Floquet, Monodromy and Poincare

(Review of Lec 17, 18)

$$(*)_1 \begin{cases} \dfrac{dX}{dt} = A(t)X, & A(t+T) = A(t) \\ X(0) = X(T) \end{cases}$$

$$X(t) = \Phi(t)X(0)$$

$$X(T) = X(0) \iff \Phi(T)X(0) = X(0)$$

$$\iff (\Phi(T) - I)X(0) = 0$$

$(*)_1$ has a non-trivial solution $(X(0) \neq 0)$

$\iff (\Phi(T) - I)$ is not invertible

$\left((\Phi(T) - I)^{-1}\right.$ does not exist$\left.\right)$

$\iff \mu = 1$ is an eigenvalue of $\Phi(T)$

($X(0)$ is an eigenvector of $\Phi(T)$
(wrt. $\mu = 1$))

[H, Thm 1.1, p. 408]

$$(*)_2 \begin{cases} \frac{dX}{dt} = A(t)X + g(t) \\ X(t+T) = X(t) \end{cases}$$

$$A(t+T) = A(t)$$
$$g(t+T) = g(t)$$

$$X(t) = \Phi(t)X(0) + \int_0^t \underbrace{\Phi(t,s)}_{=\Phi(t)\Phi(s)^{-1}} g(s)\,ds$$

$$X(T) = X(0)$$

$$\Longleftrightarrow \quad X(0) = \Phi(T)X(0) + \int_0^T \Phi(T,s)g(s)\,ds$$

$$\Longleftrightarrow \quad (\Phi(T) - I)X(0) = -\int_0^T \Phi(T,s)g(s)\,ds$$

If $(\Phi(T)-I)^{-1}$ exists, then $(*)_2$ has a soln. for any $g(t)$ and

$$X(0) = -(\Phi(T)-I)^{-1}\int_0^T \Phi(T,s)g(s)\,ds$$

and $\|X(\cdot)\|_{C^0(0,T)} \le \alpha \int_0^T \|g(s)\|\,ds$

Otherwise, for $(*)_2$ to have a solution, $g(\cdot)$ needs to satisfy some compatibility condition

[H, Thm 2.1, p.412]

$$(*)_3 \begin{cases} \dfrac{dX}{dt} = A(t)X + f(t,X) \\ X(t+T) = X(t) \end{cases}$$

$A(t+T) = A(t)$

$f(t+T, X) = f(t, X)$

$(*)_3$ has a unique solution if

(1) $(\Phi(T) - I)^{-1}$ exists

(2) $\| f(t,X) - f(t,Y) \| \leq \theta \| X - Y \|$

(3) $\alpha \theta T < 1$

[H, p.415, Thm 2.3]

$$(*)_4 \begin{cases} \dfrac{dX}{dt} = F(t, X, \mu) \\ X(t+T) = X(t) \end{cases}$$

$F(t+T, X, \mu) = F(t, X, \mu)$

Suppose (1) $(*)_4 \big|_{\mu=0}$ has a solution $\gamma(t)$

(2) Let $A(t) = D_X F(t, \gamma(t), 0) \Rightarrow \Phi(t)$

$(\Phi(T) - I)^{-1}$ exists

Then $(*)_4$ has a solution for $|\mu| \ll 1$

Floquet Theory [M, Sec 2.8]

$$\frac{dX}{dt} = A(t)X \qquad A(t+T) = A(t)$$

$$\boxed{\Phi(t) = Q(t)e^{tB}, \quad Q(t+T) = Q(t)}$$

$$\Phi(0) = I \Rightarrow Q(0) = I$$

$$\Rightarrow Q(nT) = I$$

$$\Phi(T) = e^{TB} := M \quad \text{(Monodromy matrix)}$$

$$\Phi(nT) = e^{nTB} = M^n$$

$$X(t) = Q(t)e^{tB}X(0)$$

$$X(nT) = Q(nT)e^{nTB}X(0)$$

$$= M^n X(0)$$

$$= \mu^n X(0)$$

choose $MX(0) = \mu X(0)$

$$\|X(nT)\| = |\mu|^n \|X(0)\| \to \begin{cases} 0 & \text{if } |\mu| < 1 \\ +\infty & \text{if } |\mu| > 1 \end{cases}$$

