Fundamental Solution for Periodic System

$$\frac{dX}{dt} = A(t)\,X, \qquad X(0) = X_0$$

$A(t+T) = A(t)$ T-periodic

Recall:

$$X(t) = \Phi(t)\,X_0$$

where $\Phi(t)$ satisfies:

$$\frac{d\,\Phi(t)}{dt} = A(t)\,\Phi(t), \qquad \Phi(0) = I$$

Floquet Theory

$$\boxed{\text{Structure of } \Phi(t) \text{ when } A \text{ is } T\text{-per?}}$$

(Quick review of properties of $\Phi(t)$

(1) Consider $\dfrac{dX}{dt} = A(t)\,X, \qquad X(s) = Z$

Solution $X(t) = \Phi(t,s)\,X(s) = \Phi(t,s)\,Z$

Then $\Phi(t) = \Phi(t,0)$

and $\Phi(t,s) = \Phi(t)\,\Phi(s)^{-1}$ (inverse exists) pf Let $\Psi(t) = \Phi(t)\Phi(s)^{-1}$

Then $\Psi(s) = \Phi(s)\Phi(s)^{-1} = I$

$$\frac{d\Psi(t)}{dt} = \frac{d\Phi(t)}{dt}\Phi(s)^{-1}$$

$$= A(t)\Phi(t)\Phi(s)^{-1}$$

$$= A(t)\Psi(t)$$

Hence $\Psi(t)Z_1$ solves $\frac{dX}{dt} = A(t)X,\ X(s) = Z_1$

$\Rightarrow \quad \Psi(t)Z_1 = \Phi(t,s)Z_1 \qquad \forall Z_1$

$\Rightarrow \quad \Psi(t) = \Phi(t,s)$

i.e. $\Phi(t)\Phi(s)^{-1} = \Phi(t,s)$

(2) $\Phi(t)^{-1}$ exists

Abel Formula [M, Thm 2.34]

$$\det(\Phi(t,s)) = \exp\left(\int_s^t tr(A(r))dr\right) \neq 0$$

See also Hw 1 #4.

[M, Thm 2.36 p. 62] (Floquet 1883)

Let A be T-per: $A(t+T) = A(t)$.

Then there is B and $P(\cdot)$ (T-periodic) s.t.

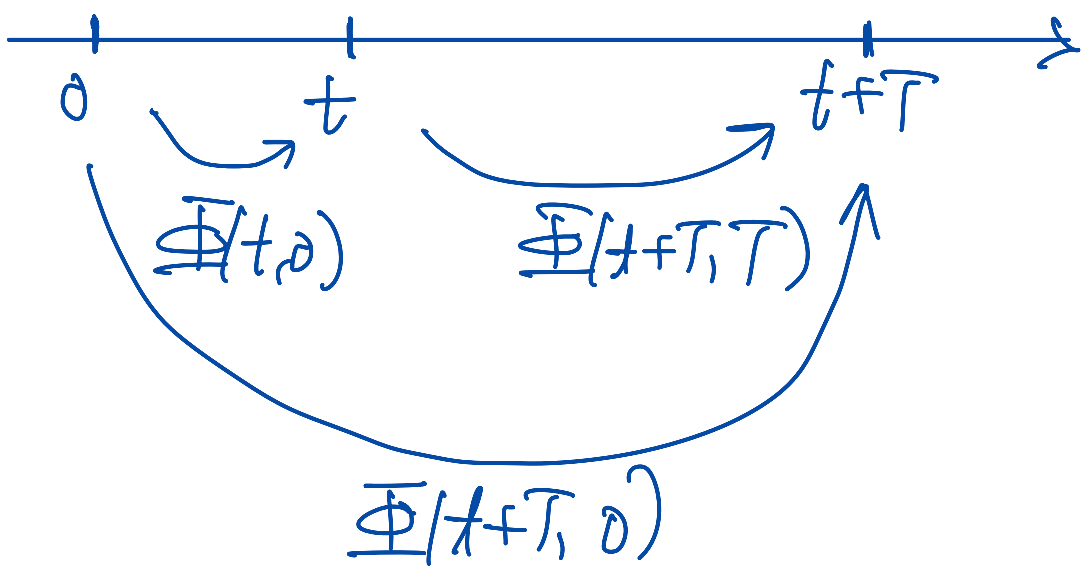
$$\boxed{\Phi(t) \ \left(:= \Phi(t,0)\right) = P(t)\, e^{tB}}$$

