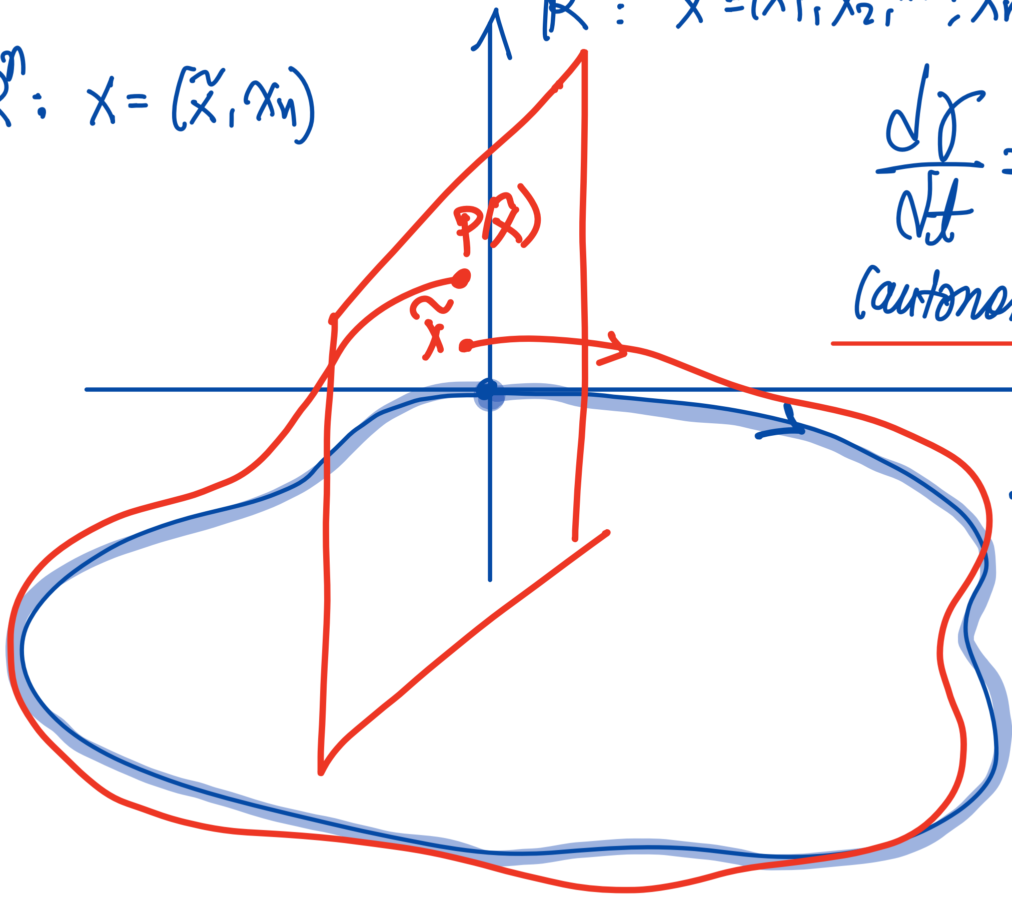
Examples of Stability Analysis of Periodic Solutions

$\mathbb{R}^{n-1}$: $\tilde{x} = (x_1, x_2, \cdots, x_{n-1})$

$\mathbb{R}^n$: $x = (\tilde{x}, x_n)$



$$\frac{d\gamma}{dt} = F(\gamma)$$

(autonomous system)

$\gamma(t)$: $\gamma(t+T) = \gamma(t)$

① Monodromy matrix: $M = \Phi(T)^{n \times n}$, $Me_n = e_n$

Poincaré Map: $P: S \cong \mathbb{R}^{n-1} \longrightarrow S \cong \mathbb{R}^{n-1}$

$$[DP(0)]^{(n-1)\times(n-1)}$$

time shift

②

$$\begin{bmatrix} & & \\ & M & \\ & & \end{bmatrix}^{n \times n} = \left[\begin{array}{c|c} DP(0)^{(n-1)\times(n-1)} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \times\times\times\times\times\times\times\times & 1 \end{array}\right]$$

③ $\boxed{\operatorname{Spec}\{M\} = \operatorname{Spec}\{DP(0)\} \cup \{1\}}$

④ Eigenvalues of DP

$\{|\mu_i|\} < 1 \Rightarrow$ asymptotically stable.

$|\mu_i| > 1 \Rightarrow$ unstable.

⑥ How to compute μ_i?

$$\det M = \det \Phi(T) = (\det DP(0))(1) = \det DP(0)$$

Abel Formula (Thm 2.34)

$$\frac{d\Phi(t,s)}{dt} = A(t)\,\Phi(t,s), \quad \Phi(s,s) = I, \quad t > s$$

$$\boxed{\det \Phi(t,s) = e^{\int_s^t \operatorname{tr} A(r)\,dr}}$$

$$\boxed{\det \Phi(T) = e^{\int_0^T \operatorname{tr} A(r)\,dr}}$$

⑦ $\dfrac{d\gamma(t)}{dt} = F(\gamma(t)),\quad A(t) = D_x F(\gamma(t))$

$$\underline{\mathrm{tr}\,A(t)} = \mathrm{tr}\,D_x F(\gamma(t)) = \sum_i \partial_{x_i} F(\gamma(t))$$