Let $X(t)=Q(t)Y(t)$, or $Y(t)=Q^{-1}(t)X(t)$

Then
$$\dot{X}(t)=\big(Q(t)Y(t)\big)^{\circ}$$
$$=\dot{Q}(t)Y(t)+Q(t)\dot{Y}(t)$$

$$\frac{d}{dt}\Phi(t)=A(t)\Phi(t)$$

i.e.
$$\big(Q(t)e^{tB}\big)^{\circ}=A(t)Q(t)e^{tB}$$
$$\dot{Q}(t)\cancel{e^{tB}}+Q(t)B\cancel{e^{tB}}=A(t)Q(t)\cancel{e^{tB}}$$
$$\underline{\dot{Q}(t)+Q(t)B=A(t)Q(t)}$$

$$\dot{X}(t)=\big(A(t)Q(t)-Q(t)B\big)Y+Q(t)\dot{Y}(t)$$
$$=A(t)\underbrace{Q(t)Y}_{X}-Q(t)BY+Q\dot{Y}$$

$$\Longrightarrow\quad \dot{X}-A(t)X=Q(t)\big(\dot{Y}-BY\big)$$

Hence
$$\underset{\underset{\Phi(t)}{\Downarrow}}{\dot{X}=A(t)X}\iff\underset{\underset{e^{tB}}{\Downarrow}}{\dot{Y}=BY}$$

$$\Phi(T) = M = e^{TB}$$

$$MX = \mu X \qquad BY = \lambda Y$$

- Eigenvalues of B (λ_i) are called characteristic exponents of B
- Eigenvalues of M (μ_i) are called characteristic multipliers of M
- $\mu_i = e^{T\lambda_i}$

$$Re(\lambda_i) < 0 \iff |\mu_i| < 1$$

Autonomous System $\frac{dX}{dt} = F(X)$ (*)

$$\boxed{\text{Suppose } (*) \text{ has a } T\text{-periodic solution } \gamma(t) \quad (\gamma(t+T) = \gamma(t))}$$

Note: T is not known apriori.

Let $A(t) = D_X F(\gamma(t))$.

Then $A(t+T) = A(t)$

Consider: $\frac{d\gamma(t)}{dt} = F(\gamma(t))$

Then $\frac{d}{dt}\Big(\underbrace{\frac{d}{dt}\gamma(t)}_{X(t)}\Big) = \frac{d}{dt} F(\gamma(t)) = \underbrace{D_X F(\gamma(t))}_{A(t)} \underbrace{\frac{d\gamma}{dt}}_{X(t)}$

Hence $X(t) = \dot{\gamma}(t) \neq 0$ solves

$$\frac{dX}{dt} = A(t) X, \quad X(t+T) = X(t)$$

$$\Longrightarrow \boxed{(\Phi(T) - I)^{-1} \text{ does not exist!}}$$

$$\boxed{\begin{array}{ll} \text{and} & X(0) = \Phi(T)\,X(0) \\ \text{i.e.} & \dot{\gamma}(0) = \Phi(T)\,\dot{\gamma}(0) \\ \text{i.e.} & \dot{\gamma}(0) \text{ is an eigenvector of } \Phi(T) \text{ w.r.t. } \mu=1 \end{array}}$$

Physical Interpretation : time shift

$$\begin{cases} \dfrac{d\gamma(t)}{dt} = F(\gamma(t)) \\ \dfrac{d}{dt}\gamma(t+\delta) = F(\gamma(t+\delta)) \end{cases}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\gamma(t+\delta) - \gamma(t)\right) = F(\gamma(t+\delta)) - F(\gamma(t))$$

$$\frac{d}{dt}\left(\frac{\gamma(t+\delta)-\gamma(t)}{\delta}\right) = \frac{F(\gamma(t+\delta)) - F(\gamma(t))}{\delta}$$

