Stability of Equilibrium Point (Lec 10)

Consider $$\boxed{\frac{dx}{dt} = F(X), \quad X(0) = X_0}$$

① X_* is called an equilibrium point if

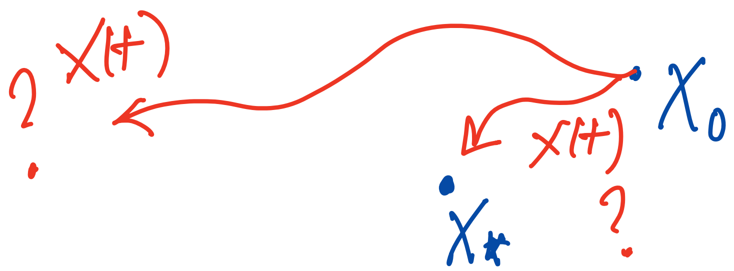
$$\boxed{F(X_*) = 0}$$

② Then $X(t) \equiv X_*$ is a solution

③ What if $X_0 \approx X_*$?

Does $X(t) \longrightarrow X_*$ as $t \to +\infty$

or $X(t)$ move away from X_* as $t \to +\infty$?



Linearize $F(x)$ around x_*

$$F(x) = \underbrace{\cancel{F(x_*)}}_{0} + DF(x_*)(x - x_*) + \frac{1}{2} D^2F(x_*)(x - x_*)^2 + \cdots\cdots$$

$$= \underbrace{[DF(x_*)]}_{A}(x - x_*) + \underbrace{\frac{1}{2} D^2F(x_*)(x - x_*)^2 + \cdots}_{O(|x - x_*|^2)}$$

$$\boxed{F(x) = A(x - x_*) + O(|x - x_*|^2)}$$

For simplicity, let $x_* = 0$

$$\boxed{\begin{aligned} &F(x) = Ax + g(x), \\ &\|g(x)\| = O(\|x\|^2) \le C\|x\|^2 \end{aligned}}$$

$\|x\|^2 \ll \|x\|$ if $\|x\| \ll 1$

Expect behavior of solution of $\frac{dX}{dt} = F(X)$ is similar to that of $\frac{dX}{dt} = AX$, at least when $X_0 \approx X_*$.

① True if A is hyperbolic

$$Re(\lambda_i) \neq 0$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdots\cdots$$

② Not true (in general) is

A is non-hyperbolic

$$Re(\lambda_i) = 0$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdots\cdots$$

$\lambda = 0, 0$ $\lambda = \pm i$

Notions of Stability (of X_*)

