Proof of Thm 5.9 (Step I)

Lemma 5.8. *Consider the affine, nonautonomous initial value problem*

$$\dot{x} = Ax + \gamma(t), \quad \pi_s x(0) = \sigma \in E^s. \tag{5.12}$$

Suppose A is hyperbolic and $\gamma(t)$ is bounded and continuous for $t \geq 0$. Then the unique solution, $x(t;\sigma)$, of (5.12) that is bounded for positive time is

$$x(t;\sigma) = e^{tA}\sigma + \int_0^t e^{(t-s)A}\pi_s\gamma(s)ds - \int_t^\infty e^{(t-s)A}\pi_u\gamma(s)ds. \tag{5.13}$$

The uniqueness of the solution (5.13) is surprising because only "half" of the initial conditions have been specified, the stable components σ. We will see that the assumption that x is bounded for $t > 0$ is enough to determine its unstable components.

Proof. Note that $\gamma(t)$, σ are given.

Need to find $X(0)$ s.t. $\pi_s X(0) = \sigma$ and $X(t)$ is bounded.

Solution of (5.13) is given by

$$X(t) = e^{At}X(0) + \int_0^t e^{A(t-s)}\gamma(s)\,ds$$

$$= e^{At}\left[\underbrace{\pi_s X(0)}_{\sigma} + \underbrace{\pi_u X(0)}_{\eta}\right] + \int_0^t e^{A(t-s)}\left[\pi_s\gamma(s) + \pi_u\gamma(s)\right]ds$$

$$= \underset{①}{e^{At}\sigma} + \int_0^t e^{A(t-s)}\pi_s\gamma(s)\,ds \quad ②$$

$$+ \underset{③}{e^{At}\eta} + \int_0^t e^{A(t-s)}\pi_u\gamma(s)\,ds \quad ④$$

① : $\|e^{At}\sigma\| \le Ce^{-Kt}\|\sigma\| \xrightarrow{t \to +\infty} 0$

② : $\left\| \int_0^t e^{A(t-s)} \Pi_s \gamma(s)\, ds \right\|$

$$\le \int_0^t \| e^{A(t-s)} \Pi_s \gamma(s) \| ds$$

$$\le \int_0^t C e^{-K(t-s)} \| \Pi_s \gamma(s) \| ds$$

$$\le CM \int_0^t e^{-K(t-s)} ds = \frac{CM[1-e^{-Kt}]}{K} \le \frac{CM}{K}$$

③ + ④ $e^{At}\eta + \int_0^t e^{A(t-s)} \Pi_u(\gamma(s))\, ds$

$$= \underbrace{e^{At}}_{\xrightarrow{t \to +\infty}\ \infty} \underbrace{\left[\underbrace{\eta}_{\in E_u} + \underbrace{\int_0^t e^{-As} \Pi_u \gamma(s)\, ds}_{\in E_u} \right]}_{\text{must go to } 0}$$

stay bounded

$$\Rightarrow \quad \eta + \int_0^\infty e^{-As}\pi_u \gamma(s)\,ds = 0$$

i.e. $$\underbrace{\boxed{\eta = -\int_0^\infty e^{-As}\pi_u \gamma(s)\,ds}}$$ ← The only choice for η

$$\left\| \int_0^\infty e^{A(t-s)}\pi_u \gamma(s)\,ds \right\| \leq \int_0^\infty C e^{-Ks} M\,ds < \infty$$

Does it work? $\infty \times 0 = ?$

Consider ③ + ④

$$= e^{At}\left[\eta + \int_0^t e^{-As}\pi_u \gamma(s)\,ds\right]$$

$$= e^{At}\left[-\int_0^\infty e^{-As}\pi_u(\gamma(s))\,ds + \int_0^t e^{-As}\pi_u \gamma(s)\,ds\right]$$

$$= -e^{At}\int_t^\infty e^{-As}\pi_u \gamma(s)\,ds$$

$$= - \int_t^\infty e^{A(t-s)} \underbrace{\Pi_u \gamma(s)}_{\in E^u} ds$$

$t-s<0$, $\in E^u$

Hence
$$\left\| \int_t^\infty e^{A(t-s)} \Pi_u \gamma(s)\, ds \right\|$$
$$\leq \int_t^\infty C e^{-K(s-t)} \| \Pi_u \gamma(s) \| ds$$
$$\leq CM \int_t^\infty e^{-K(s-t)} ds = \frac{CM}{K} < \infty$$

