Dynamics near a nonhyperbolic critical pt in $\mathbb{R}^2$

(M, Chapter 6.1–6.3)

$$\frac{dX}{dt} = AX + g(X), \qquad A^{2\times 2}, \ |g(X)| \leq C|X|^2$$

A — nonhyperbolic:

① $\lambda_1 = 0, \ \lambda_2 \neq 0$

$$A = \begin{bmatrix} 0 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

② $\lambda_1 = \lambda_2 = 0$, but only one eigenvector

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

③ $\lambda_1 = \lambda_2 = 0$, two eigenvectors

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

④ $\lambda_1 = i\omega, \quad \lambda_2 = -i\omega$

$$A = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$

$\boxed{\text{Case } ③ \quad A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}$

$$\begin{cases} \dfrac{dx}{dt} = P(x,y), \\[2ex] \dfrac{dy}{dx} = Q(x,y), \end{cases} \qquad \left| P(x,y), Q(x,y) \right| \leq c\left(|x|^2 + |y|^2\right)$$

Taylor expansion

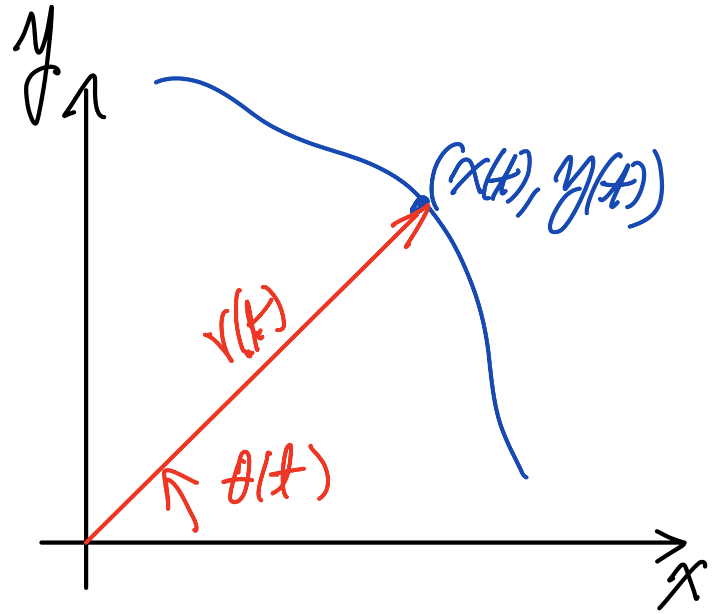
$$P(x,y) = \sum_{i=0}^{2} a_{2i} x^i y^{2-i} + \sum_{i=0}^{3} a_{3i} x^i y^{3-i} + \cdots + \sum_{i=0}^{n} a_{ni} x^i y^{n-i} + \cdots$$

$$Q(x,y) = \underbrace{\sum_{i=0}^{2} b_{2i} x^i y^{2-i}}_{} + \underbrace{\sum_{i=0}^{3} b_{3i} x^i y^{3-i}}_{} + \cdots + \underbrace{\sum_{i=0}^{n} b_{ni} x^i y^{n-i}}_{} + \cdots$$

homogeneous degree $= 2 \qquad 3 \qquad \cdots\cdots \qquad n$

Use Polar Coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta, \end{cases} \qquad \begin{cases} r^2 = x^2 + y^2 \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{cases}$$



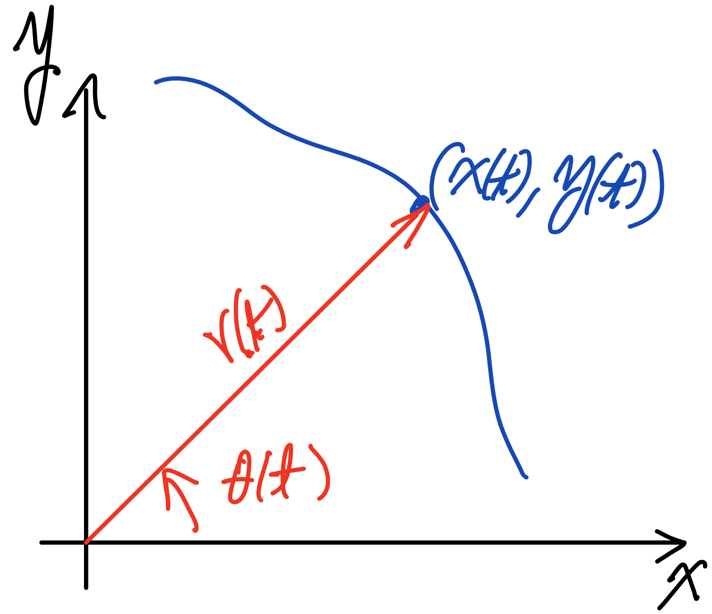
$$\underbrace{\frac{d(r^2)}{dt}}_{2r\dot{r}} = 2x\dot{x} + 2y\dot{y} \implies \dot{r} = \frac{1}{r}(x\dot{x} + y\dot{y})$$

$$\frac{dr}{dt} = \frac{1}{r}\Big[r\cos\theta\, P(r\cos\theta, r\sin\theta) + r\sin\theta\, Q(r\cos\theta, r\sin\theta)\Big]$$

$$= (\cos\theta) P(r\cos\theta, r\sin\theta) + (\sin\theta) Q(r\cos\theta, r\sin\theta)$$

Use Polar Coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta, \end{cases} \qquad \begin{cases} r^2 = x^2 + y^2 \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{cases}$$



$$\frac{d\theta}{dt} = \frac{d}{dt}\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{x\dot{y} - y\dot{x}}{x^2}\right) = \frac{x\dot{y} - y\dot{x}}{x^2+y^2}$$

$$\frac{d\theta}{dt} = \frac{(r\cos\theta)Q(\text{——}) - (r\sin\theta)P(\text{——})}{r^2}$$

$$= \frac{1}{r}\left[(\cos\theta)\,Q(r\cos\theta, r\sin\theta) - (\sin\theta)\,P(r\cos\theta, r\sin\theta)\right]$$

Equation in Polar Coordinate

$$\begin{cases} \dfrac{dr}{dt} = (\cos\theta) P(r\cos\theta, r\sin\theta) + (\sin\theta) Q(r\cos\theta, r\sin\theta) \\[2ex] \dfrac{d\theta}{dt} = \dfrac{1}{r}\left[\cos\theta\, Q(r\cos\theta, r\sin\theta) - \sin\theta\, P(r\cos\theta, r\sin\theta)\right] \end{cases}$$

$$\boxed{\frac{dr}{d\theta} = \frac{(\cos\theta) P(r\cos\theta, r\sin\theta) + (\sin\theta) Q(r\cos\theta, r\sin\theta)}{\frac{1}{r}\left[\cos\theta\, Q(r\cos\theta, r\sin\theta) - \sin\theta\, P(r\cos\theta, r\sin\theta)\right]}}$$

Equation in Polar Coordinate

$$\begin{cases} \dfrac{dr}{dt} = (\cos\theta) P(r\cos\theta, r\sin\theta) + (\sin\theta) Q(r\cos\theta, r\sin\theta) \\ \dfrac{d\theta}{dt} = \dfrac{1}{r}\left[\cos\theta\, Q(r\cos\theta, r\sin\theta) - \sin\theta\, P(r\cos\theta, r\sin\theta)\right] \end{cases}$$

Assume P, Q are homogeneous of degree n

$P(\alpha x, \alpha y) = \alpha^n P(x,y)$, $Q(\alpha x, \alpha y) = \alpha^n Q(x,y)$

Then $P(r\cos\theta, r\sin\theta) = r^n P(\cos\theta, \sin\theta)$, - - - - .