① Lyapunov Stability

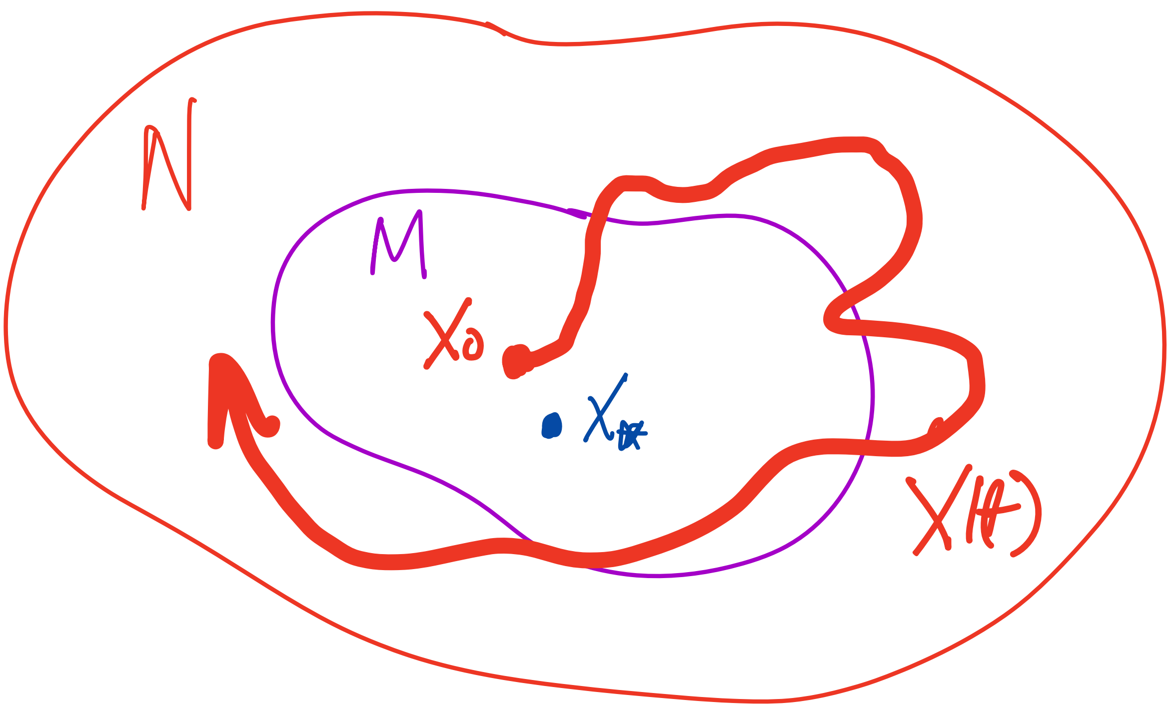
For any neighbourhood N of X_*

There is a (smaller) neighbourhood

M of X_* (usually $M \subseteq N$) s.t.

for any $X_0 \in M$, the solution

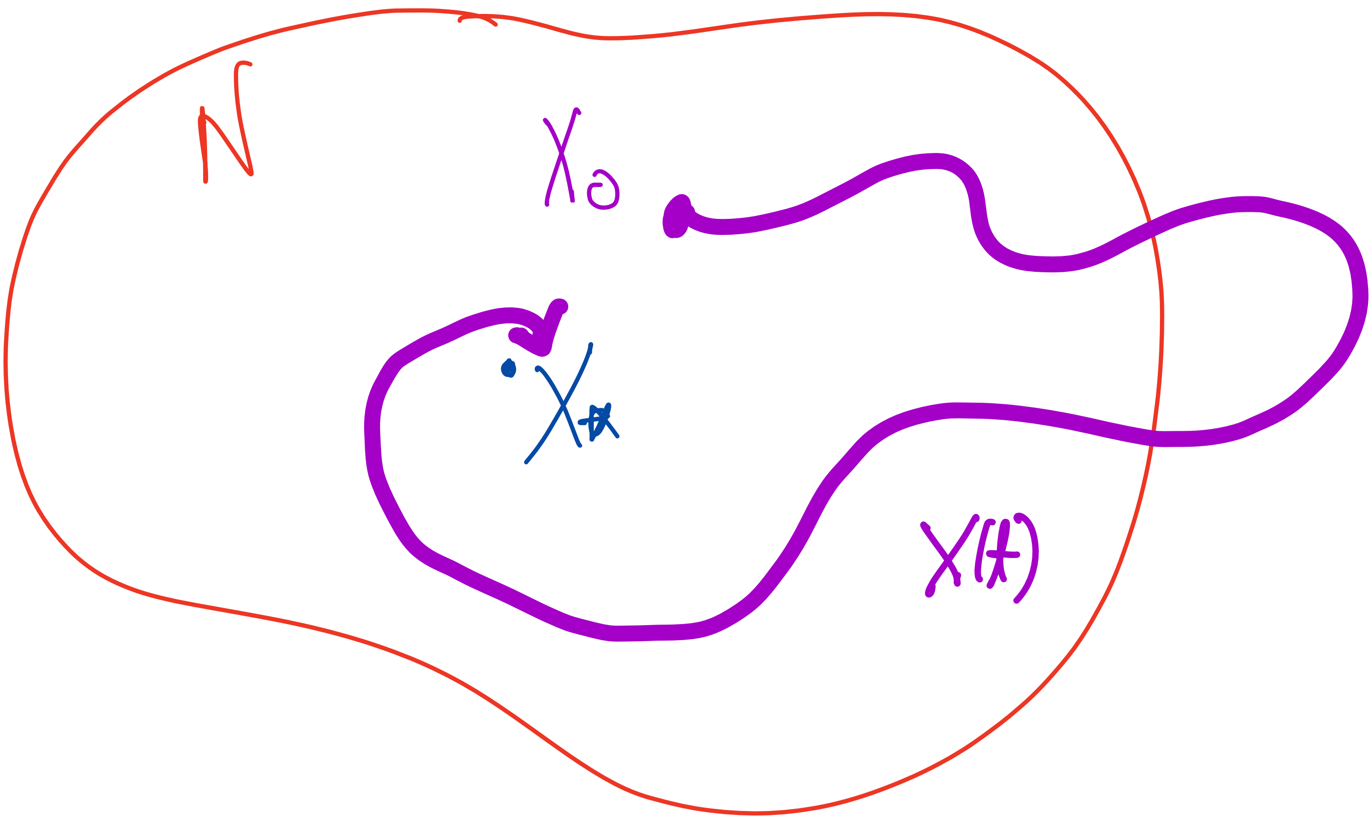
$$\phi_t(X_0) \in N \quad \text{for } t > 0$$