$\downarrow \quad \delta \to 0$

$$\boxed{\frac{d}{dt}\left(\dot{\gamma}(t)\right) = \left[D_x F(\gamma(t))\right]\dot{\gamma}(t)}$$

[H, p.416, Thm 2.4]

Consider $\frac{dx}{dt} = F(x, \mu)$ (*)

Suppose at $\mu = 0$

(1) (*) has a T-periodic solution $\gamma_0(t)$

(2) Let $A(t) = D_x F(\gamma_0(t)) \Longrightarrow \Phi(t)$

$\mu = 1$ is a simple eigenvalue of $\Phi(T)$

i.e. multiplicity of $\mu = 1$ is one

i.e. $\dim(\mathrm{Null}(\Phi(T) - I)) = 1$

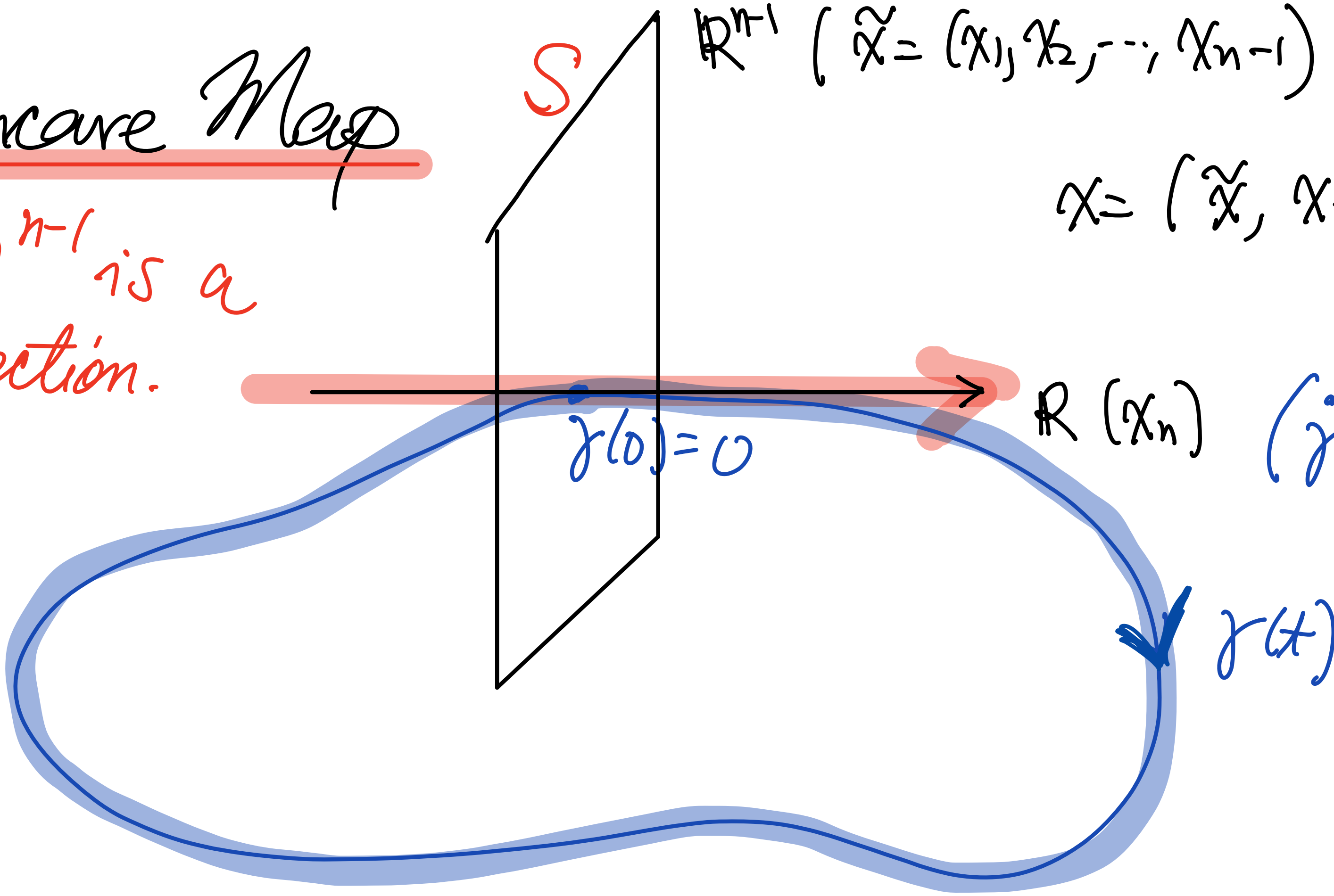
Then for $|\mu| \ll 1$, (*) has a solution $\gamma_\mu(\cdot)$ with period $T(\mu)$

