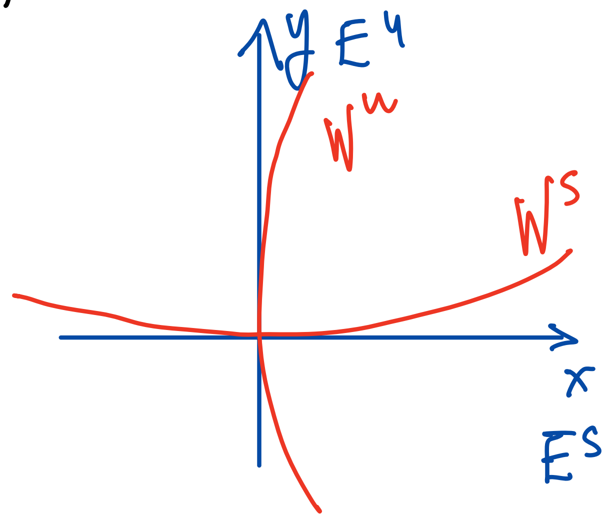
Examples of Inv. Manifolds – II

1. (Perko, p. 111)



$$\begin{cases} \dot{x} = -x - y^2 \\ \dot{y} = y + x^2 \end{cases} \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -y^2 \\ x^2 \end{pmatrix}$$

$$\frac{d}{dt} X = AX + g(X), \qquad g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y^2 \\ x^2 \end{pmatrix}$$

$$A = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$$

$E_s = x$-axis $\quad \lambda_1 = -1 < 0$

$E_u = y$-axis $\quad \lambda_2 = 1 > 0$

Formula for W^s

$$X(t) = e^{At}\begin{pmatrix} \sigma \\ 0 \end{pmatrix} + \int_0^t e^{A(t-s)} \Pi_s\, g(X(s))\,ds - \underbrace{\int_t^\infty e^{A(t-s)} \Pi_u\, g(X(s))\,ds}_{t=0}$$

$t=0$

$$\sigma = \Pi_s X(0)$$

$$\eta = \Pi_u X(0) = -\int_0^\infty e^{-As} \Pi_u\, g(X(s))\,ds$$

$$\boxed{\begin{aligned} &\text{Define: } \quad T : X(\cdot) \longrightarrow X(\cdot) \\ &(TX)(t) = e^{At}\begin{pmatrix} \sigma \\ 0 \end{pmatrix} + \int_0^t e^{A(t-s)} \Pi_s\, g(X(s))\,ds - \int_t^\infty e^{A(t-s)} \Pi_u\, g(X(s))\,ds \end{aligned}}$$

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e^{At} = \begin{pmatrix} e^{-t} & \\ & e^{t} \end{pmatrix}$$

$$g(X) = \begin{pmatrix} -y^2 \\ x^2 \end{pmatrix}, \quad \Pi_s g(X) = \begin{pmatrix} -y^2 \\ 0 \end{pmatrix}, \quad \Pi_u g(X) = \begin{pmatrix} 0 \\ x^2 \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{-t}\sigma \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} e^{-(t-s)}(-y^2(s)) \\ 0 \end{pmatrix} ds - \int_t^\infty \begin{pmatrix} 0 \\ e^{(t-s)} x^2(s) \end{pmatrix} ds$$

<u>Iteration:</u>

$$X^{(1)} = T(X^{(0)})$$

$$X^{(2)} = T(X^{(1)})$$

$$\vdots$$

$$X^{(i)} = T(X^{(i-1)})$$

$$\vdots$$

$$X^{(i)} \longrightarrow X$$

← the true solution (by Banach Fixed Theorem)

$$\begin{pmatrix} x^i(t) \\ y^i(t) \end{pmatrix} = \begin{pmatrix} e^{-t}\sigma \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} -e^{-(t-s)}(y^{i-1}(s))^2 \\ 0 \end{pmatrix} ds - \int_t^\infty \begin{pmatrix} 0 \\ e^{(t-s)}(x^{i-1}(s))^2 \end{pmatrix} ds$$

$i=0: \quad x^{(0)}(t) \equiv 0, \quad y^{(0)}(t) \equiv 0$

$i=1: \quad x^{(1)}(t) = e^{-t}\sigma, \quad y^{(1)}(t) = 0 \qquad \boxed{\begin{pmatrix} x^{(1)}(0) \\ y^{(1)}(0) \end{pmatrix} = \begin{pmatrix} \sigma \\ 0 \end{pmatrix}}$