② Asymptotic Stability

There is a neighbourhood N of X_* s.t. for any $X_0 \in N$,

$$\phi_t(X_0) \longrightarrow X_* \text{ as } t \to +\infty$$

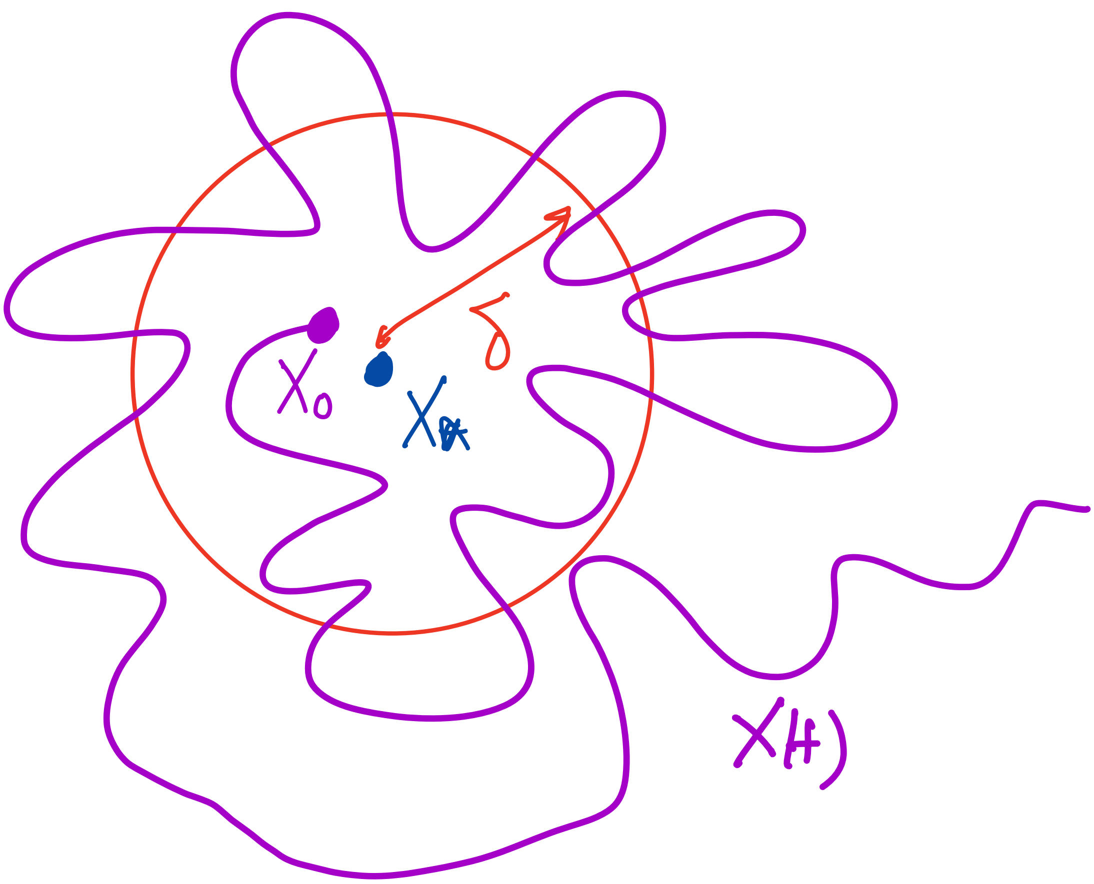


③ Instability of X_*

There is a neighbourhood N of X_* s.t. no matter how close X_0 is to X_*, there are $t_1, t_2, t_3, \cdots \longrightarrow +\infty$ s.t.