Poincare Map

$S \cong \mathbb{R}^{n-1}$ is a cross-section. $\perp e_n$

$\mathbb{R}^{n-1}$ $(\tilde{x} = (x_1, x_2, \cdots, x_{n-1}))$

$x = (\tilde{x}, x_n) \in \mathbb{R}^n$

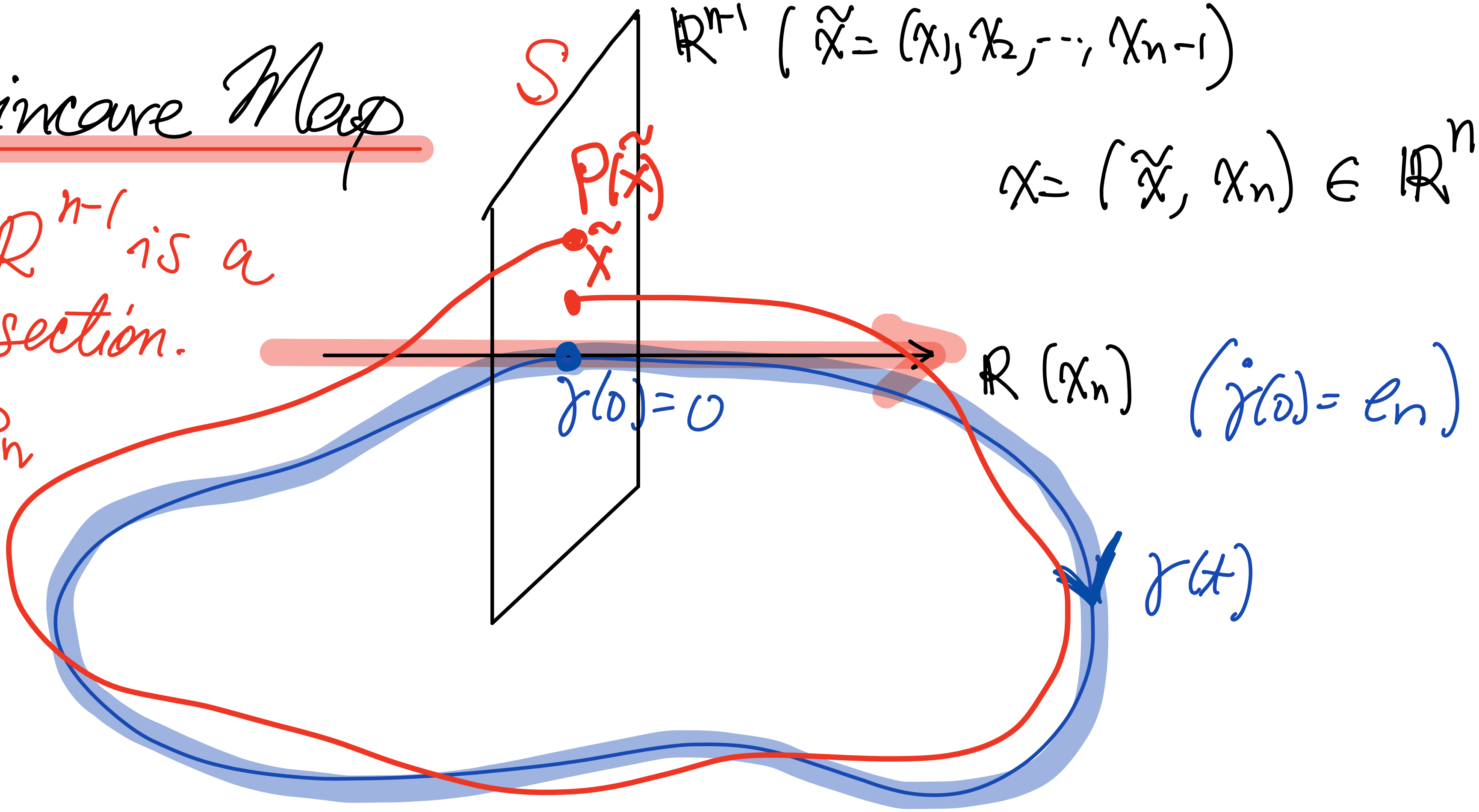


$(\dot{\gamma}(0) = e_n)$

