Center Manifolds

[M] **Theorem 4.36 (Hartman–Grobman).** *Let x^* be a hyperbolic equilibrium point of a C^1 vector field $f(x)$ with flow $\varphi_t(x)$. Then there is a neighborhood N of x^* such that φ is topologically conjugate to its linearization on N.*

Theorem 5.9 (Local Stable Manifold). *Let A be hyperbolic, $g \in C^k(U)$, $k \geq 1$, for some neighborhood U of 0, and $g(x) = o(x)$ as $x \to 0$. Denote the linear stable and unstable subspaces of A by E^s and E^u. Then there is a $\tilde{U} \subset U$ such that local stable manifold of* (5.10),

$$W^s_{loc}(0) \equiv \left\{ x \in W^s(0) : \ \varphi_t(x) \in \tilde{U}, \ t \geq 0 \right\},$$

is a Lipschitz graph over E^s that is tangent to E^s at 0. Moreover, $W^s_{loc}(0)$ is a C^k manifold.

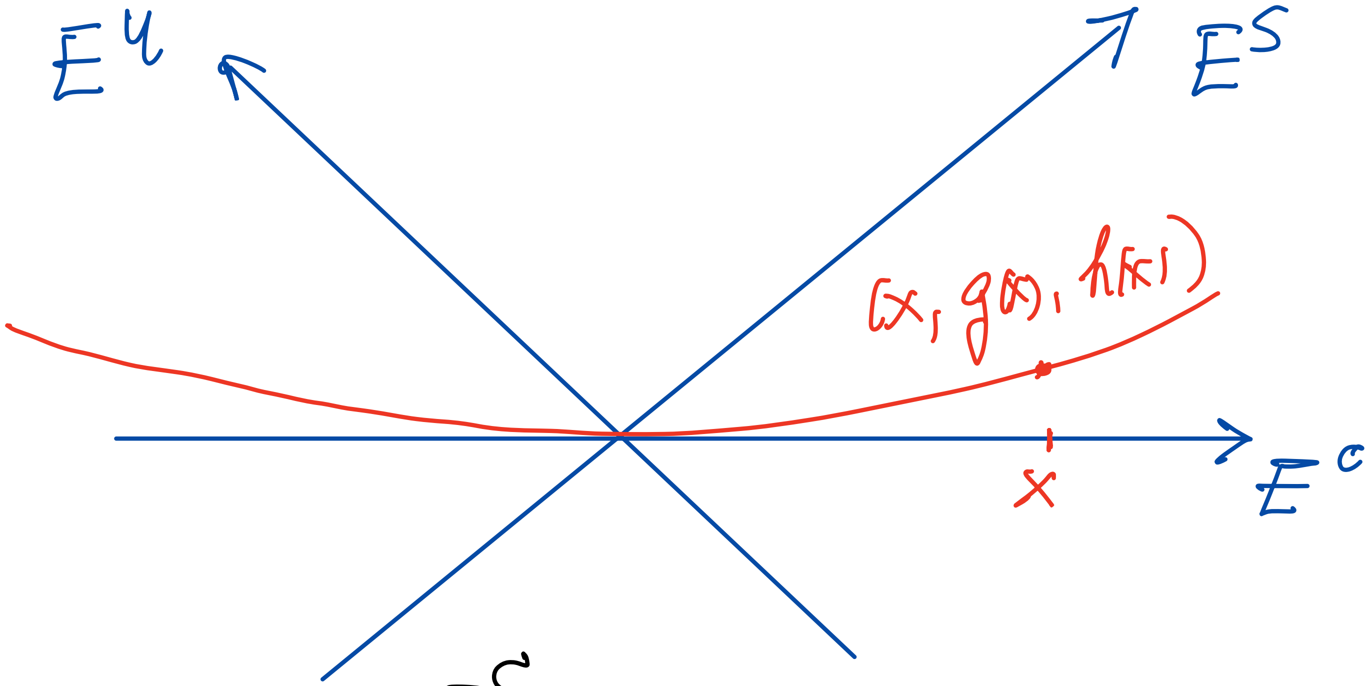
Theorem 5.21 (Center Manifold). *Suppose that f is a C^k vector field, $k \geq 1$, with an equilibrium at the origin. Let the eigenspaces of $Df(0) = A$ be written $E^u \oplus E^c \oplus E^s$. Then there is a neighborhood of the origin in which there exist C^k locally invariant manifolds: the local stable manifold, W^s_{loc}, tangent to E^s, on which $|x(t)| \to 0$ as $t \to \infty$, the local unstable manifold W^u_{loc}, tangent to E^u, on which $|x(t)| \to 0$ as $t \to -\infty$, and a local center manifold W^c, tangent to E^c.*