$$\phi_{t_i}(X_0) \notin N$$

(There is $\delta > 0$ s.t. $\|\phi_{t_i}(X_0) - X_*\| \geq \delta$)



[M, Thm 4.19] Asymptotic linear stability
$\Longrightarrow$ Asymptotic non-linear stability

Suppose A satisfies $\mathrm{Re}(\lambda_i) < 0$ $\neq 0$
(hyperbolic)
($\Longrightarrow$ there is $K, C > 0$ s.t.

$$\|e^{At}X_0\| \leq M e^{-Kt}\|X_0\|, \text{ for } t > 0)$$

Consider (1 ≤ M)

$$\frac{dX}{dt} = AX + g(X), \quad \|g(x)\| \leq C\|X\|^2$$

$$X(0) = X_0$$

Then there is $\delta > 0$ s.t. for any $\|X_0\| \leq \delta$
we have

$$\|X(t)\| \xrightarrow{t \to +\infty} 0$$

(also exp. fast, with rate asymptotically $= K$)

Pf

$$\|g(x)\| \le C\|x\|^2$$

For any $\varepsilon > 0$, there is $\delta > 0$ s.t.

if $\|x\| \le \delta$, then $\|g(x)\| \le \underline{C\|x\|}\,\|x\|$

$\le \varepsilon \|x\|$

$$\left(\begin{array}{c} C\|x\| \le C\delta \le \varepsilon \\ \text{choose } \delta = \frac{\varepsilon}{C} \end{array} \right)$$

$$\frac{dX}{dt} = AX + \underbrace{g(X)}_{h(t)}, \quad X(0) = X_0$$

$$\left(\begin{array}{l} AX + g(x) \\ \approx -\lambda X + \varepsilon X \\ \approx -(\lambda - \varepsilon)X \end{array} \right)$$

$$X(t) = e^{At}X_0 + \int_0^t e^{A(t-s)} g(X(s))\,ds$$

$$\|X(t)\| \le \|e^{At}X_0\| + \int_0^t \|e^{A(t-s)} g(X(s))\|\,ds$$

$$\le M e^{-Kt}\|X_0\| + \int_0^t M e^{-K(t-s)} \underbrace{\|g(X(s))\|}\,ds$$

if $\underline{\|X(s)\| \le \delta}$, then $\underline{\|g(X(s))\| \le \varepsilon \|X(s)\|}$

$$\leq M e^{-Kt} \|X_0\| + \int_0^t M e^{-K(t-s)} \varepsilon \|X(s)\| \, ds$$

$$\underbrace{e^{Kt} \|X(t)\|}_{g(t)} \leq M \|X_0\| + \int_0^t \varepsilon M \underbrace{e^{Ks} \|X(s)\|}_{g(s)} \, ds$$

$$0 \leq g(t) \leq M \|X_0\| + \int_0^t \varepsilon M g(s) \, ds$$

$$\Downarrow \text{ G.I.}$$

$$g(t) \leq M \|X_0\| e^{\int_0^t \varepsilon M ds} = M \|X_0\| e^{\varepsilon M t}$$

$$e^{Kt} \|X(t)\| \leq M \|X_0\| e^{\varepsilon M t}$$

$$\boxed{\|X(t)\| \leq M \|X_0\| e^{-(K - \varepsilon M)t}} \xrightarrow{t \to +\infty} 0$$