Poincare Map

$S \cong \mathbb{R}^{n-1}$ is a cross-section.

$\perp e_n$



- For any $X(0) = \tilde{x} \in S$, close to $\gamma(0)$, there is a $\tau(x)$ s.t.

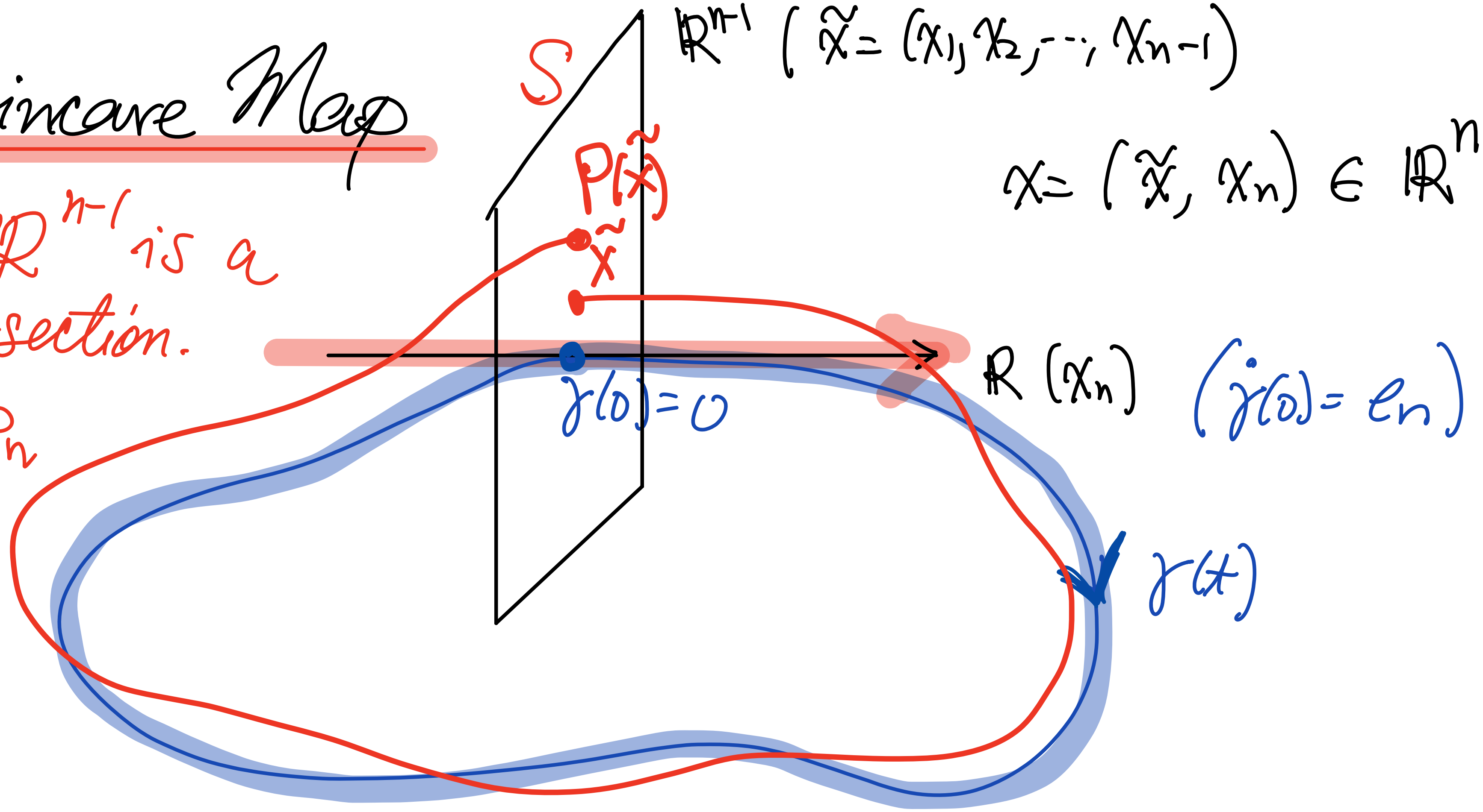
$$\phi_{\tau(\tilde{x})}(\tilde{x}) \in S$$