$$\begin{aligned} \dot{x} &= Cx + F(x,y,z), \\ \dot{y} &= Sy + G(x,y,z), \\ \dot{z} &= Uz + H(x,y,z). \end{aligned} \tag{5.34}$$

Theorem 5.23 (Nonhyperbolic Hartman–Grobman). *Suppose* (5.34) *is a C^1 vector field with equilibrium at the origin, that all the eigenvalues of C have zero real part, that S is a contracting and U is an expanding hyperbolic matrix, and that $F, G, H = o(x,y,z)$. Then there is a neighborhood N of the origin such that $W^c_{loc} = \{(x, g(x), h(x)) : x \in E^c\} \cap N$ and the dynamics in N is topologically conjugate to the system*

$$\begin{aligned} \dot{x} &= Cx + F(x, g(x), h(x)), \\ \dot{y} &= Sy, \\ \dot{z} &= Uz. \end{aligned} \tag{5.37}$$

How to find $g(x)$ & $h(x)$?



$x \in E^c \qquad x = x(t)$

$y = g(x) \in E^s \qquad y = g(x(t))$

$z = h(x) \in E^u \qquad z = h(x(t))$

$$\dot{y} = (Dg)(x)\,\dot{x}$$

$$\Rightarrow \boxed{(Dg)(x)\left[Cx + F(x, g(x), h(x))\right] = Sg(x) + G(x, g(x), h(x))}$$

(5.35)

$$\dot{z} = (Dh)(x)\,\dot{x}$$

$$\Rightarrow \boxed{(Dh)(x)\left[Cx + F(x, g(x), h(x))\right] = Uh(x) + H(x, g(x), h(x))}$$

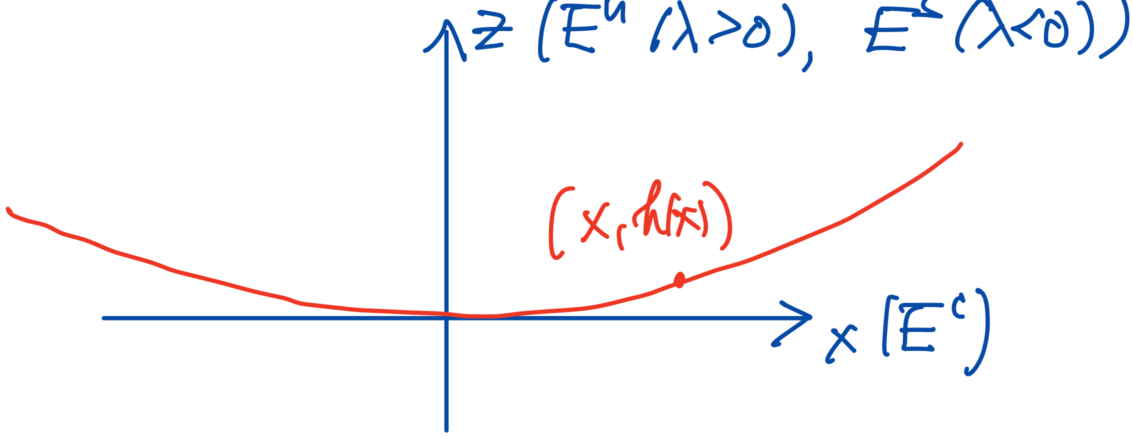
(Then use Taylor Expansion)