Equation in Polar Coordinate

$$\begin{cases} \dfrac{dr}{dt} = r^n \left[c P(c,s) + s Q(c,s) \right] \\[2ex] \dfrac{d\theta}{dt} = r^{n-1} \left[c Q(c,s) - s P(c,s) \right] \end{cases}$$

$c = \cos\theta$
$s = \sin\theta$

$$\frac{dr}{d\theta} = r \underbrace{\left[\frac{cP(c,s) + sQ(c,s)}{sQ(c,s) - cP(c,s)} \right]}_{g(\theta)}$$

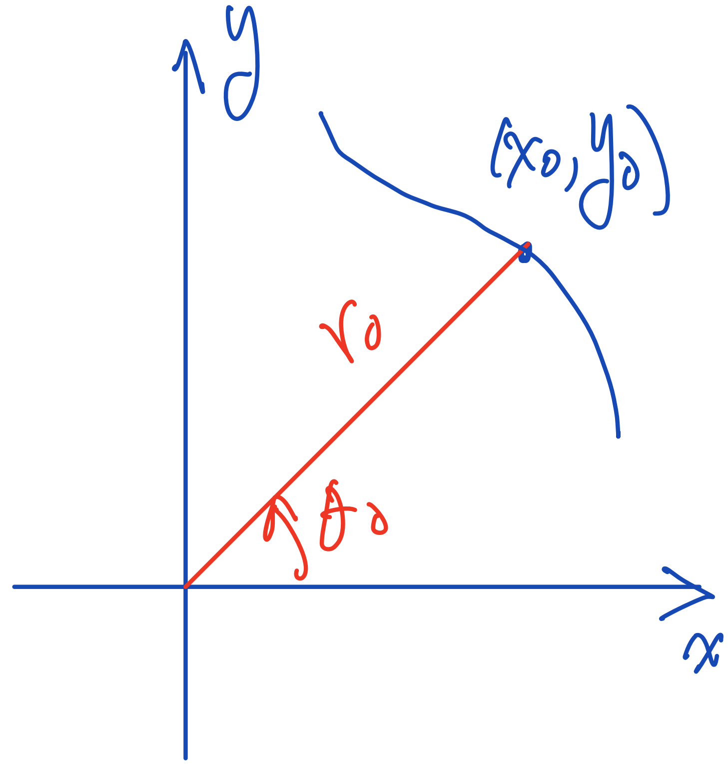
Solution in Polar Coordinate

- $$\frac{dr}{d\theta} = r\,g(\theta)$$ ← separable equation

- $$\int_{r_0}^{r} \frac{dr}{r} = \int_{\theta_0}^{\theta} g(\theta)\,d\theta$$

- $$\ln r = \ln r_0 + \int_{\theta_0}^{\theta} g(\varphi)\,d\varphi$$

- $$r(\theta) = r(\theta_0)\, e^{\int_{\theta_0}^{\theta} g(\varphi)\,d\varphi}$$



(e.g. $\theta_0 = 0$)

Solution in Polar Coordinate

- $$\frac{dr}{d\theta} = r\, g(\theta)$$ $\leftarrow$ separable equation

- $$\int_{r_0}^{r} \frac{dr}{r} = \int_{\theta_0}^{\theta} g(\theta)\, d\theta$$

- $$\ln r = \ln r_0 + \int_{\theta_0}^{\theta} g(\varphi)\, d\varphi$$

- $$r(\theta) = r(\theta_0)\, e^{\int_{\theta_0}^{\theta} g(\varphi)\, d\varphi}$$

Note:

$$g(\varphi) = \varphi(\varphi + 2\pi)$$

Define:

$$G_1 = \int_0^{2\pi} g(\varphi)\, d\varphi$$

Solution in Polar Coordinate

$$\boxed{r(\theta) = r(\theta_0)\, e^{\int_{\theta_0}^{\theta} g(\varphi)\, d\varphi}}$$

assume well defined

Note:

$$g(\varphi) = \varphi(\varphi + 2\pi)$$

Define:

$$G = \int_0^{2\pi} g(\varphi)\, d\varphi$$

$$r(\theta_0 + 2\pi) = r(\theta_0)\, e^{G}$$

$$r(\theta_0 + 2\pi(2)) = r(\theta_0)\, e^{2G}$$

$$r(\theta_0 + 2\pi 3) = r(\theta_0)\, e^{3G}$$

$$\vdots \qquad \vdots$$

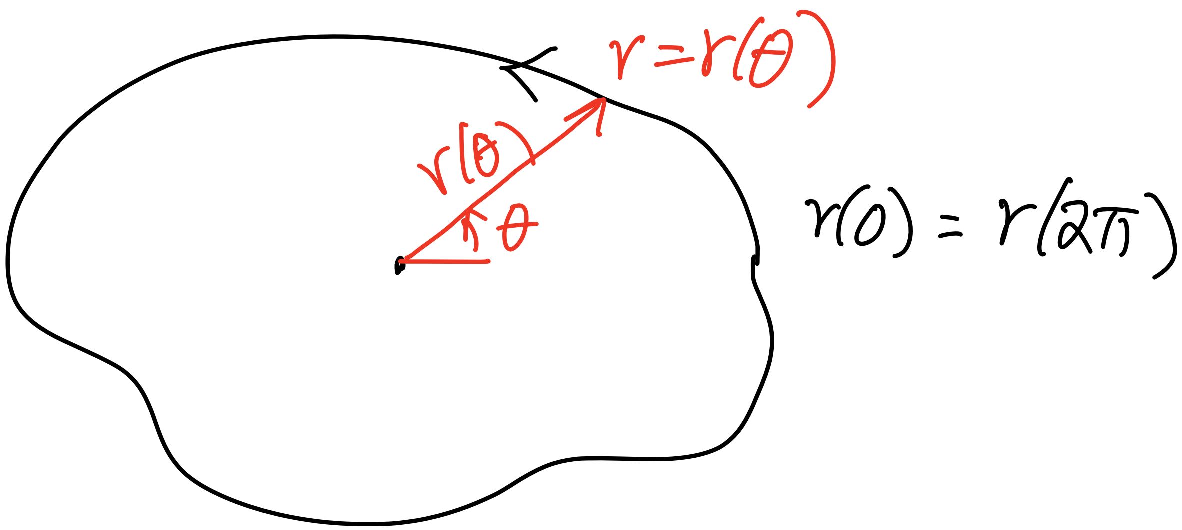
$$r(\theta_0 + 2\pi n) = r(\theta_0)\, e^{nG} \xrightarrow{n \to \infty} ?$$

Solution in Polar Coordinate

Suppose $G = \int_0^{2\pi} g(\varphi)\,d\varphi$ is well-defined.

