Explicit Computation of W^s

① $\begin{cases} \frac{dx}{dt} = -x \\ \frac{dy}{dt} = y^2 \\ \frac{dz}{dt} = z \end{cases} \iff \frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & & \\ & 0 & \\ & & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ y^2 \\ 0 \end{pmatrix}$

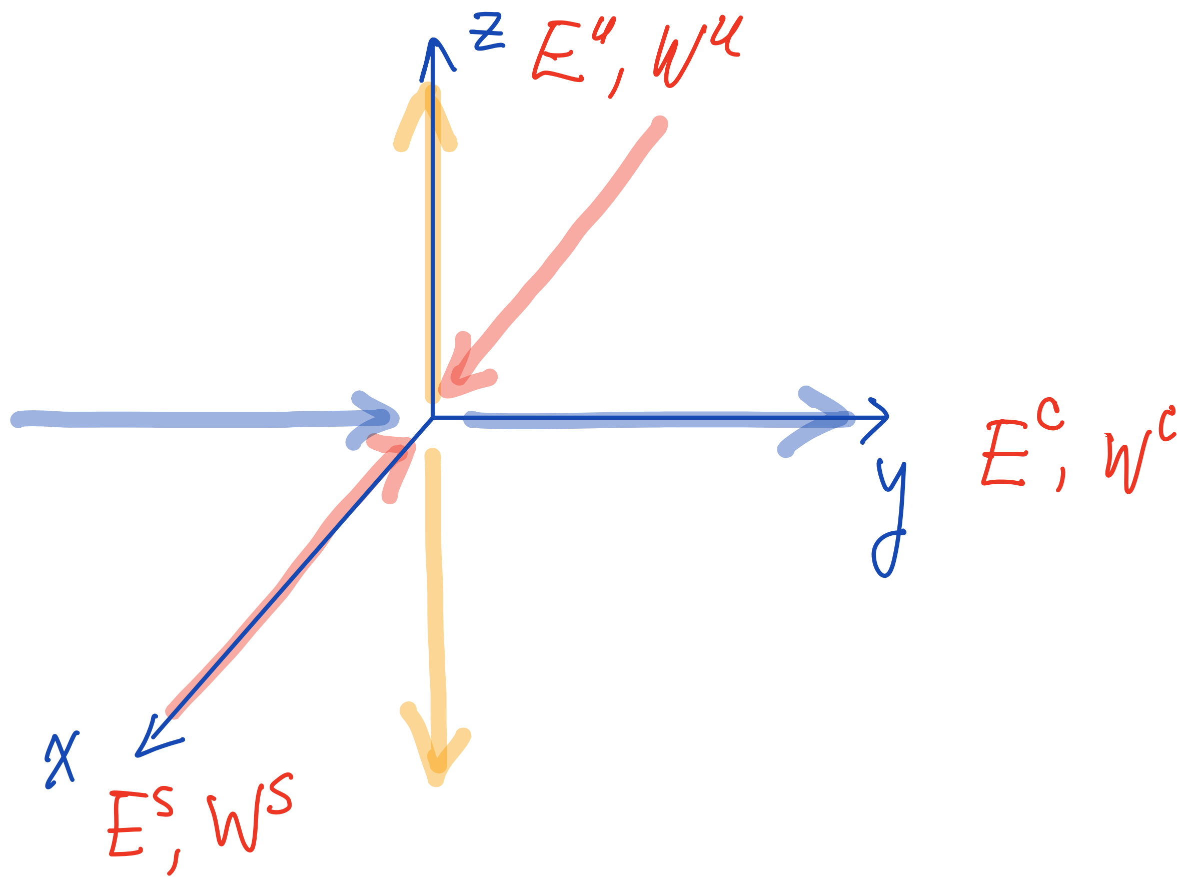
$E^s =$ x-axis

$E^c =$ y-axis

$E^u =$ z-axis

Note that the equations are decoupled. Hence

$$E^s = W^s, \quad E^c = W^c, \quad E^u = W^u$$



(2) $\begin{cases} \dot{x} = -x \\ \dot{y} = -y + x^2 \\ \dot{z} = z + x^2 \end{cases} \iff \frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ x^2 \\ x^2 \end{pmatrix}$

