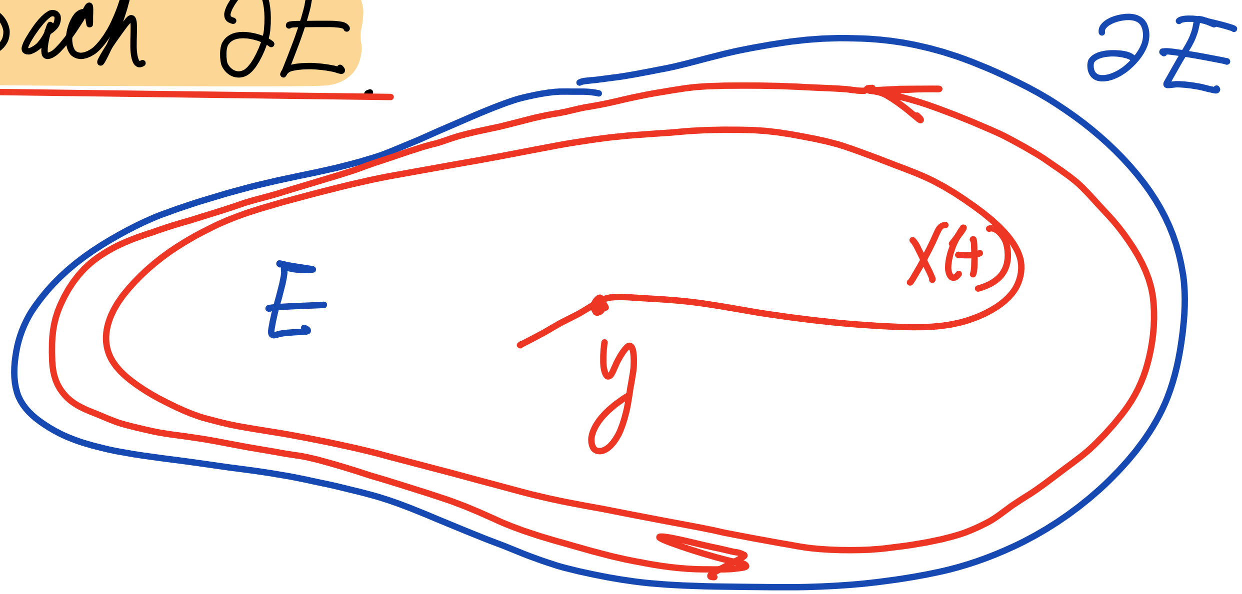
Let $F: E \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be locally Lip in E

If $\beta^* < \infty$, it is possible that as $t \to \beta^*$, $X(t)$ does not have a limit, but still $X(t)$ must approach ∂E.

Behavior of $X(t)$ as $t \to \beta^*$ (or α^*)

Let $F: E \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be locally Lip in E

If $\beta^* < \infty$, it is possible that as $t \to \beta^*$, $X(t)$ does not have a limit, but still $X(t)$ must approach ∂E.