Hence

$$\boxed{\begin{aligned} X(t) &= ① + ② + ③ + ④ \\ &= e^{At}\sigma + \int_0^t e^{A(t-s)} \Pi_s \gamma(s)\, ds - \int_t^\infty e^{A(t-s)} \Pi_u \gamma(s)\, ds \end{aligned}}$$

Uniqueness Let $X(t)$ & $Y(t)$ be two bounded solutions of $\dot{X} = AX + f(t)$, $\pi_s X(0) = \sigma$

$$\begin{cases} \dot{X}(t) = AX(t) + f(t), & \pi_s X(0) = \sigma \\ \dot{Y}(t) = AY(t) + f(t), & \pi_s Y(0) = \sigma \end{cases}$$

$$\underbrace{(X-Y)^{\cdot} = A(X-Y)}, \quad \underbrace{\pi_s(X(0) - Y(0)) = 0}$$

$$\Rightarrow (X(0) - Y(0)) \in E_u$$

$$\Downarrow$$

$$\underbrace{X(t) - Y(t) = e^{At} \underbrace{(X(0) - Y(0))}_{\in E_u}}$$

$$\to \infty \quad \text{as } t \to +\infty$$

Hence $\boxed{X(0) - Y(0) = 0} \Longrightarrow X(t) \equiv Y(t).$

Proof of Thm 5.9 (Step II)

Consider $\frac{dX}{dt} = AX + \underbrace{g(X)}_{\gamma(t)\text{ — given}}, \quad X(0) = X_0$

Suppose $X(t) \xrightarrow{t\to+\infty} 0$, in particular, $X(t)$ is bounded

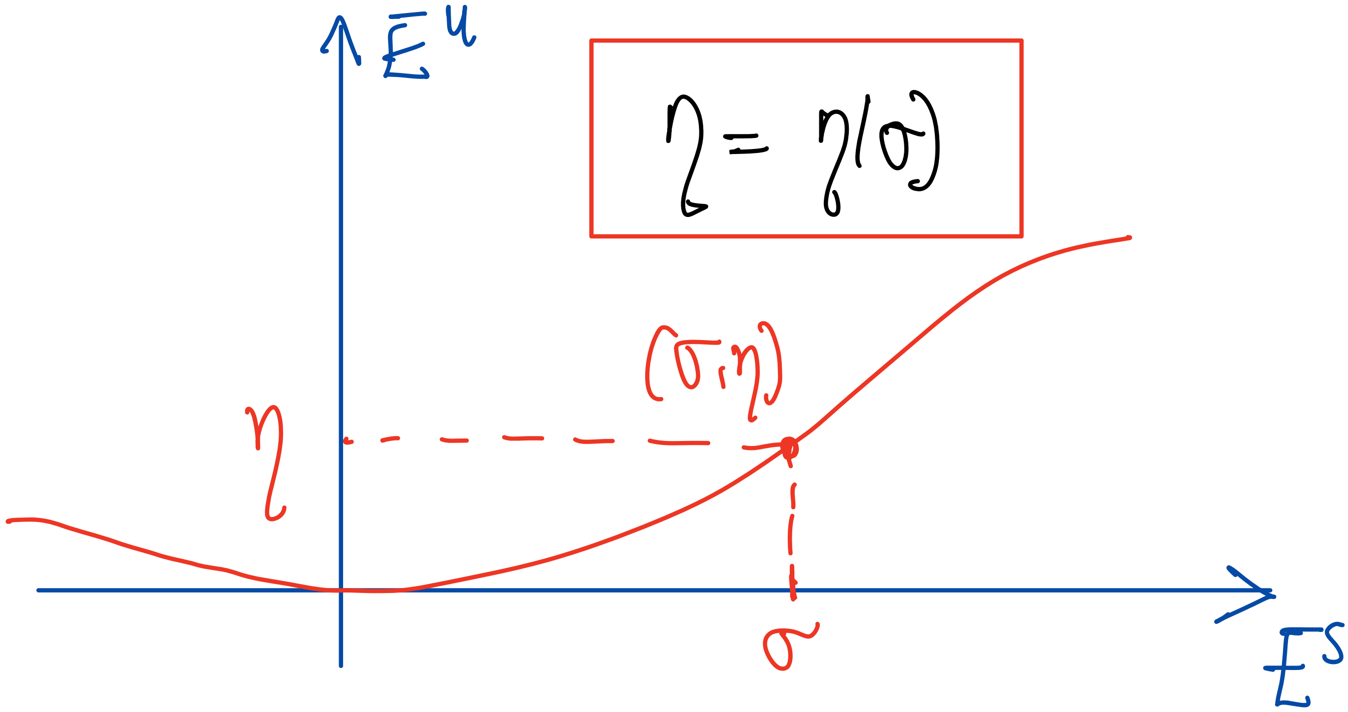
Then $X(t)$ must be given by:

$$\boxed{X(t) = e^{At}\pi_s X_0 + \int_0^t e^{A(t-s)}\pi_s g(X(s))ds - \int_t^\infty e^{A(t-s)}\pi_u g(X(s))ds}$$

$$X(0) = \underbrace{\pi_s X(0)}_{\sigma \in E^s} - \underbrace{\int_0^\infty e^{-As}\pi_u g(X(s))ds}_{\eta \in E^u}$$