$i=2:$

$$x^{(2)}(t) = e^{-t}\sigma - \int_0^t e^{-(t-s)} (\quad) ds = e^{-t}\sigma$$

$$y^{(2)}(t) = -\int_t^\infty e^{(t-s)} e^{-2s} \sigma^2 ds$$

$$= -\int_t^\infty e^t e^{-3s} \sigma^2 ds$$

$$= -e^t \sigma^2 \frac{e^{-3t}}{3} = -\frac{\sigma^2 e^{-2t}}{3}$$

$$\begin{pmatrix} x^{(2)}(t) \\ y^{(2)}(t) \end{pmatrix} = \begin{pmatrix} e^{-t}\sigma \\ -\frac{\sigma^2 e^{-2t}}{3} \end{pmatrix} \Rightarrow \boxed{\begin{pmatrix} x^{(2)}(0) \\ y^{(2)}(0) \end{pmatrix} = \begin{pmatrix} \sigma \\ -\frac{\sigma^2}{3} \end{pmatrix}}$$

$i=3$

$$x^{(3)}(t) = \begin{pmatrix} e^{-t}\sigma \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} -e^{-(t-s)} \left(y^{(2)}(s)\right)^2 \\ 0 \end{pmatrix} ds - \int_t^\infty \begin{pmatrix} 0 \\ e^{(t-s)} \left(x^{(2)}(s)\right)^2 \end{pmatrix} ds$$

$$= \begin{pmatrix} e^{-t}\sigma \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} -e^{-(t-s)} \frac{e^{-4s}\sigma^4}{9} \\ 0 \end{pmatrix} ds$$

$$- \int_t^\infty \begin{pmatrix} 0 \\ e^{(t-s)} e^{-2s}\sigma^2 \end{pmatrix} ds$$

$$= \begin{pmatrix} e^{-t}\sigma \\ 0 \end{pmatrix} - \frac{e^{-t}\sigma^4}{9} \begin{pmatrix} \int_0^t e^{-3s} ds \\ 0 \end{pmatrix} - e^{t}\sigma^2 \begin{pmatrix} 0 \\ \int_t^\infty e^{-3s} ds \end{pmatrix}$$

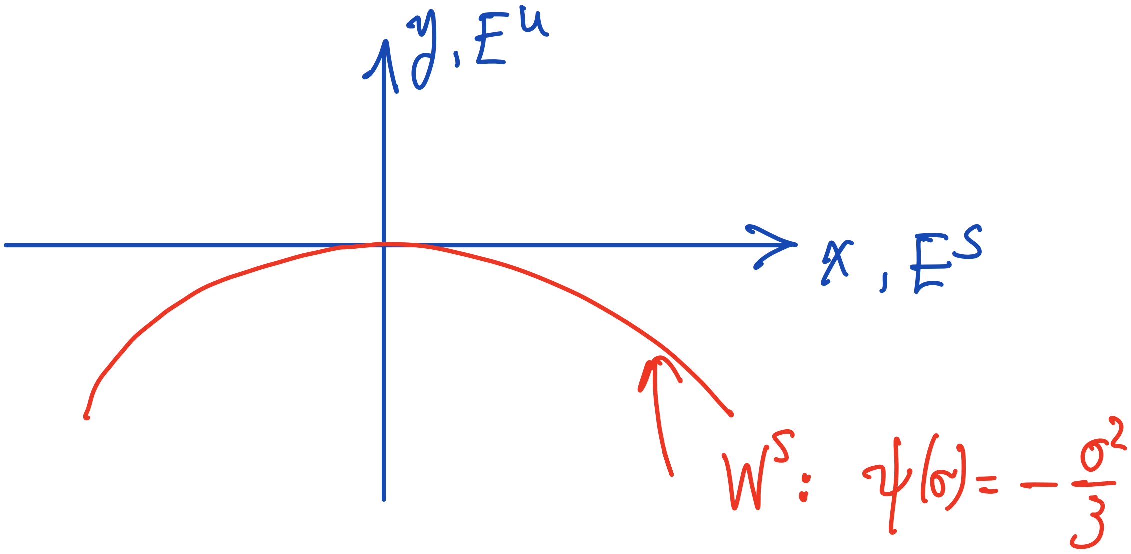
$$= \begin{pmatrix} e^{-t}\sigma \\ 0 \end{pmatrix} - \frac{e^{-t}\sigma^4}{9} \begin{pmatrix} \frac{1-e^{-3t}}{3} \\ 0 \end{pmatrix} - e^{t}\sigma^2 \begin{pmatrix} 0 \\ \frac{e^{-3t}}{3} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t}\sigma \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{\sigma^4}{27}\left(e^{-t} - e^{-4t}\right) \\ -\frac{1}{3}\sigma^2 e^{-2t} \end{pmatrix}$$