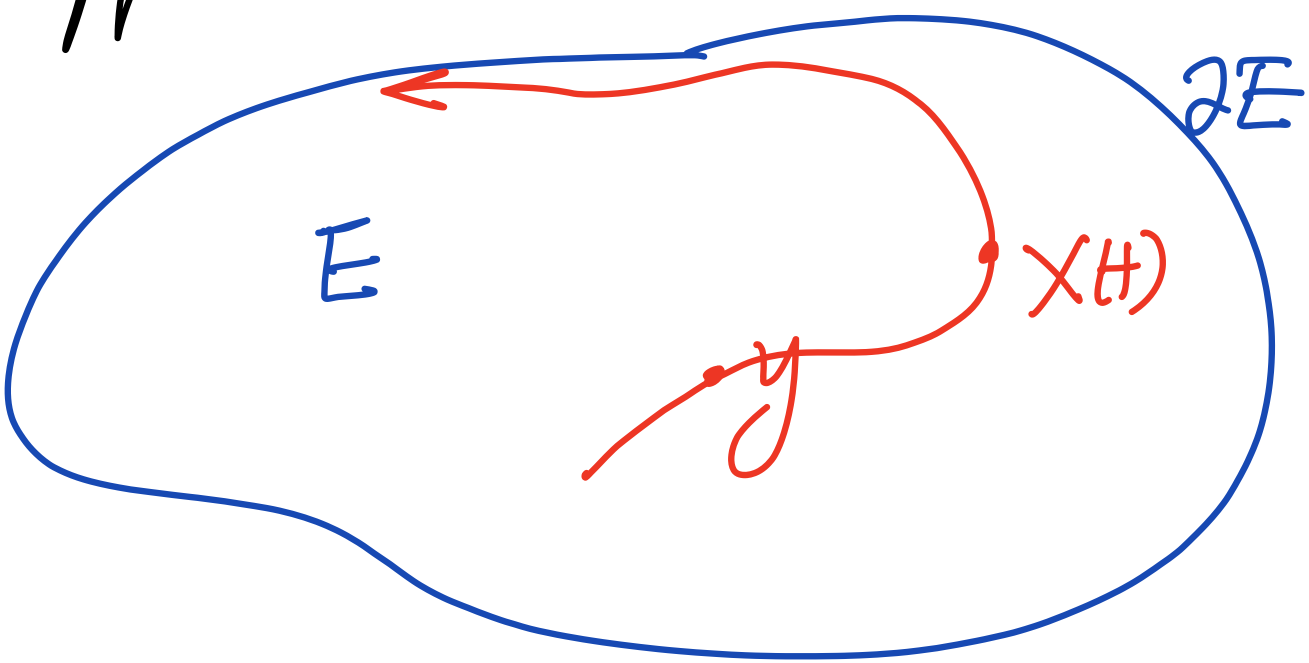


Behavior of $X(t)$ as $t \to \beta^*$ (or α^*)

Let $F : E \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be locally Lip in E

If $\beta^* = \infty$, as $t \to \beta^*$,

$X(t)$ can approach ∂E:

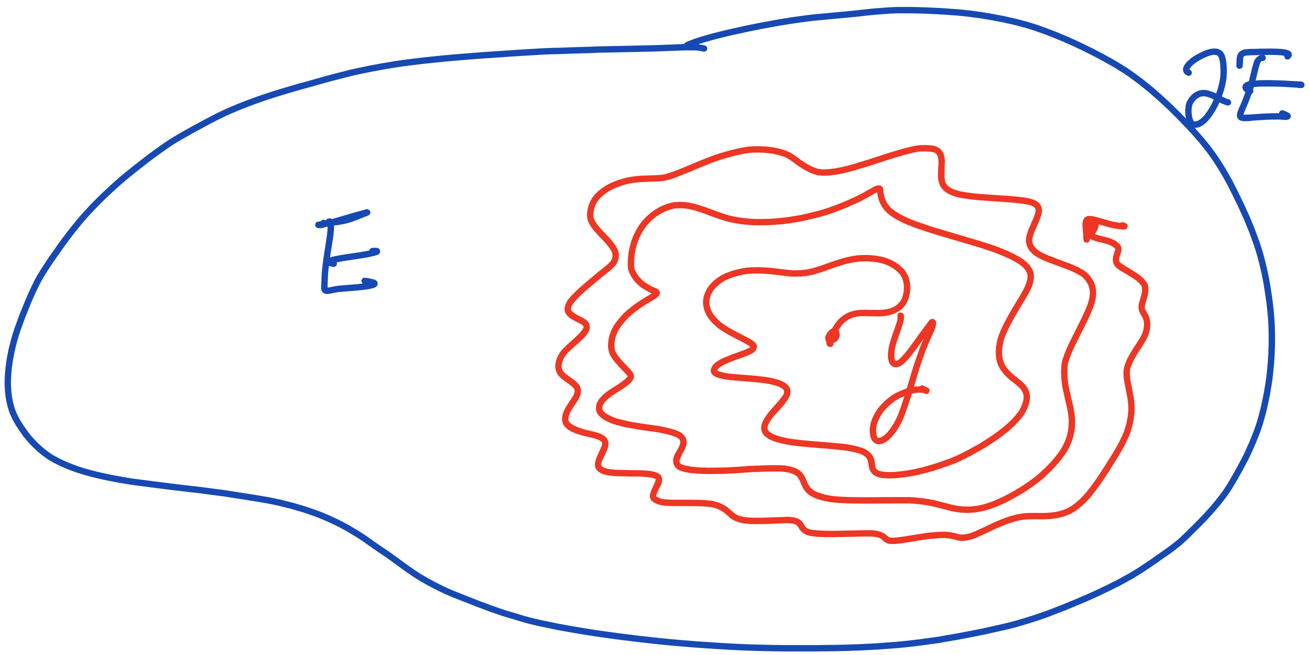


Behavior of $X(t)$ as $t \to \beta^*$ (or α^*)

Let $F: E \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be locally Lip in E

If $\beta^* = \infty$, as $t \to \beta^*$,

or $X(t)$ can remain inside E:

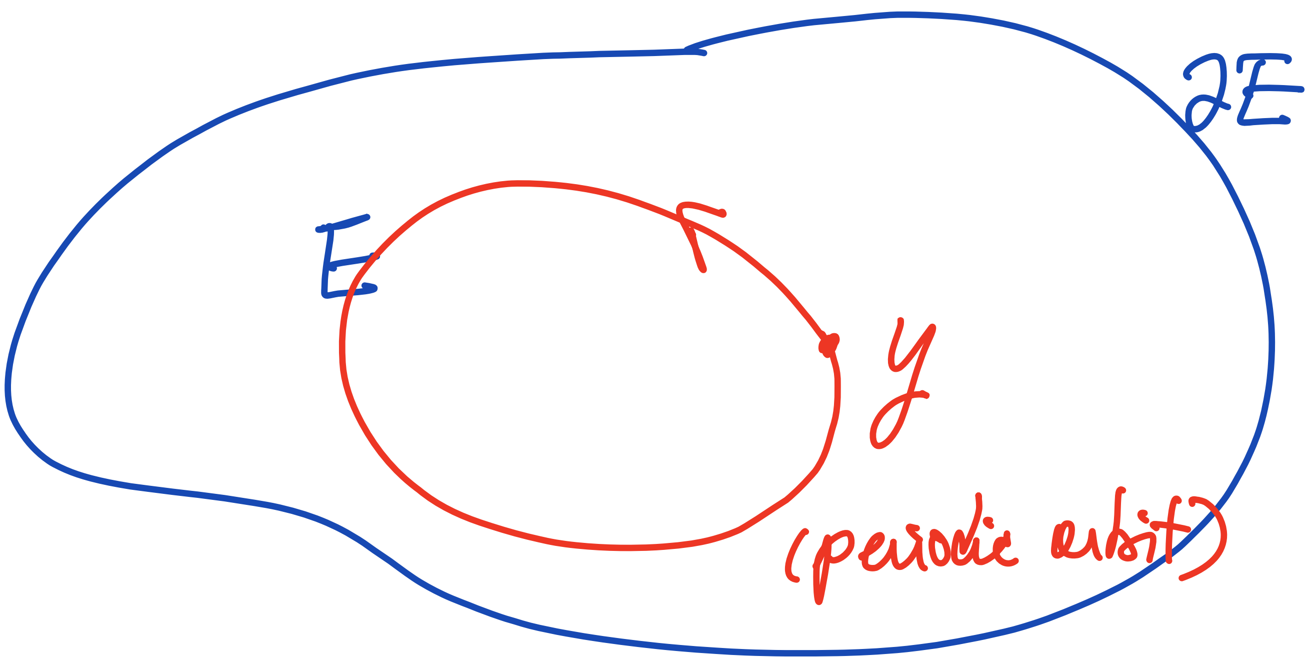


Behavior of $X(t)$ as $t \to \beta^*$ (or α^*)

Let $F: E \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be locally Lip in E

If $\beta^* = \infty$, as $t \to \beta^*$,

or $X(t)$ can remain inside E:



How to Ensure β^* (and α^*) $= \infty$?