If $\boxed{G = 0,}$ then $r(\theta = 0) = r(\theta = 2\pi)$

i.e. periodic orbit.



Solution in Polar Coordinate

Suppose $G = \int_0^{2\pi} g(\varphi)\,d\varphi$ is well-defined.

If $\boxed{G < 0}$, then $\ln r(2\pi) = \ln r(0) + G$

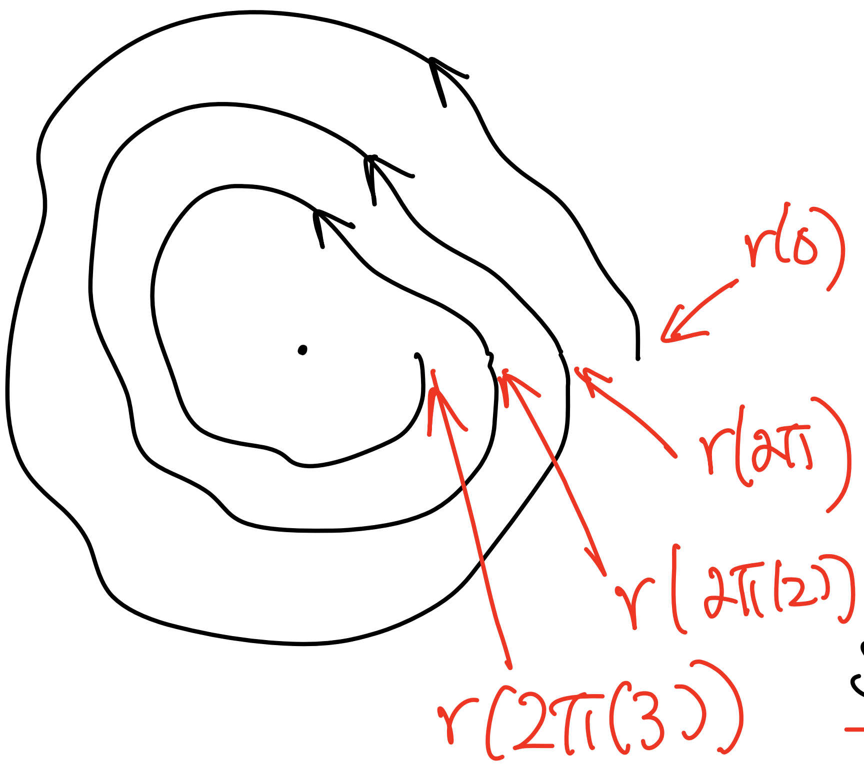
$$r(2\pi) = r(0)e^{G}$$

$$r(2\pi 2) = r(0)e^{2G}$$

$$r(2\pi 3) = r(0)e^{3G}$$

$$\vdots$$

$$r(2\pi n) = r(0)e^{nG} \xrightarrow{n \to +\infty} 0$$



Spiral in, asymptotically stable

Solution in Polar Coordinate

Suppose $G = \int_0^{2\pi} g(\varphi)\,d\varphi$ is well-defined.

If $\boxed{G > 0}$, then $\ln r(2\pi) = \ln r(0) + G$

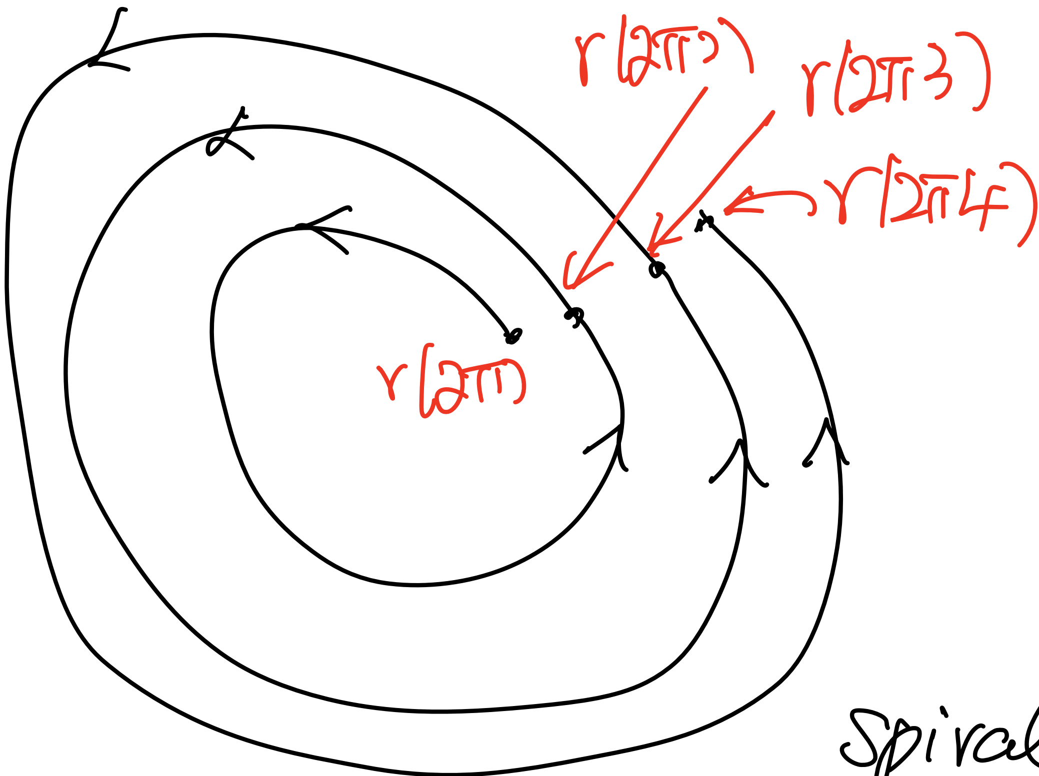
$$r(2\pi) = r(0)e^{G}$$

$$r(2\pi 2) = r(0)\,e^{2G}$$

$$r(2\pi 3) = r(0)e^{3G}$$

$$\vdots$$

$$r(2\pi n) = r(0)e^{nG} \xrightarrow{n \to +\infty} +\infty$$



Spiral out, asymptotically unstable

Solution in Polar Coordinate

One thing to be carful

$$\begin{cases} >0 \,? \\ <0 \,? \end{cases}$$

$$\frac{d\theta}{dt} = r^{n-1}\left[c\,Q(c,s) - s\,P(c,s)\right]$$

- As t increases, does $\theta(t)$ increase/decrease?
- Suppose $r(\theta) \to \begin{cases} +\infty \\ 0 \end{cases}$ as $\theta \to +\infty$

then $r(\theta(t)) \longrightarrow \begin{cases} +\infty \\ 0 \end{cases}$ as $t \to +\infty$ or $-\infty$?

Solution in Polar Coordinate

Suppose $G = \int_0^{2\pi} g(\varphi)\,d\varphi$ does not exist.