$$= \underline{\mathrm{div}\,F(\gamma(t))}$$

$$\boxed{\det \Phi(T) = e^{\int_0^T \mathrm{div}\,F(\gamma(r))\,dr}}$$

⑧ $\underline{n=2}$

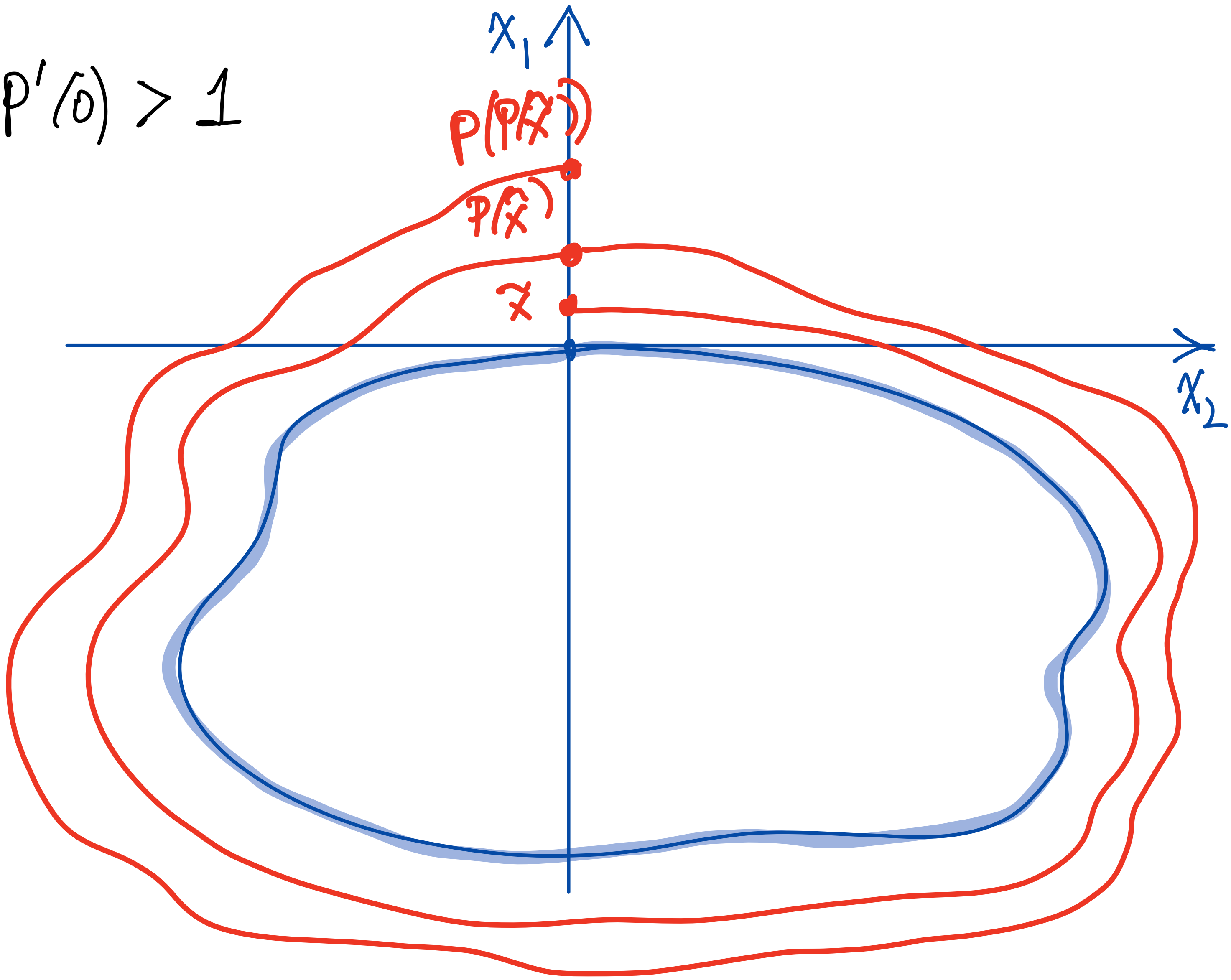
$$\det M = \det P'(0)$$

$$\boxed{\det P'(0) = e^{\int_0^T \mathrm{div}\,F(\gamma(t))\,dt}}$$

$P'(0)$

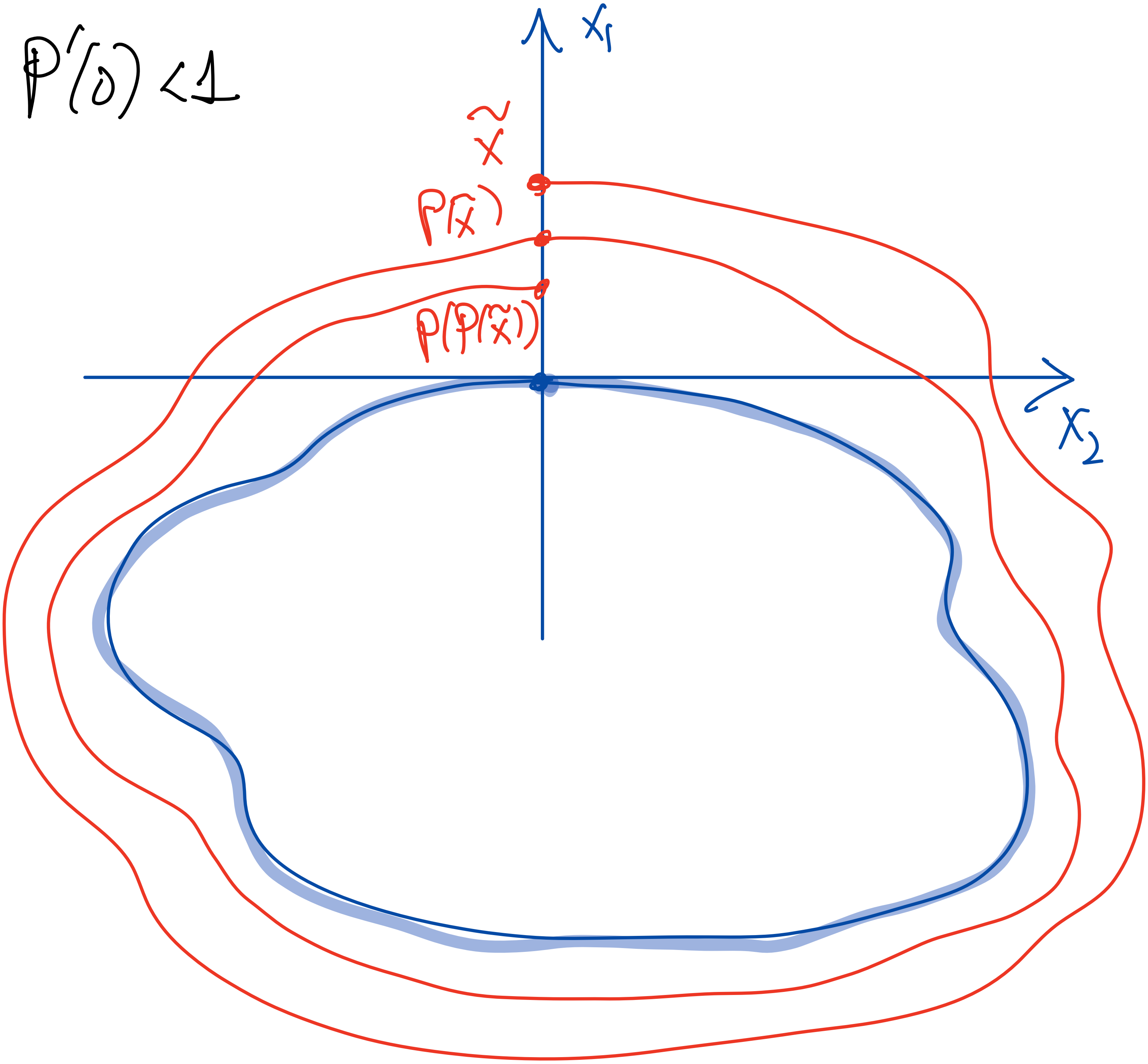
$$P'(0)\begin{Bmatrix} >1 \\ =1 \\ <1 \end{Bmatrix} \iff \int_0^T \mathrm{div}\,F(\gamma(t))\,dt \begin{Bmatrix} >0 \\ =0 \\ <0 \end{Bmatrix}$$

$P'(0) > 1$



$$\tilde{x} < P(\tilde{x}) < P^2(\tilde{x}) < P^3(\tilde{x}) < \cdots\cdots$$

$P'(0) < 1$



$$\tilde{x} > P(\tilde{x}) > P^2(\tilde{x}) > P^3(\tilde{x}) > \cdots$$

Examples

(1) $$\dot{x} = \underline{-y + x}(1 - x^2 - y^2)$$
$$\dot{y} = \underline{x + y}(1 - x^2 - y^2)$$