E^s = xy-plane E^u = z-axis

Solve the above system:

$$\dot{x} = -x \implies \underline{x(t) = c_1 e^{-t}}$$

$$\dot{y} = -y + x^2 \implies \dot{y} + y = x^2$$

$$\frac{d}{dt}(e^t y) = e^t x^2 = c_1^2 e^{-t}$$

$$e^t y = c_2 + \int_0^t c_1^2 e^{-s}\, ds$$

$$= c_2 + c_1^2(1 - e^{-t})$$

$$\underline{y(t) = c_2 e^{-t} + c_1^2 (e^{-t} - e^{-2t})}$$

$$\dot{z} = z + x^2 \implies \dot{z} - z = x^2$$

$$\frac{d}{dt}(e^{-t} z) = e^{-t} x^2 = c_1^2 e^{-3t}$$

$$e^{-t} z = c_3 + c_1^2 \int_0^t e^{-3s}\, ds = c_3 + \frac{c_1^2}{3}(1 - e^{-3t})$$

$$\underline{z(t) = c_3 e^t + \frac{c_1^2}{3}(e^t - e^{-2t})}$$

$$\begin{cases} x(t) = c_1 e^{-t} \\ y(t) = c_2 e^{-t} + c_1^2 \left(e^{-t} - e^{-2t}\right) \\ z(t) = c_3 e^{t} + \dfrac{c_1^2}{3}\left(e^{t} - e^{-2t}\right) \end{cases}$$

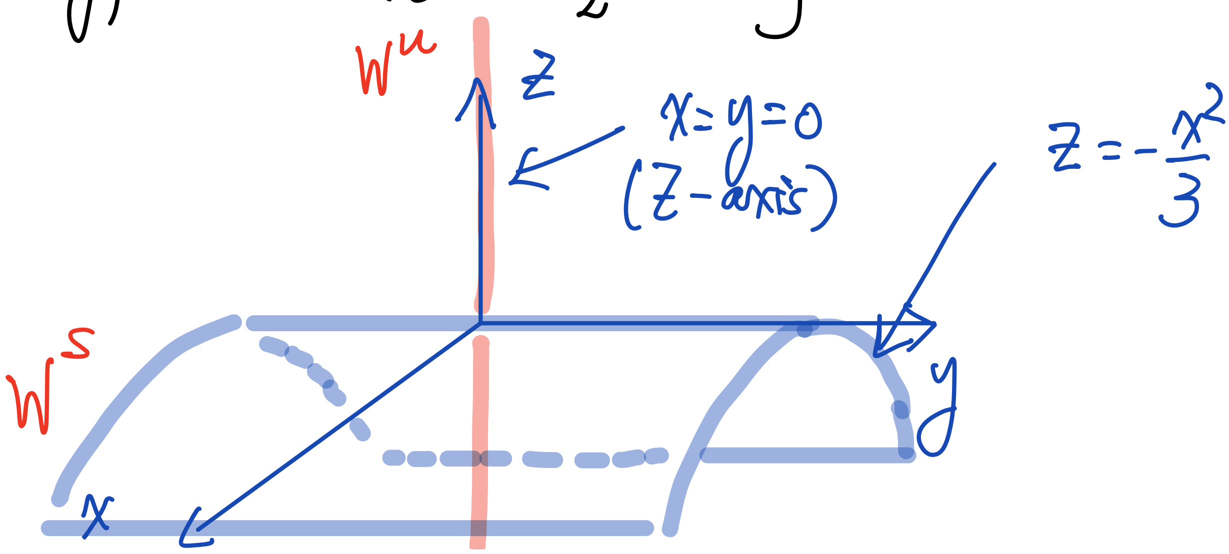
$$\underline{\left(x(0), y(0), z(0)\right) = \left(c_1, c_2, c_3\right)}$$

<u>For W^s,</u> find c_1, c_2, c_3 s.t. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \xrightarrow{t \to +\infty} 0$

From z, we need $c_3 + \dfrac{c_1^2}{3} = 0 \iff z = -\dfrac{x^2}{3}$

<u>For W^u,</u> find c_1, c_2, c_3 s.t. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \xrightarrow{t \to -\infty} 0$

$$\left.\begin{array}{l} \text{From } x, \text{ we need } c_1 = 0 \\ \text{From } y, \text{ we need } c_2 = 0 \end{array}\right\} \quad x = y = 0$$