σ is given $\Longrightarrow X(t) \Longrightarrow \eta = -\int_0^\infty e^{-As}\pi_u g(X(s))ds$

$= \eta(\sigma)$



For any $\sigma \in E^s$, define

$$T : C^0(\mathbb{R}^+, \mathbb{R}^n) \longrightarrow C^0(\mathbb{R}^+, \mathbb{R}^n)$$

$$(TX)(t) = e^{tA}\sigma + \int_0^t e^{A(t-s)}\Pi_s\, g(X(s))\,ds - \int_t^\infty e^{A(t-s)}\Pi_u\, g(X(s))\,ds$$

Find a fixed point of T: $X = T(X)$

i.e.

$$X(t) = e^{tA}\sigma + \int_0^t e^{A(t-s)}\Pi_s\, g(X(s))\,ds - \int_t^\infty e^{A(t-s)}\Pi_u\, g(X(s))\,ds$$

Let $\delta > 0$, (to be found)

$$G_\delta = \{ X : X(t) \in C^0(\mathbb{R}^+, \mathbb{R}^n), \|X(t)\| \leq \delta \}$$

(1) $T : G_\delta \longrightarrow G_\delta$

(2) $X, Y \in G_\delta$,

$$\|TX - TY\| \leq c\,\|X - Y\|, \qquad c < 1.$$

(Then by Banach-Fixed Point Thm, there is a unique fixed point in G_δ.)

Two Properties of $g(X)$

$$\boxed{\|g(X)\| \leq O(|X|^2) \ll |X| \text{ for } |X| \ll 1}$$

(1) $\forall \varepsilon > 0, \exists \delta > 0$ such that if $\|X\| \leq \delta$, then $\boxed{\|g(X)\| \leq \varepsilon \|X\|}$

(2) $\forall \varepsilon > 0, \exists \delta > 0$ such that if $\|X\|, \|Y\| \leq \delta$, then

$$\boxed{\|g(X) - g(Y)\| \leq \varepsilon \|X - Y\|}$$

Proof of $T: C_\delta \to C_\delta$

$$(TX)(t) = e^{At}\sigma + \int_0^t e^{A(t-s)}\Pi_s g(X(s))\,ds - \int_t^\infty e^{A(t-s)}\Pi_u g(X(s))\,ds$$

$$\|(TX)(t)\| \le e^{At}\sigma + \int_0^t \|e^{A(t-s)}\Pi_s g(X(s))\|\,ds - \int_t^\infty \|e^{A(t-s)}\Pi_u g(X(s))\|\,ds$$

$$\le Ce^{-Kt}\|\sigma\| + \int_0^t Ce^{-K(t-s)}\varepsilon\|X(s)\|\,ds + \int_t^\infty Ce^{K(t-s)}\varepsilon\|X(s)\|\,ds$$

$$\le C\|\sigma\| + \int_0^t Ce^{-K(t-s)}\varepsilon\delta\,ds + \int_t^\infty Ce^{K(t-s)}\varepsilon\delta\,ds$$

$$= C\left[\|\sigma\| + \varepsilon\delta\int_0^t e^{-K(t-s)}\,ds + \varepsilon\delta\int_t^\infty e^{K(t-s)}\,ds\right]$$

$$= C\left[\|\sigma\| + \varepsilon\delta\frac{(1-e^{-Kt})}{K} + \varepsilon\delta\frac{1}{K}\right]$$

$$= C\left[\|\sigma\| + \frac{2\varepsilon\delta}{K}\right]$$

$$= \underbrace{C\|\sigma\|}_{} + \underbrace{\frac{2C}{K}\varepsilon\delta}_{} \leq \delta$$