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{A,\ \lambda_1, \lambda_2 = 1 \pm i}\begin{pmatrix} x \\ y \end{pmatrix} - \underbrace{\begin{pmatrix} x(x^2+y^2) \\ y(x^2+y^2) \end{pmatrix}}_{O(x^3+y^3)}$$

(origin non-hyperbolic, unstable, $Re(\lambda_1, \lambda_2) > 0$, but other solutions?)

$$\boxed{\begin{aligned} \dot{x} &= -y + x\,h(r) \\ \dot{y} &= x + y\,h(r) \end{aligned} \qquad r = \sqrt{x^2+y^2}}$$

Use polar coord. $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\boxed{\begin{aligned} \dot{r} &= r\,h(r) \\ \dot{\theta} &= 1 \end{aligned}}$$

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{\sqrt{x^2+y^2}}$$

$$= \frac{x(\cancel{-y} + x(1-x^2-y^2)) + y(\cancel{x} + y(1-x^2-y^2))}{\sqrt{x^2+y^2}}$$

$$= \frac{x^2+y^2 - x^2(x^2+y^2) - y^2(x^2+y^2)}{\sqrt{x^2+y^2}}$$

$$\underline{\dot{r} = r(1-r^2)}$$

$$\dot{\theta} = \frac{1}{1+\frac{y^2}{x^2}}\left[\frac{x\dot{y} - y\dot{x}}{x^2}\right]$$