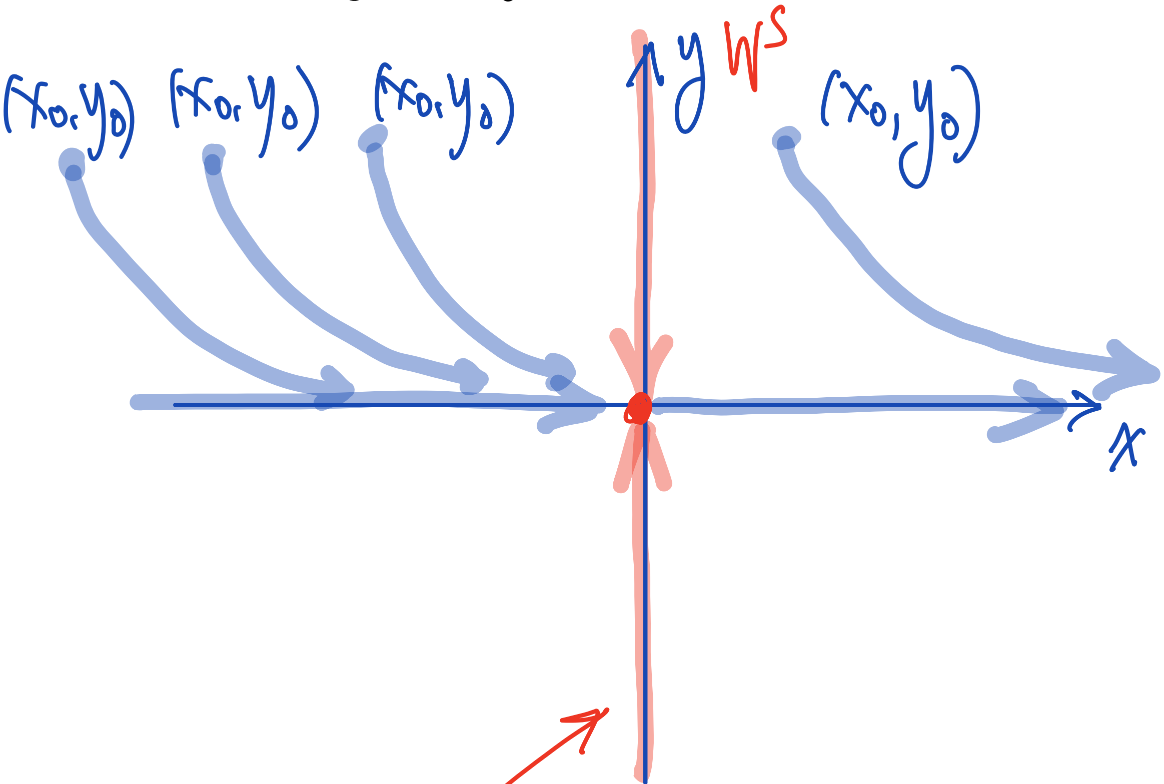


③ $\begin{cases} \dot{x} = x^2 \\ \dot{y} = -y \end{cases} \iff \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x^2 \\ 0 \end{pmatrix}$

$E^c = x\text{-axis}$ $\qquad E^s = y\text{-axis}$

$\dot{x} = x^2, \quad x(t) = \dfrac{x_0}{1 - x_0 t} \to \begin{cases} +\infty, & x_0 > 0,\ t \to \frac{1}{x_0} \\ 0, & x_0 < 0\ \ t \to +\infty \end{cases}$

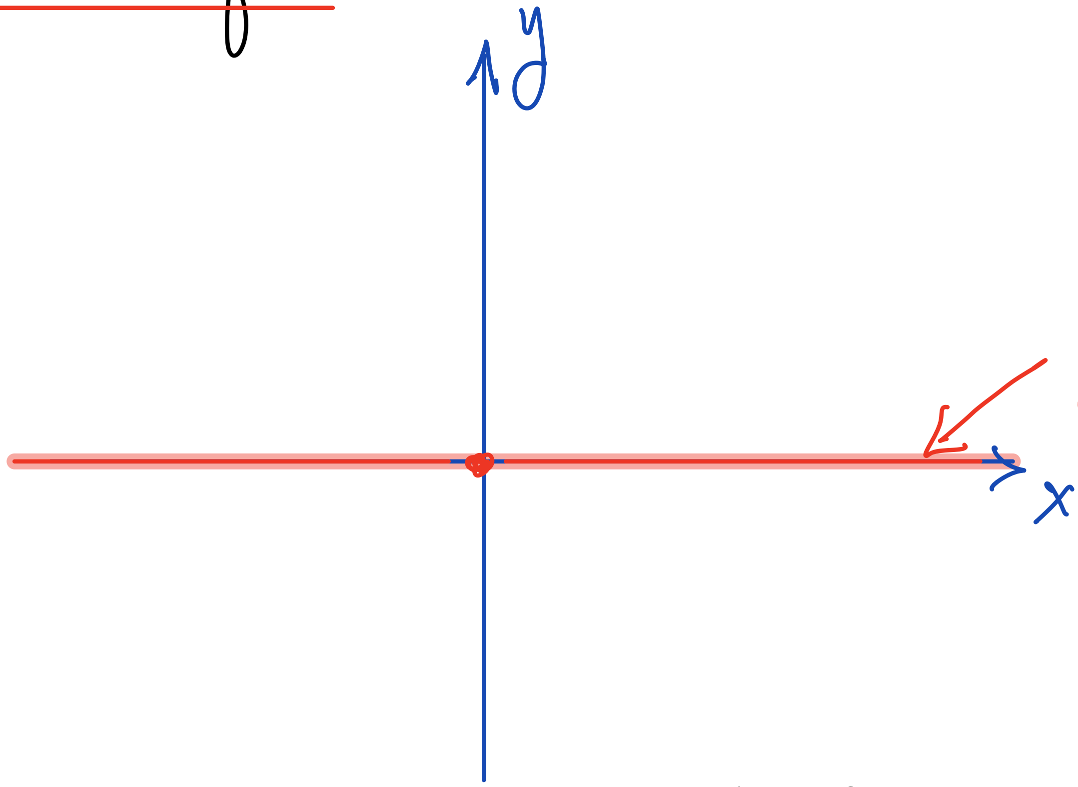
$\dot{y} = -y, \quad y(t) = y_0 e^{-t} \longrightarrow 0$ as $t \to +\infty$