How can G fail to exist?

$$g(\theta) = \frac{\cos\theta\, P(\cos\theta, \sin\theta) + \sin\theta\, Q(\cos\theta, \sin\theta)}{\sin\theta\, Q(\cos\theta, \sin\theta) - \cos\theta\, P(\cos\theta, \sin\theta)}$$

denominator becomes zero at some pt θ_c

$$\exists\, \theta_c \text{ s.t. } c(\theta_c)Q(\theta_c) - s(\theta_c)P(\theta_c) = 0$$

Solution in Polar Coordinate

Suppose $G = \int_0^{2\pi} g(\varphi)\,d\varphi$ does not exist.

How can G fail to exist?

$$g(\theta) = \frac{\cos\theta\, P(\cos\theta, \sin\theta) + \sin\theta\, Q(\cos\theta, \sin\theta)}{\sin\theta\, Q(\cos\theta, \sin\theta) - \cos\theta\, P(\cos\theta, \sin\theta)}$$

$$\int_{\theta_0}^{\theta_c} g(\varphi)\,d\varphi = \begin{cases} +\infty \implies & \ln r(\theta_c) = +\infty \\ & r(\theta_c) = +\infty \\ -\infty \implies & \ln r(\theta_c) = -\infty \\ & r(\theta_c) = 0 \end{cases}$$

Example (M, p.192, Ex 6.2)

homogeneous of degree 3

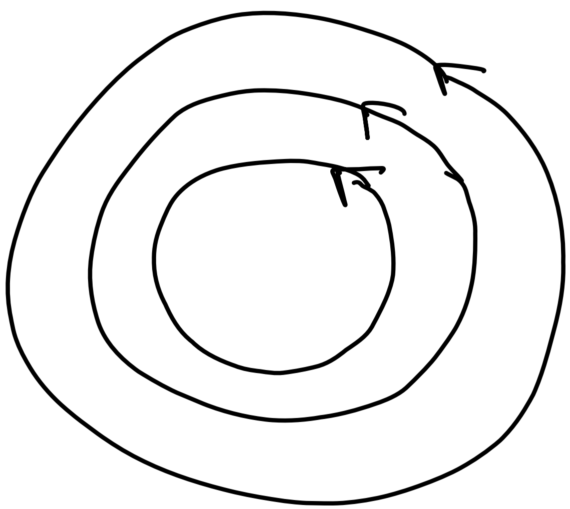
$$\dot{x} = P(x,y) = -x^2y - y^3$$
$$\dot{y} = Q(x,y) = x^3 + xy^2$$

periodic orbit

$$\dot{r} = r^3\left[c(-c^2s - s^3) + s(c^3 + cs^2)\right] = 0$$

angle rotates

$$\dot{\theta} = \frac{r^3}{r}\left[c(c^3 + cs^2) - s(-c^2s - s^3)\right] = r^2 > 0$$



$\dot{\theta} = r^2 > 0$

Periodic Orbit

Example (M, p.192, Ex 6.3)

homog. deg 2

$$\dot{x} = P(x,y) = -(x^2+y^2)(x+y)$$

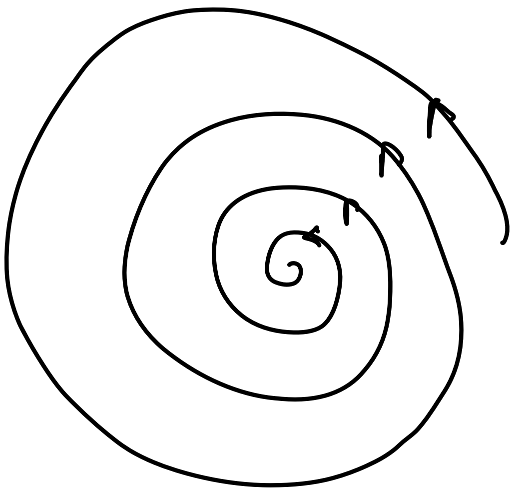
$$\dot{y} = Q(x,y) = (x^2+y^2)(x-y)$$

$r \downarrow 0$

$$\dot{r} = r^3\left[-c\,(c^2+s^2)(c+s) + s(c^2+s^2)(c-s)\right] = -r^3 < 0$$

$$\dot{\theta} = \frac{r^3}{r}\left[c\,(c^2+s^2)(c-s) + s(c^2+s^2)(c+s)\right] = r^2 > 0$$

θ rotate



Spiral in, asymptotically stable.

Example (M, p. 193, Ex 6.4)

$$\begin{cases} \dot{x} = y^2 - x^2 \\ \dot{y} = -2xy \end{cases}$$

homog. deg = 2

$$\begin{cases} \dot{r} = c[s^2 - c^2] + s[-2cs] = -r^2\cos\theta \\ \dot{\theta} = r[c(-2cs) - s(s^2 - c^2)] = -r\sin\theta \end{cases}$$

$$\frac{dr}{d\theta} = r\left(\frac{\cos\theta}{\sin\theta}\right)$$

$= 0$ at $\theta = 0, \pi$, not integrable

Example (M, p. 192, Ex 6.4)

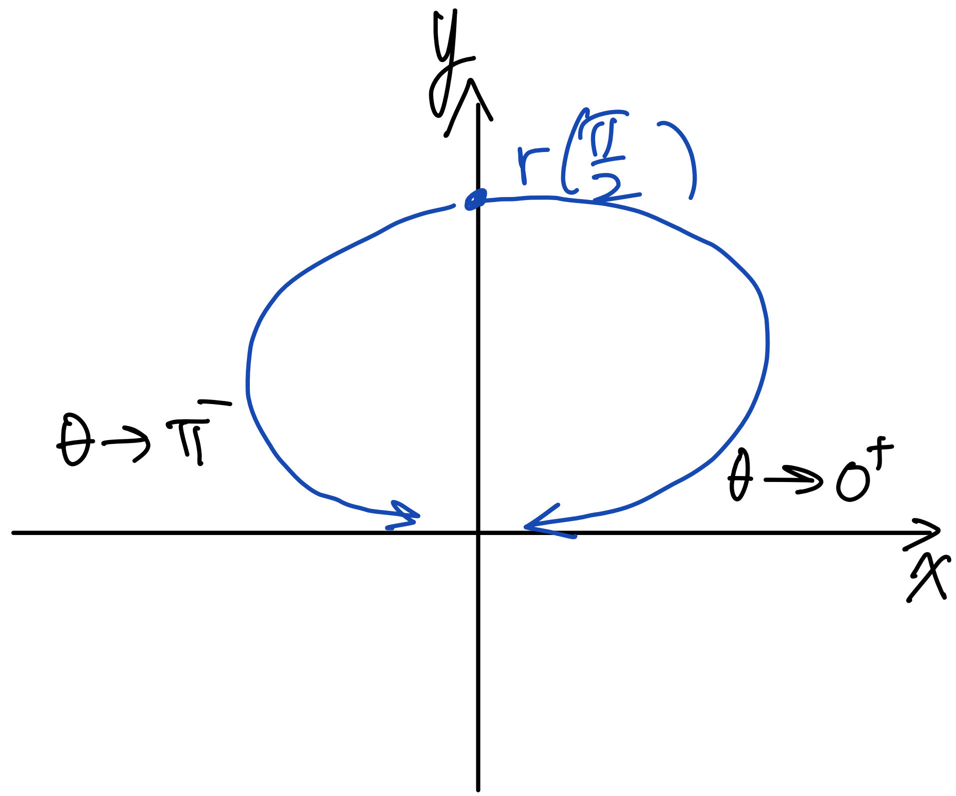
$$\ln r(\theta) = \ln r\left(\frac{\pi}{2}\right) + \int_{\frac{\pi}{2}}^{\theta} \frac{\cos\varphi}{\sin\varphi}\, d\varphi$$

$$\int_{\frac{\pi}{2}}^{0^+} \frac{\cos\varphi}{\sin\varphi}\, d\varphi = -\infty$$