Choose $\varepsilon \ll 1$ s.t. $K - \varepsilon M > 0$

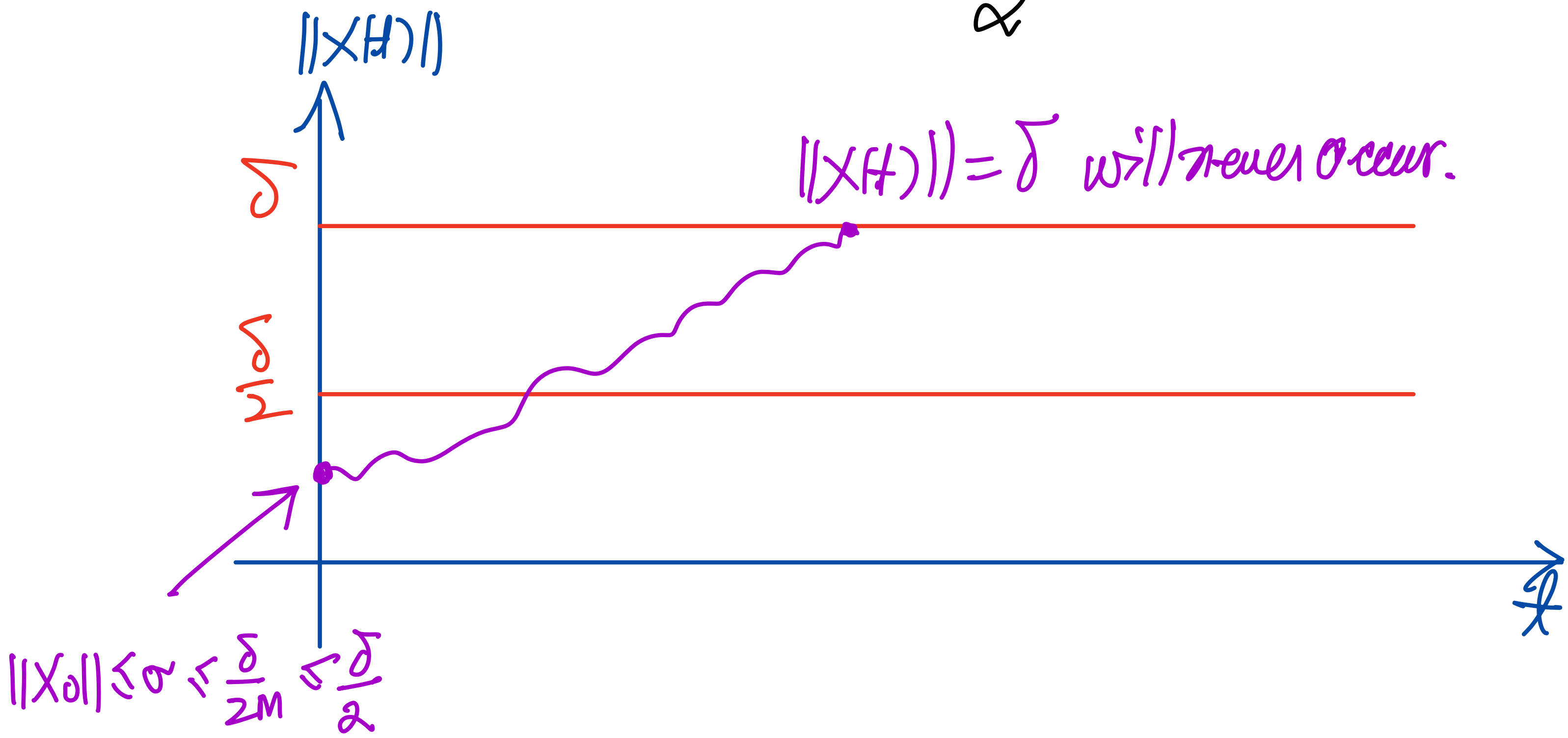
Choose $\underline{\|X_0\| \le \sigma \le \frac{\delta}{2M}}$ $\left(\le \frac{\delta}{2} \ (< \delta) \right)$

$$\Longrightarrow \quad M\|X_0\| \le M\sigma \left(\le \frac{\delta}{2} < \delta \right)$$

Then

$$\|X(t)\| \le M\|X_0\| e^{-(K-\varepsilon M)t}$$

$$\le M\|X_0\| \le M\sigma \le \frac{\delta}{2} \ (< \delta) \text{ for all } t.$$