$$= \frac{x\dot{y} - y\dot{x}}{x^2+y^2}$$

$$= \frac{x(x + y(1-x^2-y^2)) - y(-y + x(1-x^2-y^2))}{x^2+y^2}$$

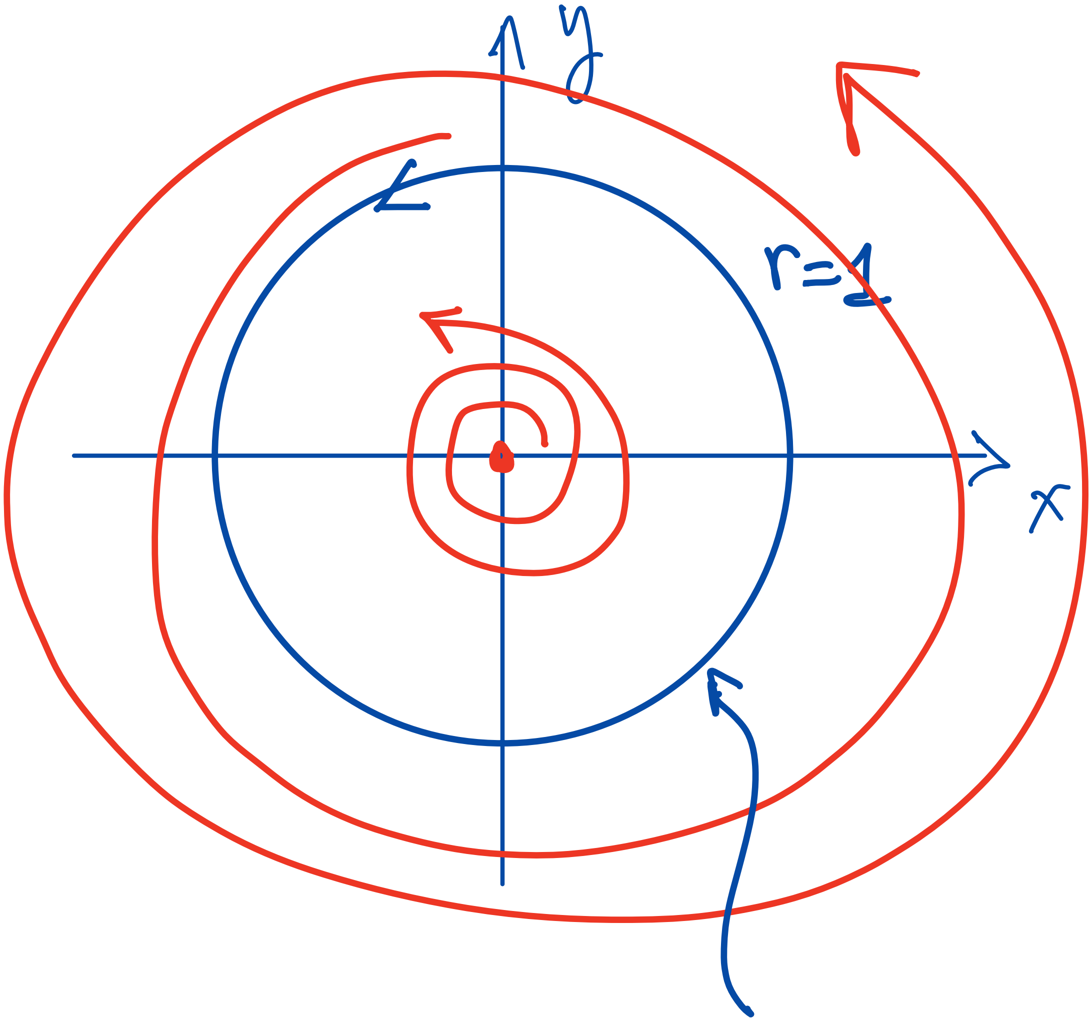
$$= \frac{x^2+y^2}{x^2+y^2}$$

$$\underline{\dot{\theta} = 1}$$

$$\dot{r} = r(1-r^2)$$

2π-per. solution
$x = \cos t$
$y = \sin t$

$r=0$, unstable $r=1$, stable



Asymptotically stable 2π-per. solution

$(1)_2$ $\quad F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y + x(1-x^2-y^2) \\ x + y(1-x^2-y^2) \end{pmatrix}$

$$DF\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1-x^2-y^2-2x^2 & -1-2xy \\ 1-2xy & 1-x^2-y^2-2y^2 \end{pmatrix}$$

$x = \cos t,\ y = \sin t$ (per. solution)

$$= \underbrace{\begin{pmatrix} -2c^2 & -1-2sc \\ 1-2sc & -2s^2 \end{pmatrix}}_{A(t)}$$

$$\operatorname{tr} A(t) = -2$$

$$e^{\int_0^{2\pi} \operatorname{tr} A(t)\, dt} = e^{-4\pi} < 1 \Rightarrow \text{asymp. stable}$$

$$[\Phi(2\pi)] = [M] = \left[\begin{array}{c|c} p'(0) & 0 \\ \hline \ast & 1 \end{array}\right] = \begin{bmatrix} e^{-4\pi} & 0 \\ \ast & 1 \end{bmatrix}$$

(2) [T, p. 322]