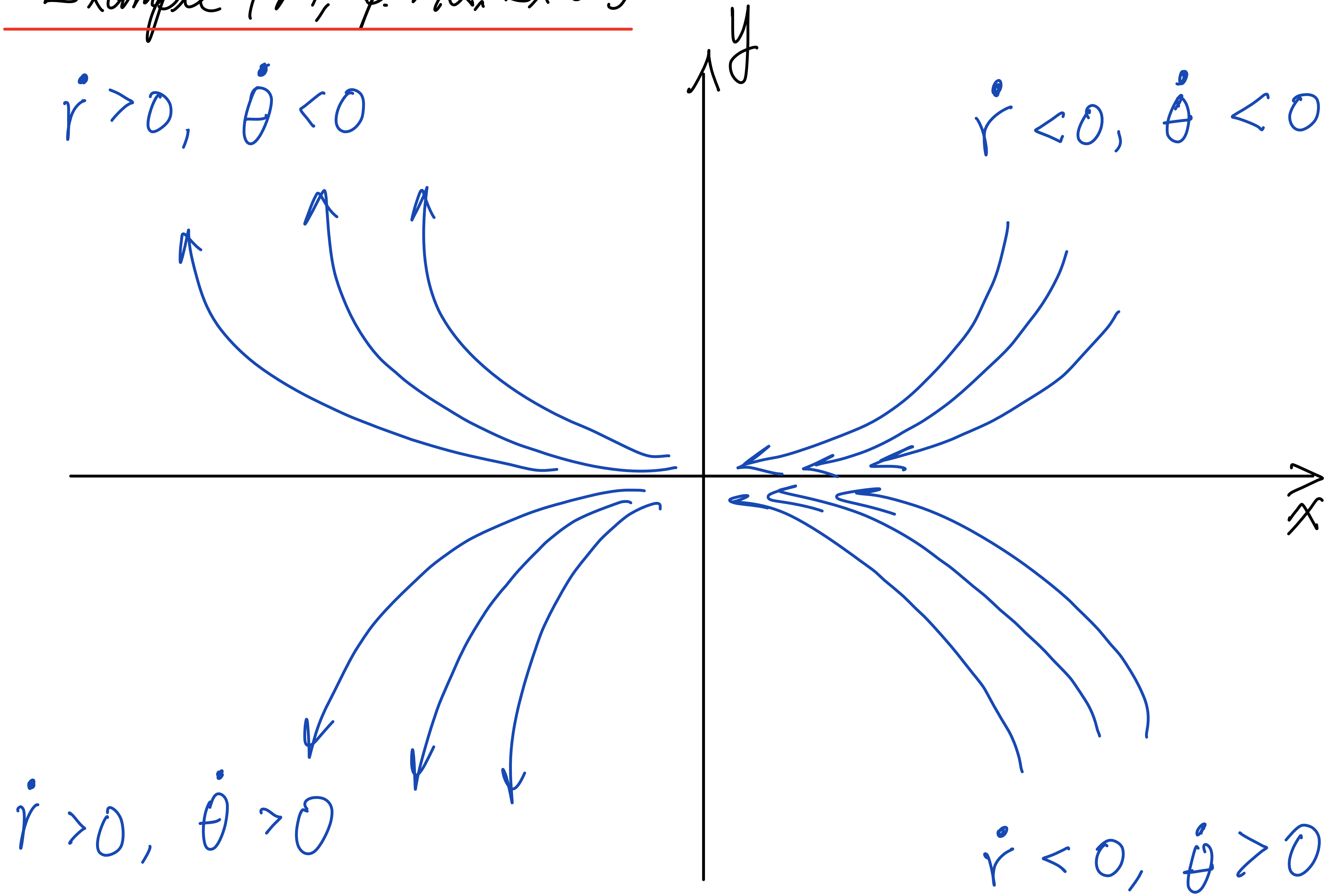
$\Longrightarrow r(0^+) = 0$

$$\int_{\frac{\pi}{2}}^{\pi^-} \frac{\cos\varphi}{\sin\varphi}\, d\varphi = -\infty$$

$\Longrightarrow r(\pi^-) = 0$



Example (M, p. 192, Ex 6.4)



Example (M, p. 192, Ex 6.4)