(If $\quad \|X_0\| \le \frac{\delta}{2M} \left(< \frac{\delta}{2} < \delta \right),$

then $\quad \|X(t)\| \le \frac{\delta}{2} \ (< \delta).$)

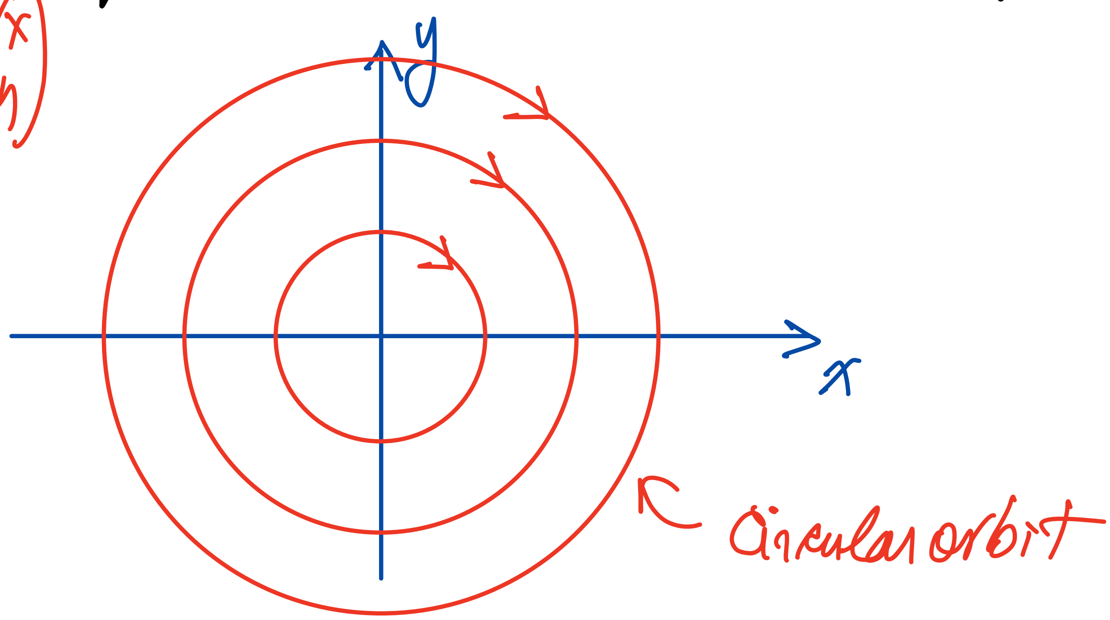
Example of Non-hyperbolic System

① linear $\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$ Harmonic oscillator (Hamiltonian system)

Solution $\begin{cases} x(t) = A\cos t + B\sin t \\ y(t) = B\cos t - A\sin t \end{cases}$ $\begin{pmatrix} x(0)=A \\ y(0)=B \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix}^{\cdot} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}\begin{pmatrix} x \\ y \end{pmatrix}$$

$\lambda = \pm i$



(Simple check:

$$\frac{d}{dt}\left(x(t)^2 + y(t)^2\right) = 2x\dot{x} + 2y\dot{y}$$

$$= 2x(y) + 2y(-x)$$

$$= 0$$

i.e. $x^2(t) + y^2(t) \equiv \text{constant} = x^2(0) + y^2(0)$)

② Nonlinear version

$$\dot{x} = y - x^3$$
$$\dot{y} = -x - y^3$$

$$\frac{d}{dt}\begin{pmatrix}x\\y\end{pmatrix} = \underbrace{\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}}_{AX,\ \text{linear}} + \underbrace{\begin{pmatrix}-x^3\\-y^3\end{pmatrix}}_{g(X) \le O(|X|^2)}$$

$$\frac{d}{dt}(x^2+y^2) = 2x\dot{x} + 2y\dot{y}$$
$$= 2x(\cancel{y} - x^3) + 2y(\cancel{-x} - y^3)$$
$$= -2x^4 - 2y^4 = -2(x^4+y^4) < 0$$

$\Longrightarrow$ $x^2(t) + y^2(t) \downarrow$ in time

as long as $(x,y) \neq (0,0)$