[M, Example 5-2F]

$$\dot{x} = x^2 - z^2$$
$$\dot{z} = \lambda z + x^2$$

(dynamics of x ?)

$$\frac{d}{dt}\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} x^2 - z^2 \\ x^2 \end{pmatrix}$$



$$z = h(x), \qquad \dot{z} = h'(x)\dot{x}$$

$$h'(x)\left[x^2 - h(x)^2\right] = \lambda h(x) + x^2$$

$$h(x) = \alpha x^2 + \beta x^3 + \gamma x^4 + \cdots$$

$$(2\alpha x + 3\beta x^2 + 4\gamma x^3 + \cdots)\left(x^2 - (\alpha x^2 + \beta x^3 + \gamma x^4 \cdots)^2\right)$$
$$= \lambda\alpha x^2 + \lambda\beta x^3 + \lambda\gamma x^4 \cdots + x^2$$

$$O(x^2) \Rightarrow \quad 0 = \lambda\alpha + 1 \quad \Rightarrow \quad \alpha = -\frac{1}{\lambda}$$

$$O(x^3) \Rightarrow \quad 2\alpha = \lambda\beta \quad \Rightarrow \quad \beta = \frac{2\alpha}{\lambda} = -\frac{2}{\lambda^2}$$

$$O(x^4) \Rightarrow \quad 3\beta = \lambda\gamma \quad \Rightarrow \quad \gamma = \frac{3\beta}{\lambda} = -\frac{6}{\lambda^3}$$

$$h(x) = -\frac{x^2}{\lambda} - \frac{2}{\lambda^2}x^3 - \frac{6}{\lambda^3}x^4 + \cdots$$

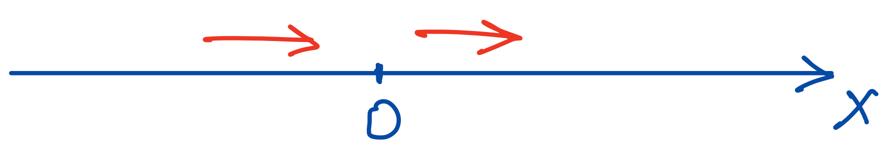
<u>Dynamics of x</u>

$$\dot{x} = x^2 - z^2$$

$$= x^2 - \left(-\frac{x^2}{\lambda} - \frac{2}{\lambda^2}x^3 - \cdots\right)^2$$

$$\boxed{\underbrace{\dot{x} = x^2}_{\dot{x} \approx x^2 \geq 0} - \frac{x^4}{\lambda^2} - \frac{4x^5}{\lambda^3} - \cdots}$$