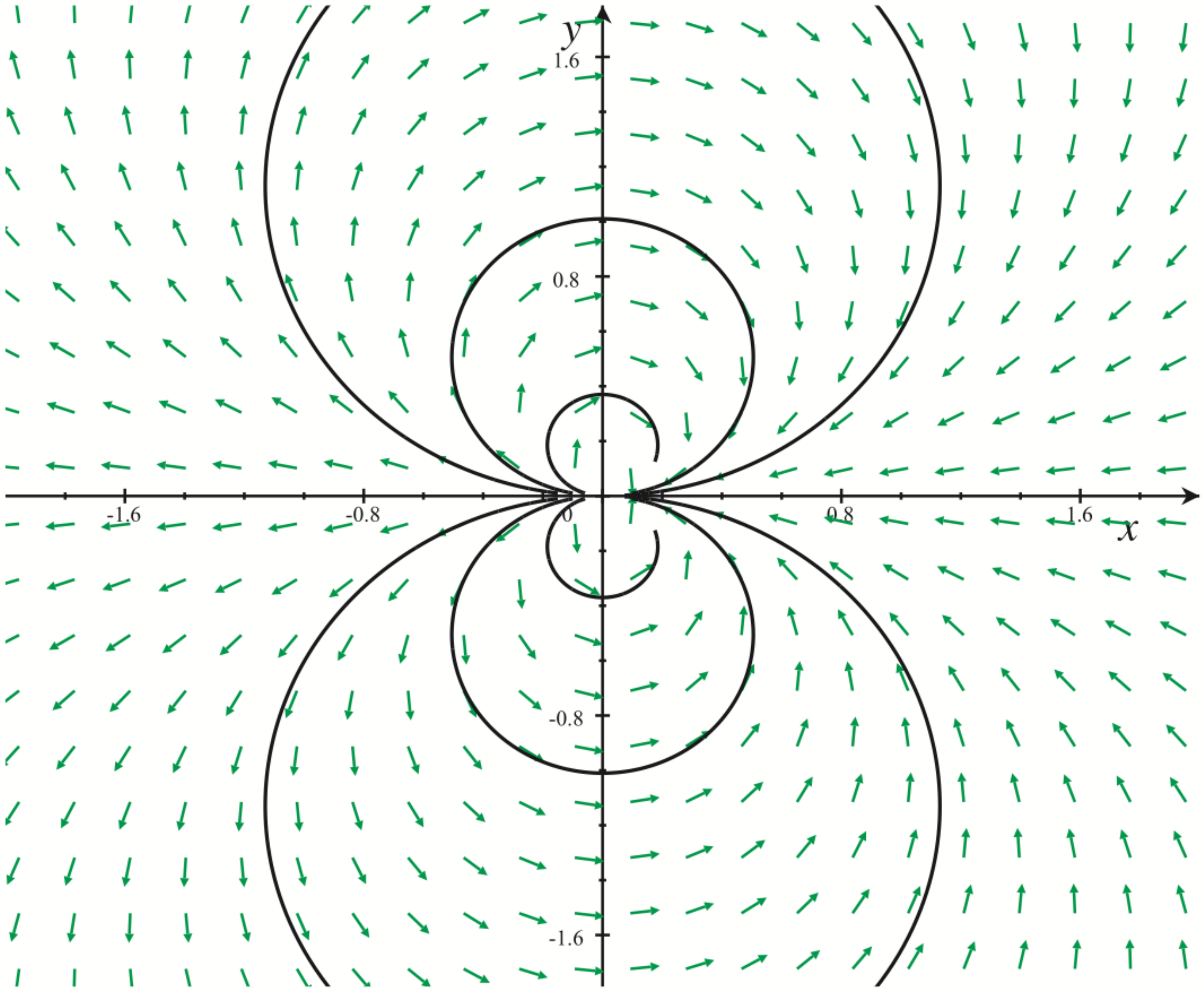


Figure 6.1. *Phase portrait of the example* (6.3).

Example (M, p.194, Ex 6.5)

$$\begin{cases} \dot{x} = y^2 x - x^2 y \\ \dot{y} = x^3 + y^3 \end{cases}$$

$$\dot{r} = r^3 [c(s^2c - c^2 s) + s(c^3 + s^3)] = r^3 \sin^2\theta > 0$$

$$\dot{\theta} = \frac{r^3}{r}[c(c^3+s^3) - s(s^2c - c^2 s)] = r^2 \cos^2\theta > 0$$

$$\frac{dr}{d\theta} = r\frac{\sin^2\theta}{\cos^2\theta}$$

$\cos^2\theta = 0$ at $\theta = \frac{\pi}{2}, \frac{3\pi}{2}.$

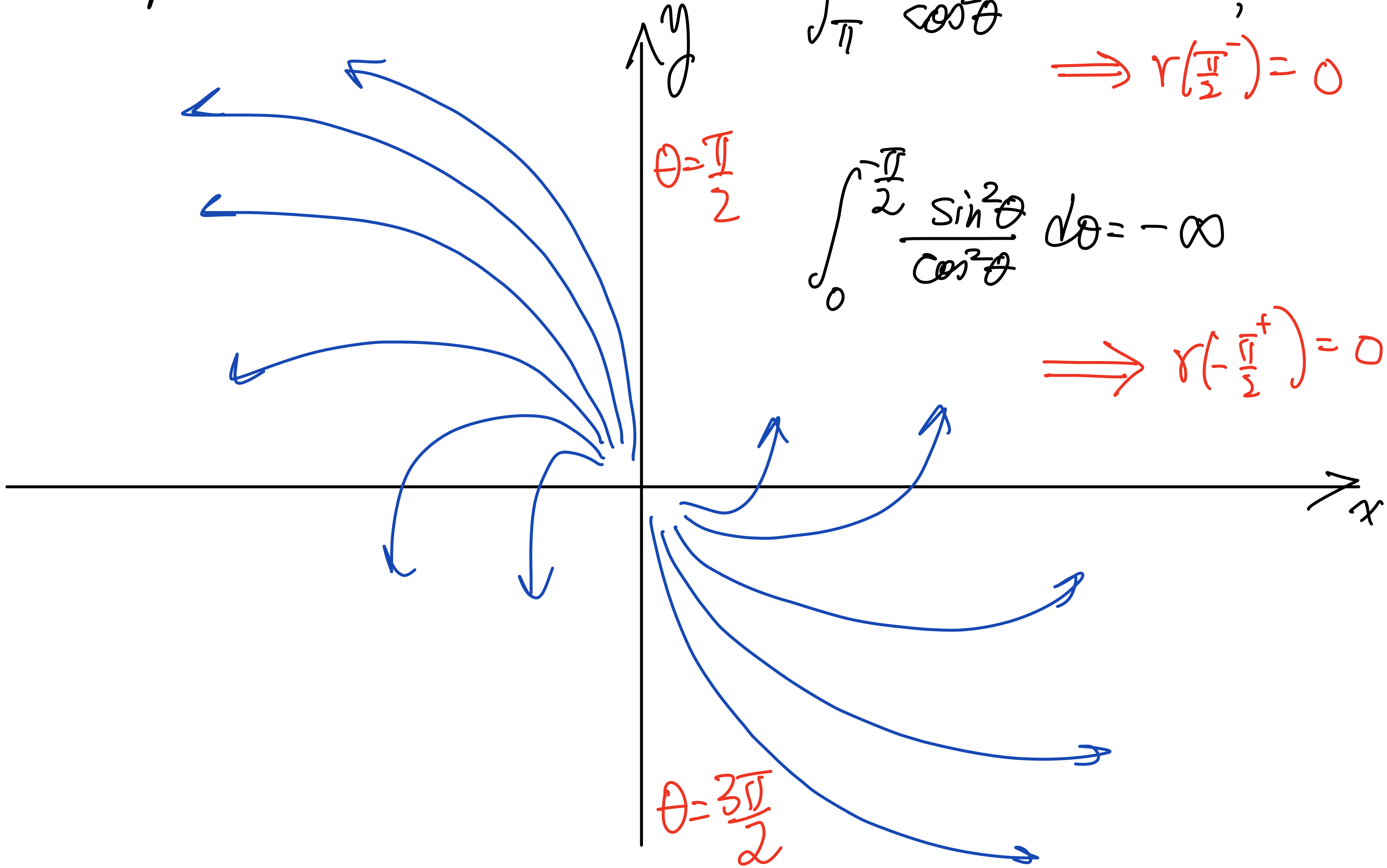
Example (M, p.194, Ex 6.5)

$$\int_{\pi}^{\frac{\pi}{2}} \frac{\sin^2\theta}{\cos^2\theta}\, d\theta = -\infty,$$

$$\Longrightarrow r\left(\tfrac{\pi}{2}^-\right) = 0$$

$$\int_{0}^{-\frac{\pi}{2}} \frac{\sin^2\theta}{\cos^2\theta}\, d\theta = -\infty$$

$$\Longrightarrow r\left(-\tfrac{\pi}{2}^+\right) = 0$$



Example (M, p.194, Ex 6.5)