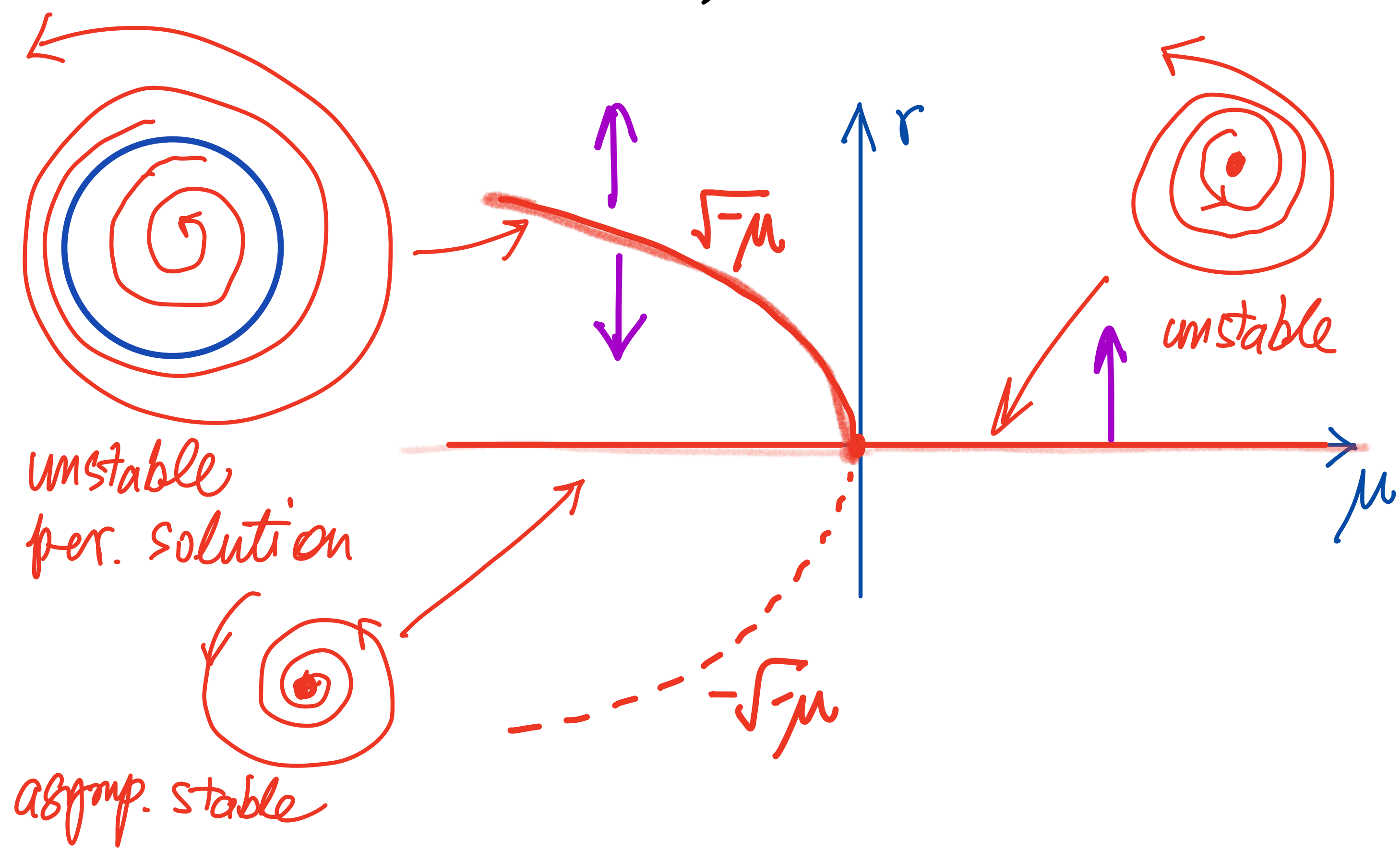
$$\dot{x} = -y + x(\mu + \sigma(x^2+y^2))$$
$$\dot{y} = x + y(\underbrace{\mu + \sigma(x^2+y^2)}_{h(r) = \mu + \sigma r^2})$$

$$\dot{r} = r\,h(r) = r(\mu + \sigma r^2)$$
$$\dot{\theta} = 1$$

$\underline{\sigma = 1,\ \mu > 0}$ $\quad \dot{r} = r(\mu + r^2) = r(r - \sqrt{-\mu})(r + \sqrt{-\mu})$

$r = 0,\ \sqrt{-\mu}$

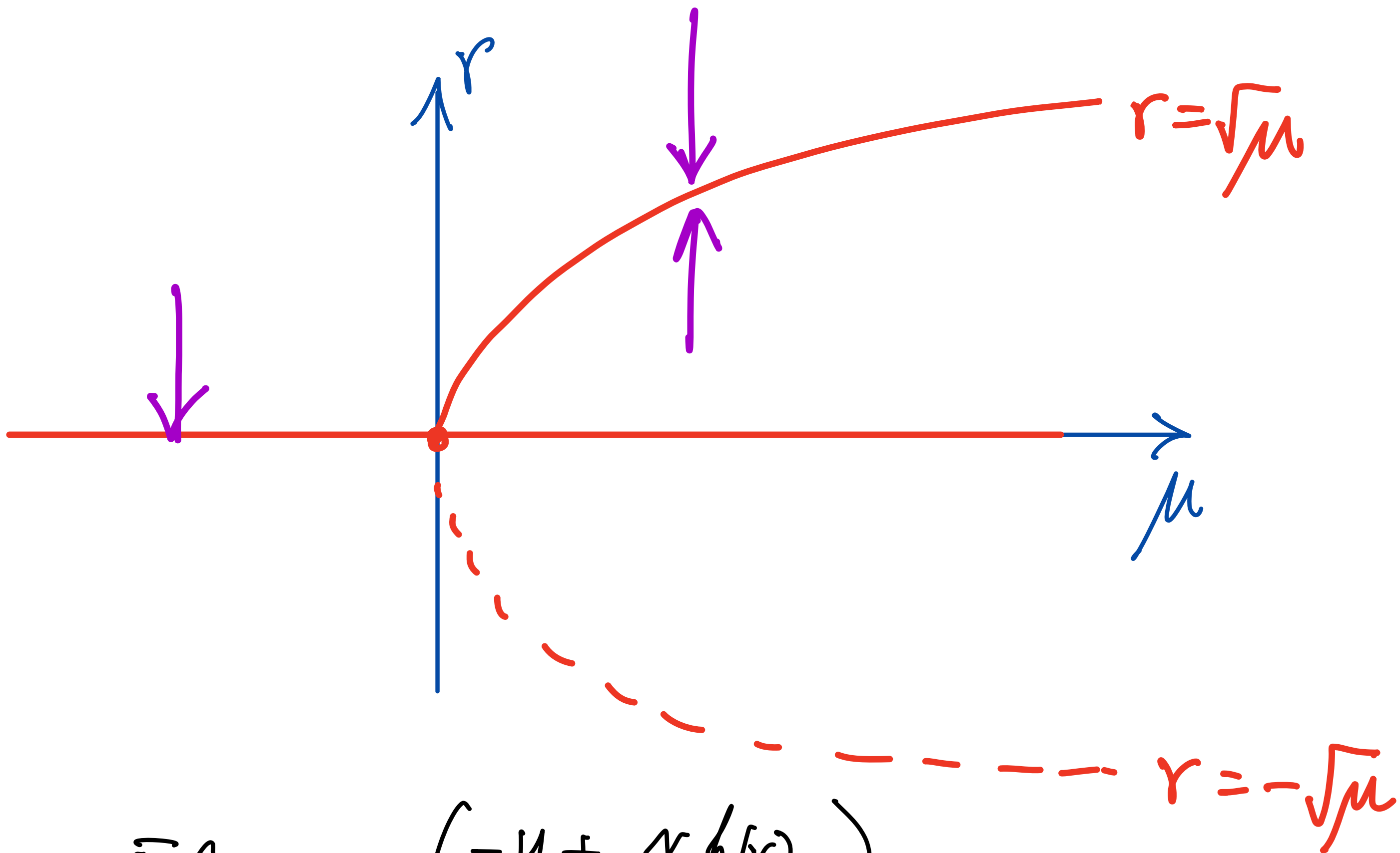
$\underline{\sigma = 1,\ \mu > 0}$ $\quad \dot{r} = r(\mu + r^2) > 0$