Pf Note:

$$\boxed{\Phi(t+T,0) = \Phi(t+T,T)\,\Phi(T,0)}$$

by uniqueness of (linear) ODE

$$\frac{dX}{dt} = A(t)\,X \qquad t \geq 0$$



Claim: $\boxed{\Phi(t+T,T)=\Phi(t,0)}$

$\Updownarrow$

Pf: $\frac{dX}{dt}=A(t)X, \quad X(T)=Z$

$$X(t+T)=\Phi(t+T,T)Z$$

Let $Y(t)=X(t+T)$

Then $Y(0)=Z$,

$$\frac{dY(t)}{dt}=\frac{dX(t+T)}{dt}=\underline{A(t+T)}X(t+T)$$
$$=\underline{A(t)}Y(t)$$

Hence $Y(t)=\Phi(t,0)Z$

i.e. $X(t+T)$

$$\Phi(t+T,T)Z=\Phi(t,0)Z \qquad (\forall Z)$$

$$\Longrightarrow \quad \Phi(t+T,T)=\Phi(t,0)$$

Claim: there is B (possibly complex) s.t. $\Phi(T,0) = e^{TB}$ (i.e. $B = \frac{1}{T}\log \Phi(T,0)$)

(Assumed claim)

Define $P(t) = \Phi(t,0)e^{-tB}$

Then

$$\underline{P(t+T)} = \Phi(t+T,0)e^{-(t+T)B}$$

$$= \Phi(t+T,T)\Phi(T,0)e^{-tB}e^{-TB}$$

$$= \Phi(t,0)\cancel{e^{TB}}e^{-tB}\cancel{e^{-TB}}$$

$$= \Phi(t,0)e^{-tB}$$

$$= \underline{P(t)}$$

Pf of Claim [M Lem 2.35 p. 61]

Let M be non-singular (M^{-1} exists)

Then there is B (possibly complex) s.t.

$$M = e^{B} \quad \text{i.e.} \quad B = \log M.$$

Note:

$$P^{-1} M P = P^{-1} e^{B} P = e^{P^{-1} B P}$$

Transform M first to its Jordan Form:

$$P^{-1} M P = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{bmatrix}$$

$$J_i = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} \qquad \lambda \neq 0$$

$$= \lambda I + N$$

$$= (\lambda I)\left(I + \frac{N}{\lambda}\right)$$

(i) commute $\quad (\lambda I)\left(I+\frac{N}{\lambda}\right) = \left(I+\frac{N}{\lambda}\right)\lambda I$

(ii) N is nilpotent, $N^m = 0$ for some m

Define

$$\boxed{\log J_i = \underbrace{(\log\lambda)I}_{a} + \underbrace{\log\left(I+\frac{N}{\lambda}\right)}_{b}}$$

(a) $\lambda \neq 0, \quad \lambda = re^{i\theta}, \quad \log \lambda = \log r + i\theta$

(b) Recall:

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots = -\sum_{j=1}^{\infty} \frac{x^j}{j}$$

$$|x| < 1$$

Def:

$$\log\left(I + \frac{N}{\lambda}\right) := -\sum_{j=1}^{\infty} \frac{1}{j}\left(-\frac{N}{\lambda}\right)^{j}$$

Only a finite sum

$$= \underbrace{-\sum_{j=1}^{m-1} \frac{1}{j}\left(-\frac{N}{\lambda}\right)^{j-1}}_{\text{well-defined.}}$$

Note:

$$e^{\log J_i} = e^{(\log\lambda)I + \log\left(I+\frac{N}{\lambda}\right)}$$

$$= e^{(\log\lambda)I}\, e^{\log\left(I+\frac{N}{\lambda}\right)}$$

$e^{A+B} = e^A e^B$ if $AB=BA$

$$= (\lambda I)\, \underbrace{e^{\log\left(I+\frac{N}{\lambda}\right)}}_{=\left(I+\frac{N}{\lambda}\right)} \quad (?)$$

$$= \lambda I + N = J_i$$

Analogy

(Within radius of convergence, composition of power series is still a valid power series.)