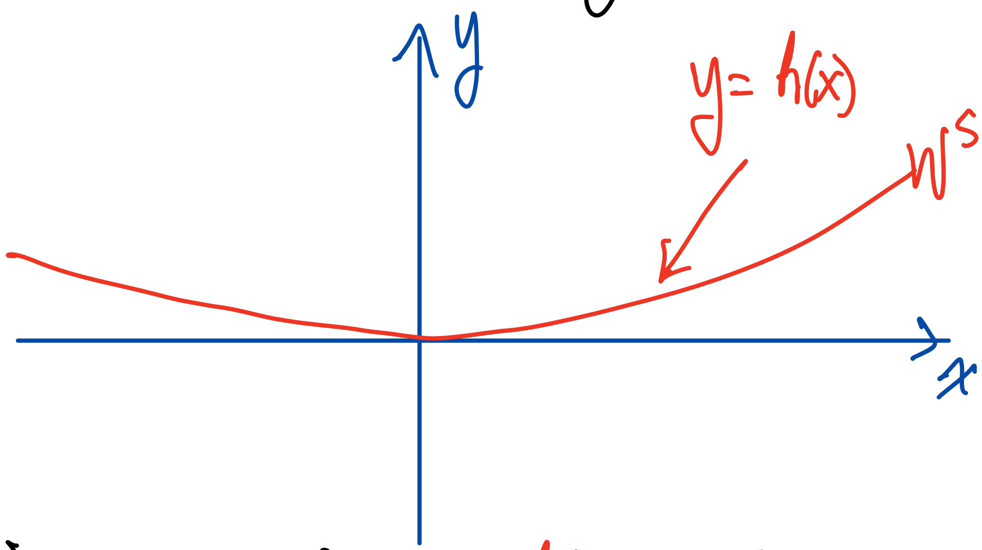
$$= \begin{pmatrix} e^{-t}\sigma - \frac{\sigma^4}{27}\left(e^{-t} - e^{-4t}\right) \\ -\frac{1}{3}\sigma^2 e^{-2t} \end{pmatrix} \Rightarrow \boxed{\begin{pmatrix} x^{(3)}(0) \\ y^{(3)}(0) \end{pmatrix} = \begin{pmatrix} \sigma \\ -\frac{\sigma^2}{3} \end{pmatrix}}$$

$$\Longrightarrow \boxed{\begin{pmatrix} x^{(4)}(0) \\ y^{(4)}(0) \end{pmatrix} = \begin{pmatrix} \sigma \\ -\frac{\sigma^2}{3} + O(\sigma^4) \end{pmatrix}}$$

$$\psi(\sigma) = -\frac{\sigma^2}{3} + O(\sigma^4)$$



Another approach: Taylor Expansion



$$\begin{cases} \dot{x} = -x - y^2 \\ \dot{y} = y + x^2 \end{cases} \quad \overset{y = h(x)}{\Longleftrightarrow} \quad \begin{aligned} \dot{x} &= -x - h(x)^2 \\ \dot{y} &= h(x) + x^2 \end{aligned}$$

$$(h(x))^{\cdot} = h(x) + x^2$$

$$h'(x)\,\dot{x} = h(x) + x^2$$

$$h'(x)\left(-x - h(x)^2\right) = h(x) + x^2$$

Let $h(x) = ax^2 + bx^3 + cx^4 + dx^5 + \cdots$

no constant and linear terms.