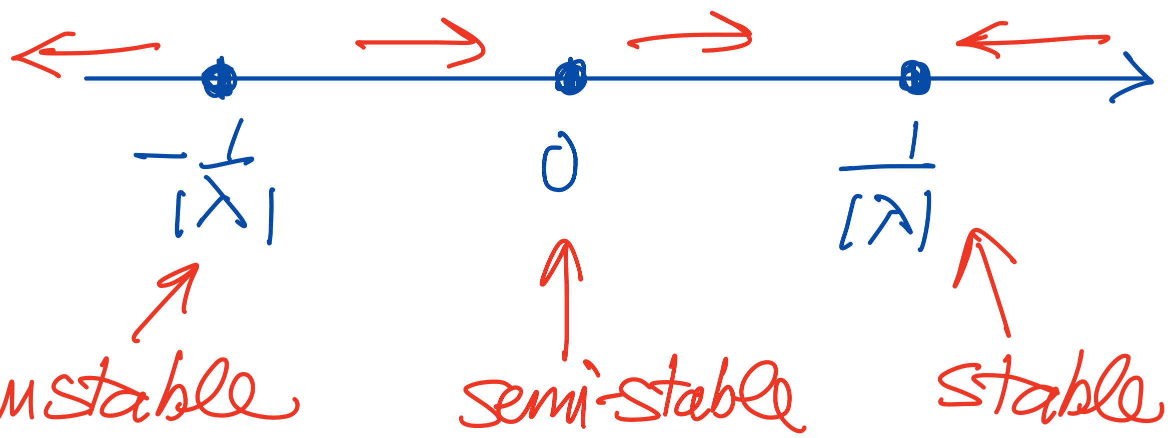
Semi-stable



$$\dot{x} \cong x^2 - \frac{x^4}{\lambda^2} + \cdots$$

$$= x^2\left(1 - \frac{x^2}{\lambda^2}\right) + \cdots$$

$$x = 0, \ \pm \frac{1}{\lambda^2}$$



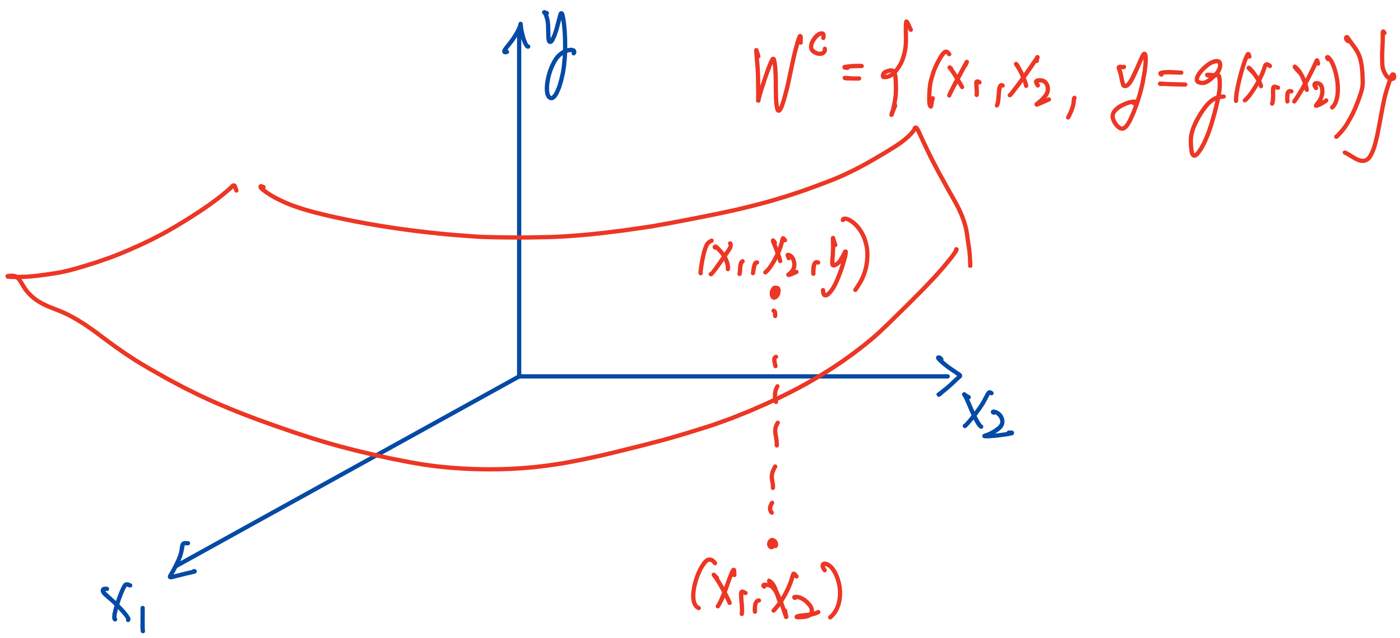
[M, Example 5.25]

$$\dot{x}_1 = -x_2 + x_1 y$$
$$\dot{x}_2 = x_1 + x_2 y$$
$$\dot{y} = -y - x_1^2 - x_2^2 + y^2$$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} = \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & -1 \end{array}\right)\begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} + \begin{pmatrix} x_1 y \\ x_2 y \\ -x_1^2 - x_2^2 + y^2 \end{pmatrix}$$