(i) $1+x = e^{\log(1+x)}$

$$= \sum_{i=0}^{\infty} \frac{(\log(1+x))^i}{i!} = \sum_{i=0}^{\infty} \frac{1}{i!}\left(-\sum_{j=1}^{\infty} \frac{(-x)^j}{j}\right)^i$$

$$= 1+x \qquad (|x|<1)$$

(ii)

$$e^{\log\left(I+\frac{N}{\lambda}\right)} = \sum_{i=0}^{\infty} \frac{1}{i!}\left(\log\left(I+\frac{N}{\lambda}\right)\right)^i$$

$$= \sum_{i=0}^{\infty} \frac{1}{i!}\left(-\sum_{j=1}^{\infty}\left(-\frac{N}{\lambda}\right)^j\right)$$

$$= \sum_{i=0}^{\infty} \frac{1}{i!}\left(\underbrace{-\sum_{j=1}^{m}\left(-\frac{N}{\lambda}\right)^j}_{\text{no problem of radius of convergence}}\right)$$

it is possible to check/compare term by term

$$= I + \frac{N}{\lambda}$$

Define:

$$\boxed{\log\left(P^{-1} M P\right) = \begin{bmatrix} \log J_1 & & & \\ & \log J_2 & & \\ & & \ddots & \\ & & & \log J_k \end{bmatrix}}$$

$$\Downarrow$$

$$P^{-1} M P = \exp \begin{bmatrix} \log J_1 & & & \\ & \log J_2 & & \\ & & \ddots & \\ & & & \log J_k \end{bmatrix}$$

$$M = \exp\left(\underbrace{P \begin{bmatrix} \ddots & & \\ & \log J_i & \\ & & \ddots \end{bmatrix} P^{-1}}_{} \right)$$

$$\boxed{B = P \begin{bmatrix} \log J_1 & & \\ & \ddots & \\ & & \log J_k \end{bmatrix} P^{-1}}$$

eg $M = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $\log M = \begin{bmatrix} \log 2 & 0 \\ 0 & \log 3 \end{bmatrix}$

(Convince yourself: $e^{\begin{bmatrix} \log 2 & \\ & \log 3 \end{bmatrix}} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$)

$$M = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = 2\underbrace{I}_{= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} + \underbrace{N}_{= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$$

$IN = NI$, $N^2 = 0$

$$\log M = \log(2I+N) = \log\left(2I\left(I+\frac{N}{2}\right)\right)$$

$$= \log(2I) + \log\left(I+\frac{N}{2}\right)$$

$$= (\log 2)I - \sum_{j=1}^{\infty}\left(-\frac{N}{2}\right)^{j}$$

$$= (\log 2)I - \left(-\frac{N}{2}\right)^{1}$$

$$= (\log 2)I + \frac{1}{2}N$$

$$= \begin{bmatrix} \log 2 & \frac{1}{2} \\ 0 & \log 2 \end{bmatrix}$$

Convince yourself that

$$e^{\begin{bmatrix} \log 2 & \frac{1}{2} \\ 0 & \log 2 \end{bmatrix}} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

Remarks

(1) Given M, $B = \log M$ is not unique

$$M = e^{B} = e^{(B + 2\pi i I)} = e^{\tilde{B}}$$

$$\left(\tilde{B} = B + 2\pi i I \right)$$

(2) Given M – real matrix, $B = \log M$ might be complex. (Nothing special: this can happen even for real number:

$$-2 = 2e^{\pi i}$$

$$\log(-2) = \log\left(2e^{\pi i}\right) = (\log 2) + \pi i$$

But for M^2, $B = \log(M^2)$ can be chosen to be real. (Sec 2.9 Prob. #21)

$$\|$$

$\Downarrow$

[M, Thm 2.37, p. 62]

$\Phi(t)$ – fundamental matrix for T-per linear system.