$\sigma = -1,\ \mu > 0$ $\qquad \dot r = r(\mu - r^2) = r(\sqrt{\mu} - r)(\sqrt{\mu} + r)$

$r = 0,\ \sqrt{\mu}$

$\sigma = -1,\ \mu < 0,$ $\qquad \dot r = r(\mu - r^2) < 0$



$$F(x,y) = \begin{pmatrix} -y + x\,h(r) \\ x + y\,h(r) \end{pmatrix}$$

$$\operatorname{div} F = h(r) + x\,h'(r)\frac{x}{r} + h(r) + y\,h'(r)\frac{y}{r}$$

$$= 2h(r) + r\,h'(r) \qquad h(r) = \mu + \sigma r^2$$

$$= 2\mu + 2\sigma r^2 + r\,2\sigma r = 2\mu + 4\sigma r^2$$

$$= 2(\mu + 2\sigma r^2)$$

$\sigma = 1, \ \mu < 0 \Rightarrow r = \sqrt{-\mu}$

$$\operatorname{div} F = 2(\mu + 2(-\mu)) = -2\mu > 0$$

unstable

$\sigma = -1, \ \mu > 0 \Rightarrow r = \sqrt{\mu}$

$$\operatorname{div} F = 2(\mu - 2\mu) = -2\mu < 0$$

stable

(3) Example from ecology (population growth)

Lotka–Volterra (Predator–Prey)

$$\dot{x} = (A - By)x \quad \text{— prey (food)}$$
$$\dot{y} = (Cx - D)y \quad \text{— predator}$$

$A, B, C, D = 1$

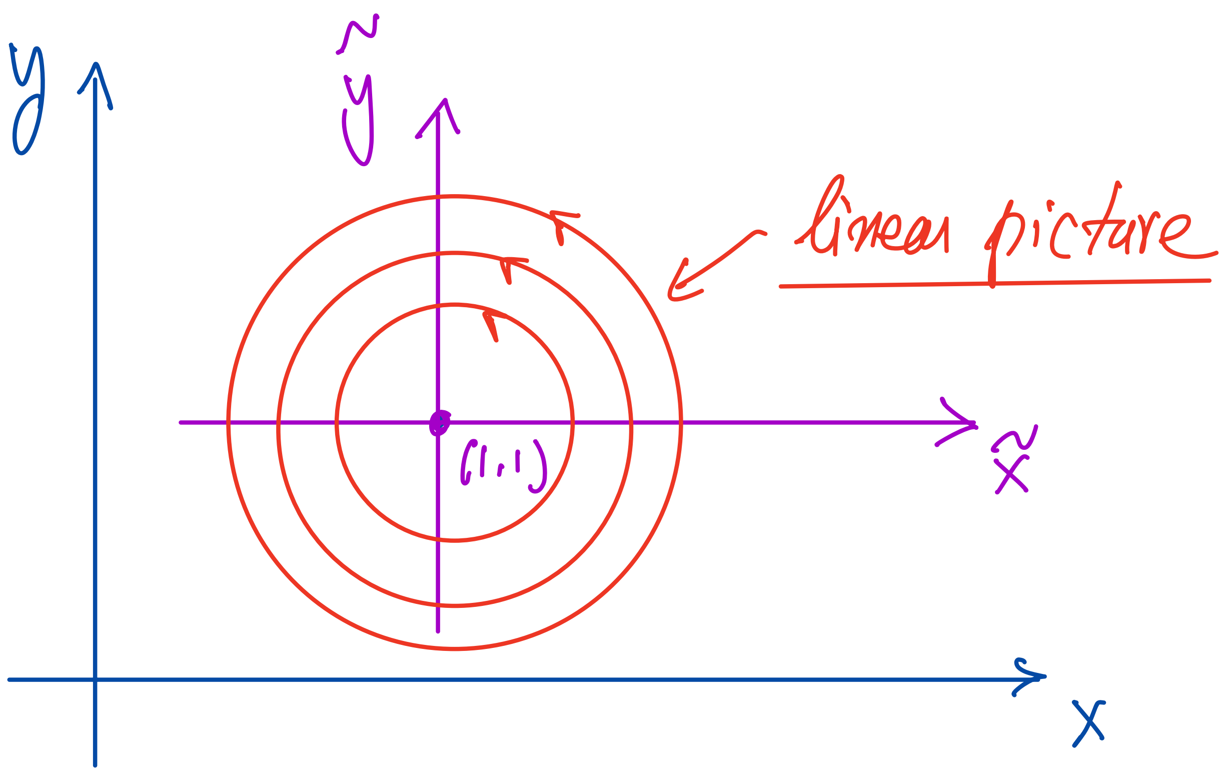
what type?