Choose $\|\sigma\| \leq \frac{\delta}{2C}$, Choose $\varepsilon = \frac{K}{4C}$

Proof of Contraction Property of T

$$(TX)(t) = e^{At}\sigma + \int_0^t e^{A(t-s)}\Pi_s g(X(s))\,ds - \int_t^\infty e^{A(t-s)}\Pi_u g(X(s))\,ds$$

$$(TY)(t) = e^{At}\sigma + \int_0^t e^{A(t-s)}\Pi_s g(Y(s))\,ds - \int_t^\infty e^{A(t-s)}\Pi_u g(Y(s))\,ds$$

$$\|(TX)(t) - (TY)(t)\|$$

$$= \Big\| \int_0^t e^{A(t-s)}\left(\Pi_s g(X(s)) - \Pi_s g(Y(s))\right)ds$$

$$- \int_t^\infty e^{A(t-s)}\left(\Pi_u g(X(s)) - \Pi_u g(Y(s))\right)ds \Big\|$$

$$\leq \int_0^t Ce^{-K(t-s)} \varepsilon \|X(s) - Y(s)\| ds$$

$$+ \int_t^\infty Ce^{K(t-s)} \varepsilon \|X(s) - Y(s)\| ds$$

$$\leq \left(\int_0^t Ce^{-K(t-s)} \varepsilon \, ds + \int_t^\infty Ce^{K(t-s)} \varepsilon \, ds \right) \|X - Y\|$$

$$\leq C\varepsilon \left(\frac{1}{K} + \frac{1}{K} \right) \|X - Y\|$$

$$= \underbrace{\left(\frac{2C\varepsilon}{K} \right)} \|X - Y\|$$

choose $\varepsilon = \dfrac{K}{4C}$ (same as before)

Then $\left(\dfrac{2C\varepsilon}{K} \right) < 1$

Proof of Thm 5.9 (Step III)

Properties of fixed pt :

1 $\sigma \longrightarrow X(t) \longrightarrow \underbrace{\eta = -\int_0^\infty e^{-As} \Pi_u g(X(s))\,ds}$

$\left(\|\sigma\| \le \frac{\delta}{2C}\right) \quad \left(\|X(t)\| \le \delta\right) \quad \left(\eta = \eta(\sigma)\right)$

$g(x) = O(x^2)$: $\|g(x)\| \le D\|x\|^2$

$$\|\eta\| \le \int_0^\infty C e^{-Ks} D \|X(s)\|^2\,ds$$

$$\le CD\delta^2 \int_0^\infty e^{-Ks}\,ds = \left(\frac{CD}{K}\right)\delta^2$$

$$\boxed{\|\sigma\| \le \frac{1}{2C}\delta \Longrightarrow \|\eta = \eta(\sigma)\| \le \frac{CD}{K}\delta^2 \simeq O(\|\sigma\|^2)}$$

$$\left(\|\eta\| \lesssim \frac{4C^3D}{K}\|\sigma\|^2\right)$$

② $X(t) \xrightarrow{t \to +\infty} 0$ (exponentially fast)

Lemma 5.10 (Generalized Grönwall). *Suppose α, M, and L are nonnegative, $L < \alpha/2$, and there is a nonnegative, bounded, continuous function $u : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying*

$$u(t) \le e^{-\alpha t} M + L \int_0^t e^{-\alpha(t-s)} u(s)\,ds + L \int_t^\infty e^{\alpha(t-s)} u(s)\,ds; \tag{5.22}$$

$L < \alpha/3$?

then $u(t) \le \frac{M}{\beta} e^{-(\alpha - L/\beta)t}$, where $\beta = 1 - 2\frac{L}{\alpha}$.

$$X(t) = e^{At}\sigma + \int_0^t e^{A(t-s)} \Pi_s\, g(X(s))\,ds - \int_t^\infty e^{A(t-s)} \Pi_u\, g(X(s))\,ds$$

$$\|X(t)\| \le Ce^{-Kt}\|\sigma\| + \int_0^t C\varepsilon e^{-K(t-s)} \|X(s)\|\,ds$$

$$+ \int_t^\infty C\varepsilon e^{K(t-s)} \|X(s)\|\,ds$$

Choose ε s.t. $C\varepsilon \le \frac{K}{3}$ i.e. $\varepsilon < \frac{K}{3C}$

Then $$\|X(t)\| \le \frac{C\|\sigma\|}{\left(1 - \frac{2C\varepsilon}{K}\right)} e^{-\left(K - \frac{C\varepsilon}{1 - \frac{2C\varepsilon}{K}}\right)t}$$

$$= \frac{C\|\sigma\|}{\left(1 - \frac{2C\varepsilon}{K}\right)} e^{-(K - 3C\varepsilon)t} \xrightarrow[t \to +\infty]{} 0$$