$\exists$ real R and $2T$-per $Q(t)$ s.t.

$$\boxed{\Phi(t) = Q(t)e^{tR}}$$

Monodromy Matrix $\quad M = \Phi(T) = e^{TB}$

$$\dot{X} = A(t)X, \quad X(0) = X_0$$

$$X(t) = \Phi(t)X_0 = P(t)e^{tB}X_0$$

$$X(nT) = P(nT)\,e^{nTB}X_0$$

$$= \underbrace{\cancel{P(0)}}_{I}\,(e^{TB})^n X_0$$

$$= M^n X_0$$

Eigenvalues of M are called Floquet Multipliers

They determine the stability of solutions.

Let $\quad MX_0 = \mu X_0$ $\quad$ X_0 – eigenvector, with μ as eigenvalue

Then $X(nT) = M^n X_0 = \mu^n X_0$

Hence $\|X(nT)\| \longrightarrow 0$ as $n \to +\infty$ if $|\mu| < 1$.

$\longrightarrow +\infty$ as $n \to \infty$ if $|\mu| > 1$

[M, Ex 2.32, p. 58] (Markus-Yamabe)

$$A(t) = \begin{bmatrix} -1+\alpha\cos^2 t & 1-\alpha\cos t\sin t \\ -1-\alpha\cos t\sin t & -1+\alpha\sin^2 t \end{bmatrix}$$

Note: for each t fixed

$$\lambda_1, \lambda_2 \text{ of } A(t) = \frac{1}{2}\left(\alpha-2 \pm \sqrt{\alpha^2-4}\right)$$

$$\left(\operatorname{tr} A = \alpha-2, \quad \det A = 2-\alpha\right)$$

Hence if $\alpha<2 \Longrightarrow A(t)$ is "stable" for all t fixed

However, consider the time dependent lin system:

$$\frac{dX}{dt} = A(t)X$$

It has 2 simple explicit solutions:

$$\boxed{X_1(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} e^{(\alpha-1)t}, \quad X_2(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} e^{-t}}$$

Hence general solution is given by

$$X(t) = c_1 X_1(t) + c_2 X_2(t)$$

For $\underline{1 < \alpha}$, the solution can go to "$+\infty$".

Note $A(t)$ is π-periodic.

$$(\cos^2 t,\ \sin^2 t,\ \cos t \sin t \sim \cos 2t,\ \sin 2t)$$

The fundamental solution is simply given by

$$\Phi(t) = [X_1(t)\ \ X_2(t)]$$

$$\left(\Phi(0) = [X_1(0)\ \ X_2(0)] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

The monodromy matrix

$$M = \Phi(\pi) = \begin{bmatrix} -e^{(\alpha-1)\pi} & 0 \\ 0 & e^{-\pi} \end{bmatrix}$$

μ_1, μ_2 of $M = \underbrace{-e^{(\alpha-1)T_1}}_{|\mu_1|>1 \text{ if } \alpha>1}, \underbrace{e^{-T_1}}_{|\mu_2|<1}$

unstable direction

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Stable direction

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Proof. If all solutions of (2) are bounded, we must have $\|e^{Ct}\|$ bounded as $t \to \infty$; if all solutions of (2) tend to zero, then $\|e^{Ct}\| \to 0$ and does so exponentially, which is to say, $\|e^{Ct}\| \leq c_1 e^{-at}$, with $a > 0$. Since

$$z = y + \int_0^t Y(t)Y^{-1}(t_1)B(t_1)z(t_1)\,dt_1 \tag{6}$$
$$= y + \int_0^t P(t)e^{C(t-t_1)}P(t_1)^{-1}B(t_1)z(t_1)\,dt_1$$

we have

$$\|z\| \leq \|y\| + \int_0^t \|P(t)\|\,\|e^{C(t-t_1)}\|\,\|P(t_1)^{-1}\|\,\|B(t_1)\|\,\|z(t_1)\|\,dt_1 \tag{7}$$
$$\leq c_1 + c_1 \int_0^t \|B(t_1)\|\,\|z(t_1)\|\,dt_1$$

whence boundedness follows, as before. The second part of the statement is also derived along previous lines.