- Poincare Map: $P(\tilde{x}) = \phi_{\tau(\tilde{x})}(\tilde{x})$

$P : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$

$(S \to S)$

Poincare Map



$S \cong \mathbb{R}^{n-1}$ is a cross-section. $\perp e_n$

- The solution $X(t)$ starting at $\tilde{x} \in S$ is a periodic orbit if and only if x is a fixed point of P, ie. $P(\tilde{x}) = \tilde{x}$

e.g. $P(0) = 0$

Relationship between M and $DP(0)$

[M, Thm 4.55]

collection of eigenvalues

$$\mathrm{Spec}(M) = \mathrm{Spec}(DP(0)) \cup \{1\}$$

$n \times n$ matrix　　$(n-1)\times(n-1)$ matrix　　Comes from time shift.

$$\begin{bmatrix} & & \\ & M & \\ & & \end{bmatrix}^{n\times n} = \left[\begin{array}{c|c} \overset{(n-1)\times(n-1)}{DP(0)} & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \times\ \times \cdots \times & 1 \end{array}\right]$$

- $e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector of M w.r.t. $\mu = 1$ (time shift)

- $M\big|_S = DP(0)$

  S: $\mathbb{R}^{n-1}$　　$DP(0)$: $(n-1)\times(n-1)$

Pf (Outline)

$$\tilde{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} \in \mathbb{R}^{n-1} = S, \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

$e_n \,//\, F(0) = \dot{\gamma}(0)$

①

$$\frac{d}{dt}\varphi_t(x) = F(\varphi_t(x))$$

$D_x \downarrow$

$$\frac{d}{dt}\left[D_x\varphi_t(x)\right] = \left[D_xF(\varphi_t(x))\right]\left[D_x\varphi_t(x)\right]$$

$$\frac{d}{dt}\left[D_x\varphi_t(0)\right] = \left[A(t)\right]\left[D_x\varphi_t(0)\right]$$

$$D_x\varphi_t(0) = \Phi(t) \qquad (D_x\varphi_0(0) = I)$$

$$\Rightarrow \boxed{D_x\varphi_T(0) = \Phi(T) = M}$$

$$[M] = \left[\begin{array}{c|c} \tilde{M} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline * * * * * * & 1 \end{array}\right]$$

Recall: $M\dot{\gamma}(0) = \dot{\gamma}(0)$, i.e. $Me_n = e_n$

(2) Let $Q: \mathbb{R}^n \longrightarrow S \subseteq \mathbb{R}^n$

$x \to \varphi_{\tau(x)}(x) \in S$

$$Q(x) = \varphi_{\tau(x)}(x)$$

$$D_x Q(x) = D_x \varphi_{\tau(x)}(x) + \frac{d\varphi_{\tau(x)}(x)}{dt} \nabla_x \tau(x)$$