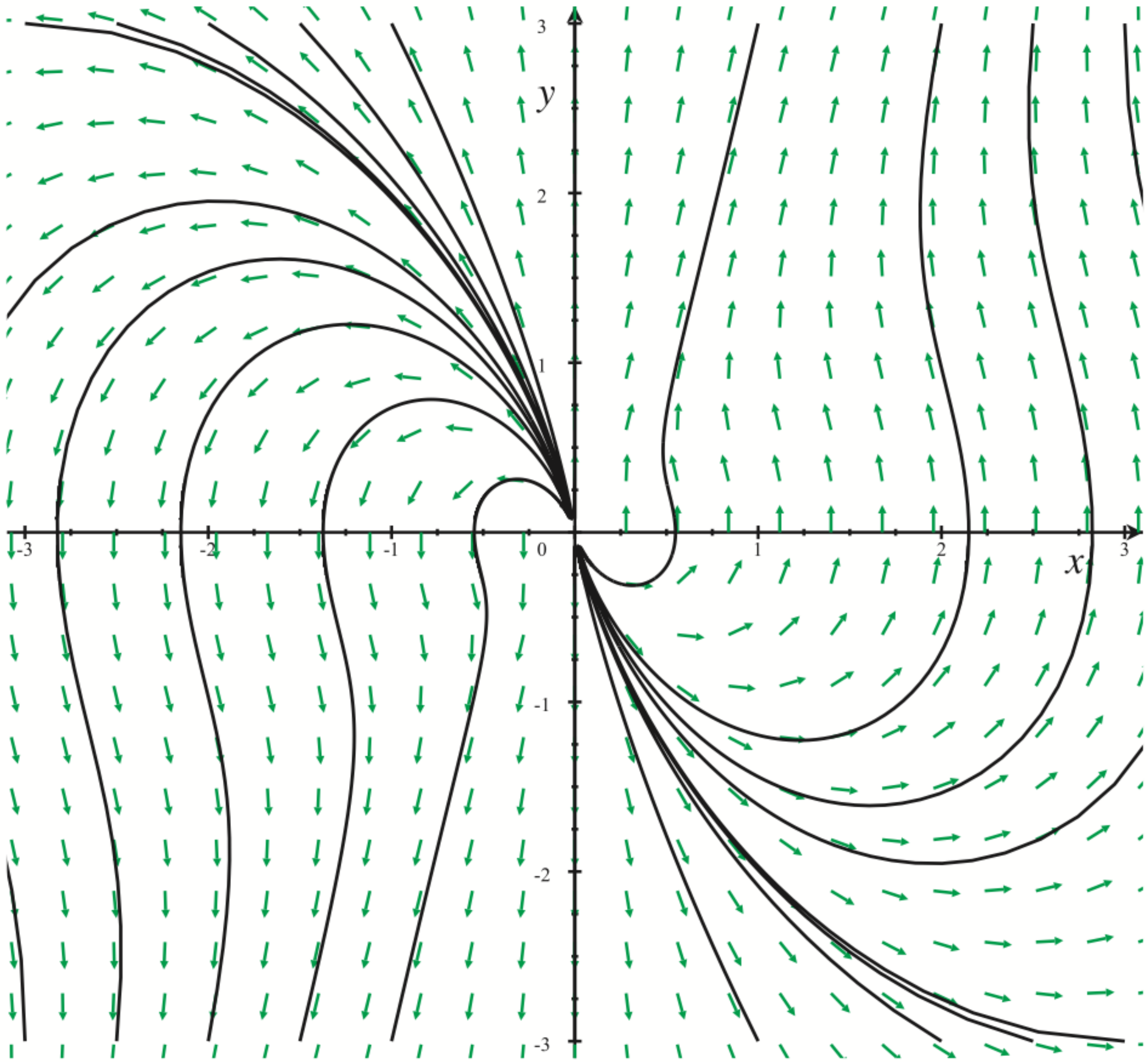


Figure 6.3. *Phase portrait of* (6.11).

Complete characterization (Case 3)

(M p. 192) Near 0 (inside $B_\delta(0)$ for some δ)

1. Topological center: every oribit is periodic enclosing 0.

2. Nonhyperbolic focus: $r(t) \to 0$, $|\theta(t)| \to +\infty$ as either $t \to +\infty$ or $t \to -\infty$

3. Nonhyperbolic node: $\exists$ some θ_c s.t. $g(\theta_c) = 0$

$$r(\theta) \longrightarrow +\infty \text{ or } -\infty$$

$$\text{as } \theta \longrightarrow \theta_c^{\pm}$$

$$\boxed{\text{Case } ④, \text{ linear center } \quad A = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}}$$

$$\begin{cases} \dot{x} = -\omega y + p(x,y) \\ \dot{y} = \omega x + q(x,y) \end{cases}, \qquad |p, q| \leq C(x^2+y^2)$$

$$\underline{\dot{r}} = c[\cancel{-\omega r s} + p(rc, rs)] + s[\cancel{\omega r c} + q(rc, rs)]$$

$$= c\, p(rc, rs) + s\, q(rc, rs) \leq \underline{O(r^2)}$$

$$\underline{\dot{\theta}} = \frac{1}{r}\left[c(\cancel{\omega r c} + q(rc, rs)) - s(\cancel{-\omega r s} + p(rc, rs))\right]$$

$$= \omega + \frac{1}{r}\left[c\, q(rc, rs) - s\, p(rc, rs)\right] = \underline{\omega + O(r)}$$

$\boxed{\text{Case (4), linear center } A = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}}$

$$\begin{cases} \dot{x} = -\omega y + p(x,y) \\ \dot{y} = \omega x + q(x,y), \end{cases} \quad |p,q| \le C(x^2+y^2)$$

$$\underline{\dot{r}} = c\,p(rc,rs) + s\,q(rc,rs) \le \underline{O(r^2)}$$

$$\underline{\dot{\theta}} = \omega + \frac{1}{r}\left[c\,q(rc,rs) - s\,p(rc,rs)\right] = \underline{\omega + O(r)}$$

<u>Note</u>: for $r \ll 1$, $\dot{\theta} \approx \omega$, i.e. θ rotates with speed ω

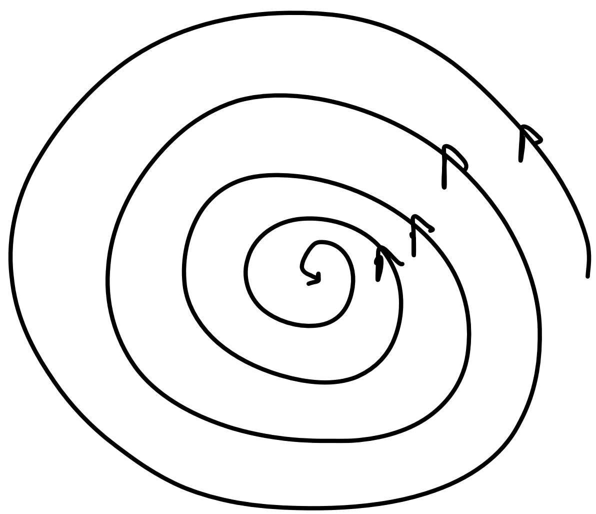
Example (p. 197, Ex 6.8)

$$\begin{cases} \dot{x} = -\omega y - x^3 \\ \dot{y} = \omega x - y x^2 \end{cases}$$