$\underline{W^s = y\text{-axis}}$

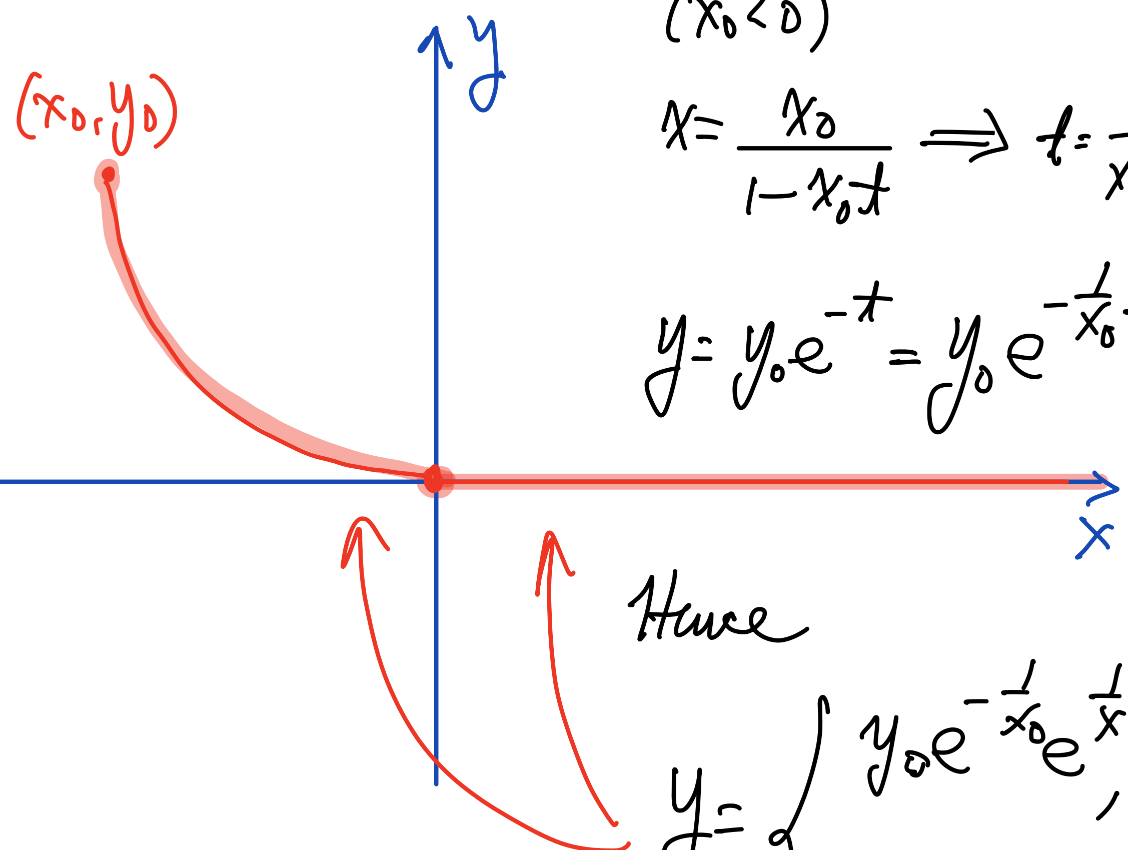
For W^c, it depends on initial condition. It is non-unique

(1)



(2)

$(x_0 < 0)$



$(x_0 < 0)$

$$x = \frac{x_0}{1 - x_0 t} \Longrightarrow t = \frac{1}{x_0} - \frac{1}{x}$$

$$y = y_0 e^{-t} = y_0 e^{-\frac{1}{x_0} + \frac{1}{x}}$$

Hence

$$y = \begin{cases} y_0 e^{-\frac{1}{x_0}} e^{\frac{1}{x}}, & x < 0 \\ 0, & x > 0 \end{cases}$$