$x=0$

$$D_x Q(0) = D_x \varphi_{\tau(0)}(0) + \frac{d}{dt}\varphi_{\tau(0)}(0)\, \nabla_x \tau(0)$$

$$\left(\tau(0) = T,\ D_x \varphi_T(0) = M,\ \frac{d}{dt}\varphi_T(0) = F(0) \parallel e_n\right)$$

$$D_x Q(0) = M + F(0)\nabla_x \tau$$

$$= \left[\begin{array}{ccc|c} & \tilde{M} & & 0 \\ & & & \vdots \\ & & & 0 \\ \hline x & x \cdots & x & 1 \end{array}\right] + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hline x \end{bmatrix} \begin{bmatrix} \nabla_x \tau \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} & \tilde{M} & & 0 \\ & & & \vdots \\ & & & 0 \\ \hline x & x \cdots & x & x \end{array}\right]$$

③ $P : S \longrightarrow S$

$\tilde{x} \longrightarrow \varphi_{\tau(\tilde{x})}(\tilde{x})$

$P(\tilde{x}) = Q(\tilde{x})$

$$\begin{bmatrix} P(\tilde{x}) \\ \hline 0 \end{bmatrix} = Q\begin{pmatrix} \tilde{x} \\ \hline 0 \end{pmatrix}$$

$$\begin{bmatrix} D_{\tilde{x}} P(\tilde{x}) \\ 0 \end{bmatrix} = \left[D_x Q\begin{pmatrix} \tilde{x} \\ 0 \end{pmatrix} \right] \left[D_{\tilde{x}} \begin{pmatrix} \tilde{x} \\ 0 \end{pmatrix} \right]$$

$\tilde{x} = 0$

$$\begin{bmatrix} \underset{(n-1)\times(n-1)}{D_{\tilde{x}} P(0)} \\ 0 \end{bmatrix} = \left[\underset{n\times n}{D_x Q(0)} \right] \begin{bmatrix} \underset{(n-1)\times(n-1)}{I} \\ \hline 0 \end{bmatrix}$$

$$= \underbrace{\left(M + F(0) \nabla_x \tau \right)}\begin{pmatrix} I \\ \hline 0 \end{pmatrix}$$

$$\left[\begin{array}{c|c} \tilde{M} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \times \times \cdots \times & \times \end{array} \right] = \begin{bmatrix} \underset{(n-1)\times(n-1)}{\tilde{M}} \\ \hline \times \times \times \cdots \times \end{bmatrix}$$

Hence $D_{\tilde{x}} P(0) = \tilde{M}$

④ $$\begin{bmatrix} & & \\ & M & \\ & & \end{bmatrix}^{n\times n} = \left[\begin{array}{c|c} \boxed{\quad \tilde{M} \quad} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline x - - - - x & \boxed{1} \end{array}\right] = \left[\begin{array}{c|c} \boxed{D_{\tilde{x}} P(0)} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline x - - - - x & \boxed{1} \end{array}\right]$$

$$\Downarrow$$

$$\boxed{\operatorname{spec}(M) = \operatorname{spec}(D_{\tilde{x}} P(0)) \cup \{1\}}$$

Proof of [H, p.416, Thm 2.4] (Outline)

Need: $\varphi_{\tau(\tilde{x})}(\tilde{x}, \mu) - \tilde{x} = 0$

i.e. $P(\tilde{x}, \mu) - \tilde{x} = 0$

At $\mu = 0$, $P(0,0) - 0 = 0$

as $\operatorname{Spec} M = \underbrace{\operatorname{Spec} DP(0)}_{1 \notin} \cup \{1\}$

Hence $D_{\tilde{x}} P(0,0) - I$ is invertible

By Implicit Function Thm,

$P(\tilde{x}, \mu) - \tilde{x} = 0$ has a solution $\tilde{x}$ for $|\mu| \ll 1$.