$g(x,y) = -x^2$

$$\begin{pmatrix} \dot{x} = -\omega y + x\, g(x,y) \\ \dot{y} = \omega x + y\, g(x,y) \end{pmatrix}$$

$$\begin{cases} \dot{r} = c(-r^3 c^3) + s(-r^3 s c^2) = -r^3 \cos^2\theta \le 0 \\ \dot{\theta} = \omega + \frac{r^3}{r}\left[c(-sc^2) - s(-c^3)\right] = \omega \end{cases}$$



spiral in,

asymptotically stable

Example (p.198 Ex 6.10)

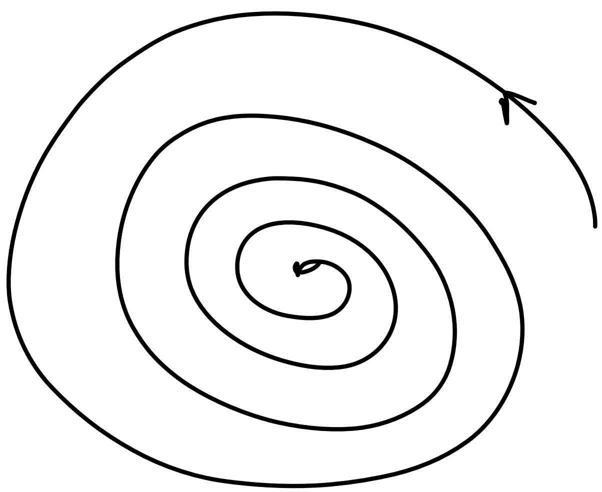
$$\begin{cases}\dot{x} = -y - x^3 - xy^2 \\ \dot{y} = x - yx^2 - y^3\end{cases}$$

$$g(x,y) = -x^2 - y^2$$

$$\left(\begin{array}{l}\dot{x} = -\omega y + x\,g(x,y) \\ \dot{y} = \omega x + y\,g(x,y)\end{array}\right)$$

$$\dot{r} = r^3\left[c(-c^3 - cs^2) + s(-sc^2 - s^3)\right] = -r^3 < 0$$

$$\dot{\theta} = 1 + r^2\left[c(-sc^2 - s^3) - s(-c^3 - cs^2)\right] = 1$$



Spiral in,

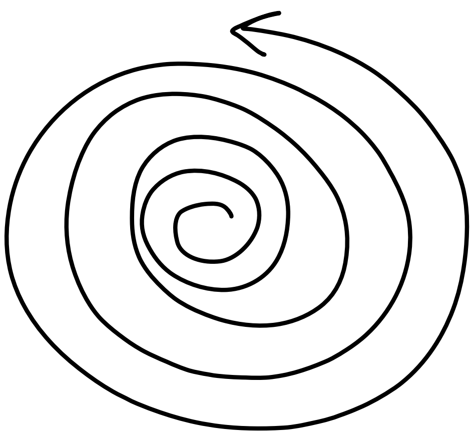
asymptotically stable

Example (p. 199, Ex 6.11)

$$\begin{cases} \dot{x} = -\omega y + xy^2 + x^2 y + y^3 \\ \dot{y} = \omega x + y^3 - x^3 - xy^2 \end{cases}$$

$$\dot{r} = r^3\left[c(cs^2 + c^2 s + s^3) + s(s^3 - c^3 - cs^2)\right] = r^3 \sin^2\theta \geq 0$$

$$\dot{\theta} = \omega + r^2\left[c(s^3 - c^3 - cs^2) - s(cs^2 + c^2 s + s^3)\right] = \underbrace{\omega - r^2}_{>0 \text{ for } r \ll 1}$$



spiral out,

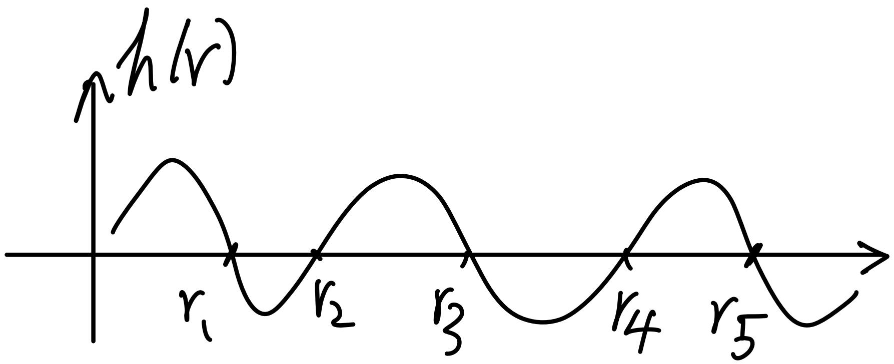
unstable

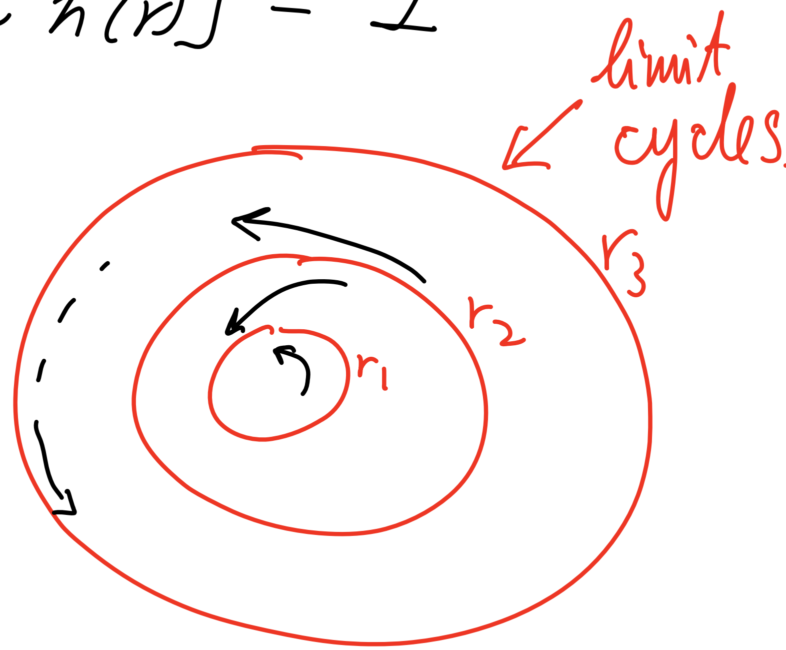
Example (p. 201, Ex 6.14)

$$\begin{cases} \dot{x} = -y + x h(r) \\ \dot{y} = x + y h(r) \end{cases} \qquad \left(r = \sqrt{x^2+y^2}\right)$$

$$\dot{r} = c[rc\,h(r)] + s[rs\,h(r)] = r h(r)$$

$$\dot{\theta} = 1 + c[s\,h(r)] - s[c\,h(r)] = 1$$





Complete characterization of linear center 4

(M p. 197) Near 0 (inside $B_\delta(0)$ for some δ)

1. Topological center: every orbit is periodic enclosing 0.
2. Nonhyperbolic focus: $r(t) \to 0$, $|\theta(t)| \to +\infty$ as either $t \to +\infty$ or $t \to -\infty$
3. Center-focus: there is an infinite nested limit cycles $\gamma_n \xrightarrow{n\to\infty} 0$ and every orbit between γ_n & γ_{n+1} cycles toward γ_n, γ_{n+1} as $t \to \pm\infty$