$\Rightarrow$ eq. pt $x = 1, \ y = 1$

$$\tilde{x} = x-1, \quad \tilde{y} = y-1$$

$$\begin{cases} \dot{x} = (1-y)\,x \\ \dot{y} = (x-1)\,y \end{cases} \iff \begin{cases} \dot{\tilde{x}} = -\tilde{y}\,(\tilde{x}+1) \\ \dot{\tilde{y}} = \tilde{x}\,(\tilde{y}+1) \end{cases}$$

$$\frac{d}{dt}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\lambda_1, \lambda_2 = \pm i, \ \text{center}} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} -\tilde{x}\tilde{y} \\ +\tilde{x}\tilde{y} \end{pmatrix}$$



Nonlinear picture?

$$\dot{x} = (1-y)\,x \qquad\qquad \frac{dx}{dy} = \frac{(1-y)\,x}{(x-1)\,y}$$

$$\dot{y} = (x-1)\,y$$

$$\frac{x-1}{x}\,dx = -\left(\frac{y-1}{y}\right)dy$$

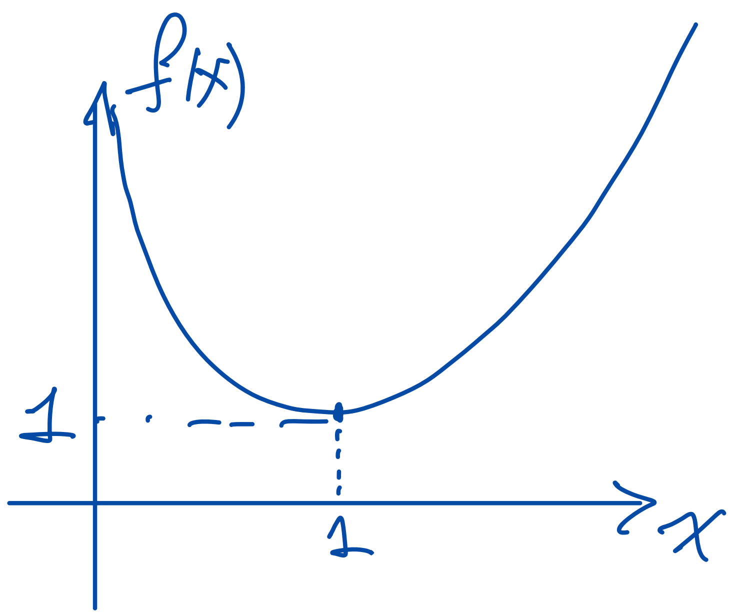
$$\int \left(1 - \frac{1}{x}\right) dx = \int -\left(1 - \frac{1}{y}\right) dy$$