[M, p105–107]

$$\frac{dX}{dt} = F(X) \; ; \quad X(0) = X_0 \qquad \text{on } \mathbb{R}^n$$

Thm 4.6 If F is (globally) bounded,

i.e. $\boxed{\|F(X)\| \leq M < \infty}$

Thm 4.8 If F has (at most) linear growth,

i.e. $\boxed{\|F(X)\| \leq A + B\|X\|}$

Thm 4.7 Do a change of time scale

How to Ensure $\beta^* \,(\text{and } \alpha^*) = \infty$?

[M, p105-107]

$$\frac{dX}{dt} = F(X) \;;\quad X(0) = X_0$$

on $\mathbb{R}^n$

Thm 4.7

Do a change of time scale

$$\boxed{\frac{dX}{dt} = F(X)}$$

$\tau = \tau(t)$

$$\boxed{\frac{dY}{d\tau} = \frac{F(Y)}{1 + \|F(Y)\|}}$$

$\leftarrow$ bounded vector field $\Rightarrow$ global solution

How to Ensure β^* (and α^*) $= \infty$?

$$Y(\tau(t)) = X(t) \quad ? \qquad \tau(t) = ?$$

$$\frac{dY}{dt}\,\frac{d\tau}{dt} = \frac{dX}{dt}$$

$$\frac{\cancel{F(Y(\tau))}}{1+\|F(Y(\tau))\|}\,\frac{d\tau}{dt} = \cancel{F(X(t))}$$

i.e.

$$\boxed{\frac{dt}{d\tau} = \frac{1}{1+\|F(Y(\tau))\|}, \qquad t(\tau) = \int_0^{\tau} \frac{dr}{1+\|F(Y(r))\|}}$$

How to Ensure $\beta^* \text{ (and } \alpha^*) = \infty$?

$$Y(\tau(t)) = X(t) \quad ? \qquad \tau(t) = ?$$

$$\frac{dY}{dt}\frac{d\tau}{dt} = \frac{dX}{dt}$$

$$\frac{\cancel{F(Y(\tau))}}{1+\|F(Y(\tau))\|}\frac{d\tau}{dt} = \cancel{F(X(t))}$$

or

$$\boxed{\frac{d\tau}{dt} = 1+\|F(X(t))\|, \quad \tau(t) = \int_0^t \left(1+\|F(X(r))\|\right)dr}$$

Flow Map: $\varphi_t(x)$

$$\frac{dX}{dt} = F(X), \qquad X(0) = x$$

$$X(t) := \varphi_t(x)$$

time

initial data

$$\boxed{\begin{aligned} \frac{d}{dt}\varphi_t(x) &= F(\varphi_t(x)) \\ \varphi_0(x) &= x \end{aligned}}$$