5. Equations with General Variable Coefficients. We already know that the boundedness of the solutions of

$$\frac{dy}{dt} = A(t)y \tag{1}$$

together with the condition $\|B(t)\| \to 0$ as $t \to \infty$ is not sufficient to ensure the boundedness of all solutions of

$$\frac{dz}{dt} = (A(t) + B(t))z \tag{2}$$

[compare (10) of Sec. 1]. We might be tempted, in the light of preceding results, to state that the result will be valid provided that we amend the condition $\|B(t)\| \to 0$ to read $\int^{\infty} \|B(t)\|\,dt < \infty$. Let us show by a counterexample that no such general theorem can hold.

Theorem 5. *There is an equation of type* (1) *with the property that all solutions approach zero as* $t \to \infty$, *and a matrix* $B(t)$ *for which* $\int^{\infty} \|B(t)\|\,dt < \infty$, *such that all solutions of* (2) *are not bounded.*

Proof. Consider the equation

$$\frac{dy_1}{dt} = -ay_1 \tag{3}$$
$$\frac{dy_2}{dt} = (\sin \log t + \cos \log t - 2a)y_2$$

whose general solution is

$$y_1 = c_1 e^{-at} \tag{4}$$
$$y_2 = c_2 e^{t \sin \log t - 2at}$$

If $a > \frac{1}{2}$, every solution approaches zero as $t \to \infty$. If we choose as our perturbing matrix

$$B(t) = \begin{pmatrix} 0 & 0 \\ e^{-at} & 0 \end{pmatrix} \tag{5}$$

the perturbed equation has the form

$$\begin{aligned} \frac{dz_1}{dt} &= -az_1 \\ \frac{dz_2}{dt} &= (\sin \log t + \cos \log t - 2a)z_2 + z_1 e^{-at} \end{aligned} \tag{6}$$

The solution of this system is

$$\begin{aligned} z_1 &= c_1 e^{-at} \\ z_2 &= e^{t \sin \log t - 2at}\left(c_2 + c_1 \int_0^t e^{-t_1 \sin \log t_1}\, dt_1\right) \end{aligned} \tag{7}$$

Let $t = e^{(2n+\frac{1}{2})\pi}$. Since

$$\begin{aligned} \int_0^t e^{-t_1 \sin \log t_1}\, dt_1 &> \int_{te^{-\pi}}^{te^{-2\pi/3}} e^{-t_1 \sin \log t_1}\, dt_1 \\ &> t(e^{-2\pi/3} - e^{-\pi}) \exp\left(-\frac{e^{-\pi} t}{2}\right) \end{aligned} \tag{8}$$

we see that, if

$$1 < 2a < 1 + e^{-\pi/2} \tag{9}$$

the solutions of (6) will be bounded only if $c_1 = 0$. This condition is fulfilled only for those solutions for which $z_1(0) = 0$.

Let us now turn to the problem of seeing what we can salvage from this. We can prove

Theorem 6. *If all the solutions of* (1) *are bounded, then all the solutions of* (2) *are bounded, provided that*

$$\begin{aligned} &(a) \quad \int^{\infty} \|B(t)\|\, dt < \infty \\ &(b) \quad \varliminf_{t \to \infty} \int^t \operatorname{tr}(A)\, dt > -\infty \end{aligned} \tag{10}$$

or, in particular, that

$$(b') \quad \operatorname{tr}(A) = 0$$

Condition (b') is relevant to the important equation

$$u'' + a(t)u = 0 \tag{11}$$

which is equivalent to a two-dimensional system satisfying (b').