$$\left(2ax+3bx^2+4cx^3+5dx^4\cdots\right)\times$$

$$\left(-x-\left(ax^2+bx^3+cx^4+dx^5+\cdots\right)^2\right)$$

$$=ax^2+bx^3+cx^4+dx^5+\cdots+x^2$$

Compare coefficients

$O(x^2)$: $\quad -2a = a+1 \Rightarrow a = -\frac{1}{3}$

$O(x^3)$: $\quad 3b = b \Rightarrow b = 0$

$O(x^4)$: $\quad -4c = c \Rightarrow c = 0$

$O(x^5)$: $\quad -5d - 2a^3 = d$

$$d = -\frac{2}{6}a^3 = \frac{1}{81}$$

$$h(x) = -\frac{1}{3}x^2 + \frac{1}{81}x^5 + \cdots$$

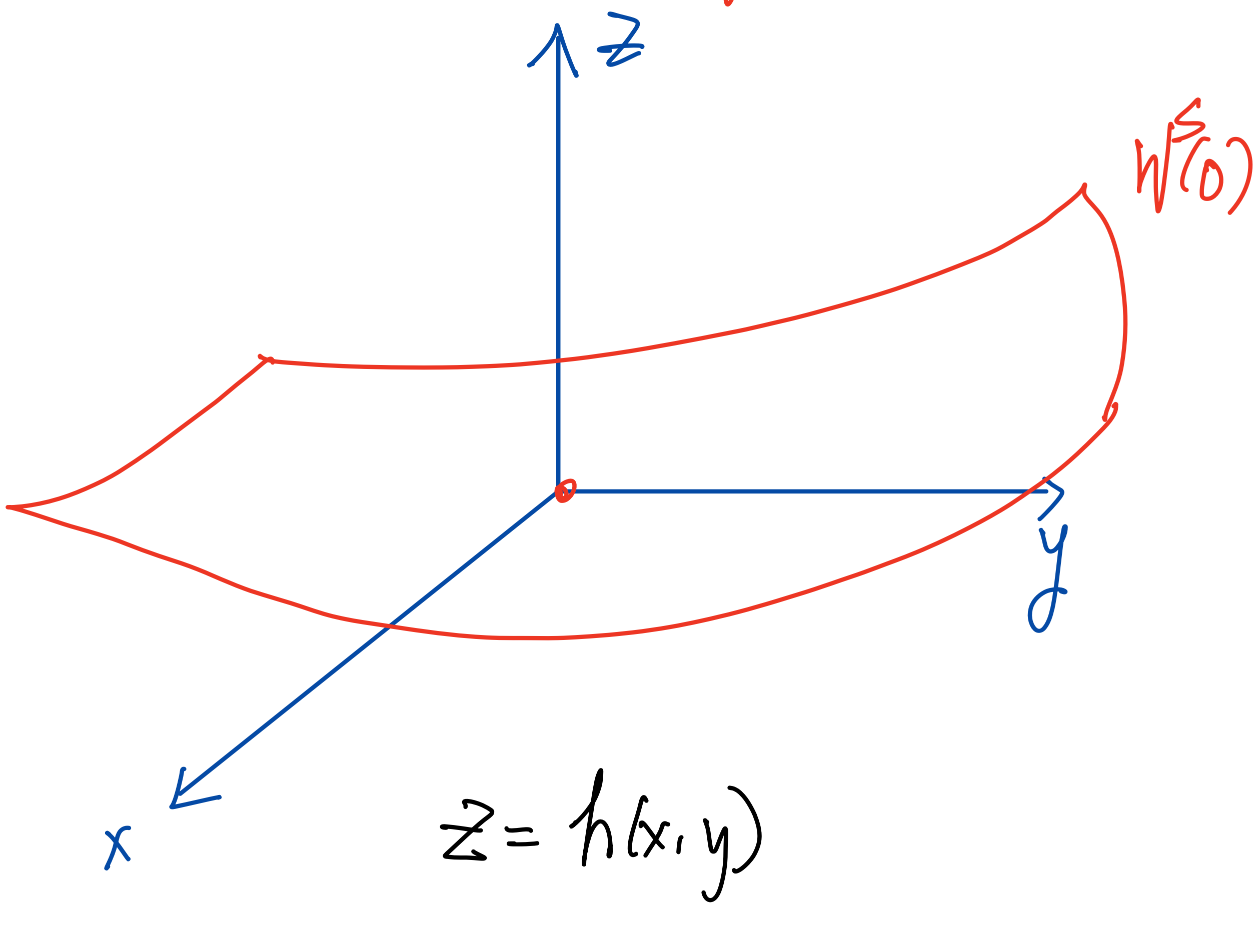
$$\boxed{W^s(0) = \left\{(x,y):\ y = -\frac{1}{3}x^2 + \frac{1}{81}x^5 + \cdots\right\}}$$

② [Perko, p. 105]

$$\begin{cases} \dot{x} = -x \\ \dot{y} = -y + x^2 \\ \dot{z} = z + x^2 \end{cases}$$

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ x^2 \\ x^2 \end{pmatrix}$$

$\lambda = -1, -1, 1.$ E^s = xy-plane, E^u = z-axis



$z = h(x, y)$

$$\dot{z} = z + x^2$$

$$h(x,y)^{\cdot} = h(x,y) + x^2$$

$$h_x \dot{x} + h_y \dot{y} = h(x,y) + x^2$$

$$h(x,y) = \underbrace{ax^2 + 2bxy + cy^2}_{\text{quadratic}} + \underbrace{O(|x|,|y|)^3}_{\text{cubic}}$$

$$(2ax + 2by)(-x) + (2bx + 2cy)(-y + x^2)$$

$$= x^2 + ax^2 + 2bxy + cy^2$$

$$(-2a)x^2 - 4bxy - 2cy^2$$

$$= (1+a)x^2 + 2bxy + cy$$

$$-2a = 1 + a \Rightarrow a = -1/3$$

$$-4b = 2b \Rightarrow b = 0$$

$$-2c = c \Rightarrow c = 0$$

$$\boxed{z = h(x,y) = -\frac{1}{3}x^2 + \cdots}$$