$\lambda = \pm i$ $\qquad\qquad$ $\lambda = -1$

$E^c = x_1 x_2$-plane, $\quad E^s = y$-axis



$$g(x_1, x_2) = \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2 + \cdots$$

$$y = g(x_1, x_2) \Rightarrow \dot{y} = g_{x_1} \dot{x}_1 + g_{x_2} \dot{x}_2$$

$$g_{x_1}(-x_2 + x_1 g) + g_{x_2}(x_1 + x_2 g)$$
$$= -g - x_1^2 - x_2^2 + g^2$$

$$(2\alpha x_1 + \beta x_2 + \cdots)(-x_2 + x_1 g) + (\beta x_1 + 2\gamma x_2 + \cdots)(x_1 + x_2 g)$$
$$= -\alpha x_1^2 - \beta x_1 x_2 - \gamma x_2^2 - x_1^2 - x_2^2 + g^2$$

$$\left.\begin{aligned} O(x_1^2) &\Rightarrow & \beta &= -\alpha - 1 \\ O(x_1 x_2) &\Rightarrow & -2\alpha + 2\gamma &= -\beta \\ O(x_2^2) &\Rightarrow & -\beta &= -\gamma - 1 \end{aligned}\right\} \Rightarrow \begin{aligned} \beta &= 0 \\ \alpha &= -1 \\ \gamma &= -1 \end{aligned}$$

$$g(x_1, x_2) = -x_1^2 - x_2^2 + \cdots$$

$$\boxed{\begin{aligned} \dot{x}_1 &= -x_2 + x_1(-x_1^2 - x_2^2) + \cdots \\ \dot{x}_2 &= x_1 + x_2(-x_1^2 - x_2^2) + \cdots \end{aligned}}$$

dynamics=?

$$\boxed{\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\text{linear}} + \underbrace{\begin{pmatrix} -x_1^3 - x_1 x_2^2 \\ -x_2 x_1^2 - x_2^3 \end{pmatrix}}_{\text{degree 3 polynomials}} + \cdots}$$

[Perko, p. 158 Example 2]

$$\dot{x}_1 = x_1 y - x_1 x_2^2$$
$$\dot{x}_2 = x_2 y - x_2 x_1^2$$
$$\dot{y} = -y + x_1^2 + x_2^2$$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & -1 \end{array}\right)\begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} + \begin{pmatrix} x_1 y - x_1 x_2^2 \\ -x_2 x_1^2 \\ x_1^2 + x_2^2 \end{pmatrix}$$

$E^c = x_1x_2$-plane $\qquad E^s = y$-axis

$$y = g(x_1, x_2) = \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2 + \cdots$$

$$g_{x_1}\dot{x}_1 + g_{x_2}\dot{x}_2 = -g + x_1^2 + x_2^2$$

$$\underbrace{\begin{aligned}&(2\alpha x_1 + \beta x_1 + \cdots)(x_1 g - x_1 x_2^2) \\ &+ (\beta x_1 + 2\gamma x_2 + \cdots)(x_2 g - x_2 x_1^2)\end{aligned}}_{O(x_1, x_2)^3} = \underbrace{-g + x_1^2 + x_2^2}_{O(x_1, x_2)^2}$$

Hence

$$\boxed{g \cong x_1^2 + x_2^2}$$

Dynamics on W^c:

$$\begin{cases} \dot{x}_1 = x_1(x_1^2 + x_2^2 + \cdots) - x_1 x_2^2 = x_1^3 + \cdots \\ \dot{x}_2 = x_2(x_1^2 + x_2^2 + \cdots) - x_2 x_1^2 = x_2^3 + \cdots \end{cases}$$

$(0,0)$ is unstable