Flow Map: $\varphi_t(x)$

$$\boxed{\frac{d}{dt}\varphi_t(x) = F(\varphi_t(x)), \qquad \varphi_0(x) = x}$$

$$\boxed{\varphi_t \circ \varphi_s = \varphi_{t+s}}$$

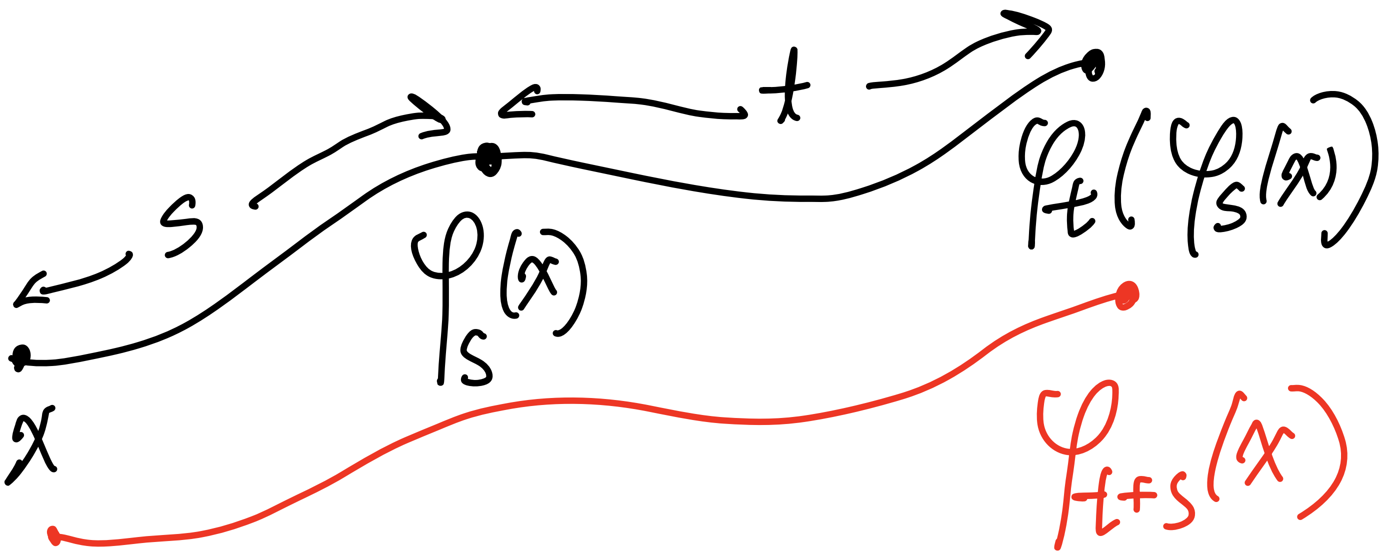
Group property

composition of maps

$$\varphi_t(\varphi_s(x)) = \varphi_{t+s}(x)$$

Flow Map: $\varphi_t(x)$

Concatenation of Solutions



Flow Map: $\varphi_t(x)$

Lipschitz continuity of $\varphi_t(x)$ on $[0, T]$

$$\|\phi_t(x) - \phi_t(y)\| \leq C_T \|x-y\|$$

$$\|\phi_t(x) - \phi_s(x)\| \leq C_T |t-s|$$

$$\|\phi_t(x) - \phi_s(y)\| \leq \|\phi_t(x) - \phi_t(y)\| + \|\phi_t(y) - \phi_s(y)\|$$

$$\leq C_T \left(\|x-y\| + |t-s| \right)$$

Flow Map: $\varphi_t(x)$

Lipschitz continuity of $\varphi_t(x)$ on $[0,T]$

$$\|\phi_t(x)-\phi_t(y)\| \leq C_T\|x-y\|$$

$$\|\phi_t(x)-\phi_s(x)\| \leq C_T|t-s|$$

$$\phi_t(x)-\phi_s(x) = \int_s^t F(\phi_r(x))\,dr$$

$$\|\phi_t(x)-\phi_s(y)\| \leq \|\phi_t(x)-\phi_t(y)\|+\|\phi_t(y)-\phi_s(y)\|$$

$$\leq C_T\left(\|x-y\|+|t-s|\right)$$

Flow Map: $\varphi_t(x)$

The flow/orbit/trajectory

$$x \longrightarrow \phi_t(x)$$

is called complete if $\phi_t(x)$ exists for all $t \in \mathbb{R}$, i.e.

$$\alpha^*(x) = -\infty, \quad \beta^*(x) = +\infty$$