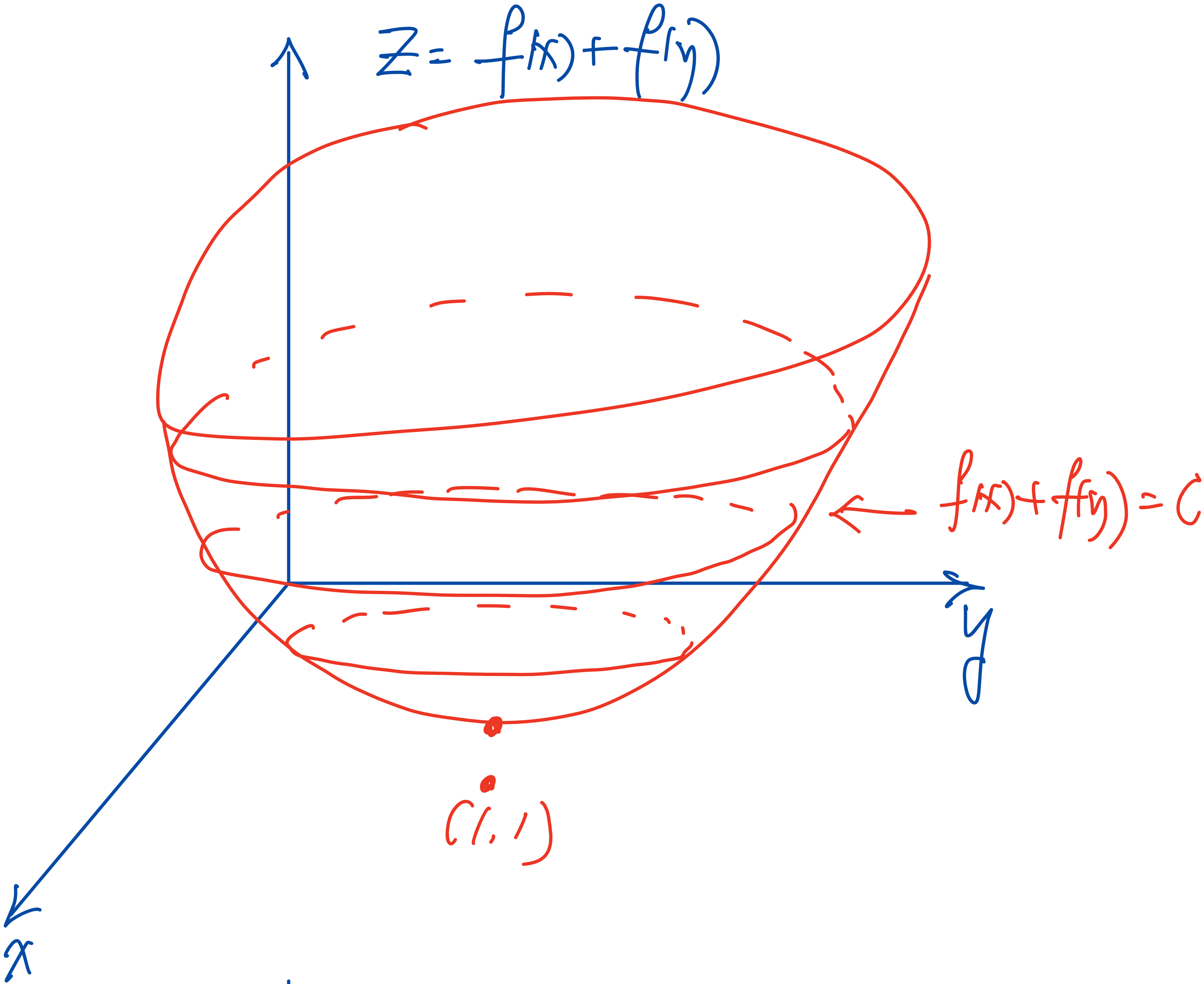
$$x - \log x = -(y - \log y) + C$$

$$\underbrace{(x - \log x)} + (y + \log y) = C$$

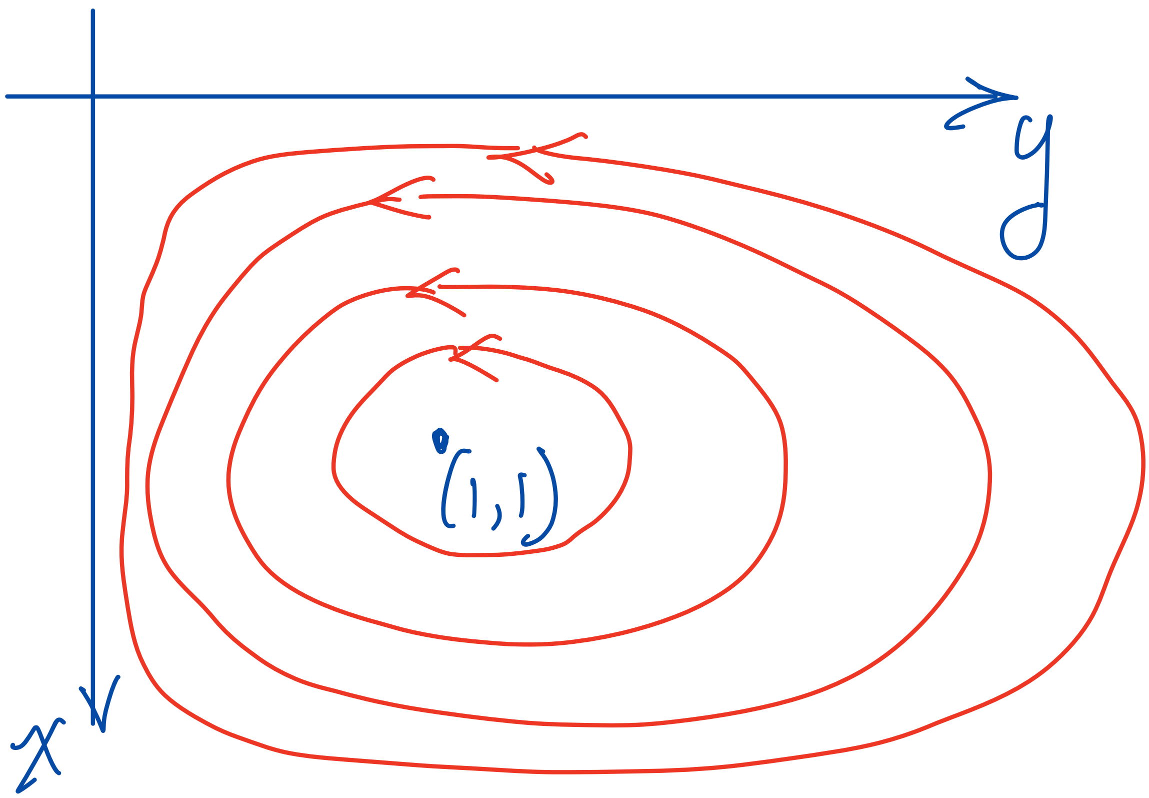
$$f(x) = x - \log x$$

$$> 0$$





Z = f(x) + f(y)
f(x) + f(y) = C
y
(1, 1)
x



y
(1, 1)
x

(4) Hamiltonian System

$$H : (X, Y) \in \mathbb{R}^{2n} \longrightarrow H(X, Y) \in \mathbb{R}$$

$$\left.\begin{aligned} \dot{X} &= \nabla_Y H(X, Y) \\ \dot{Y} &= -\nabla_X H(X, Y) \end{aligned}\right\} F(X, Y)$$

$$\frac{d}{dt} H(X(t), Y(t)) = 0$$

$$H(X(t), Y(t)) = \text{Const.}$$

$$\operatorname{div}_{(X, Y)} F(X, Y)$$

$$= \partial_{x_i} \partial_{y_i} H - \partial_{y_i} \partial_{x_i} H$$

$$= 0$$

$n = 1$

$$\dot{x} = \partial_y H(x, y)$$

$$\dot{y} = -\partial_x H(x, y)$$

$$\operatorname{div}_{(x, y)} F = \partial_x \partial_y H - \partial_y \partial_x H = 0$$

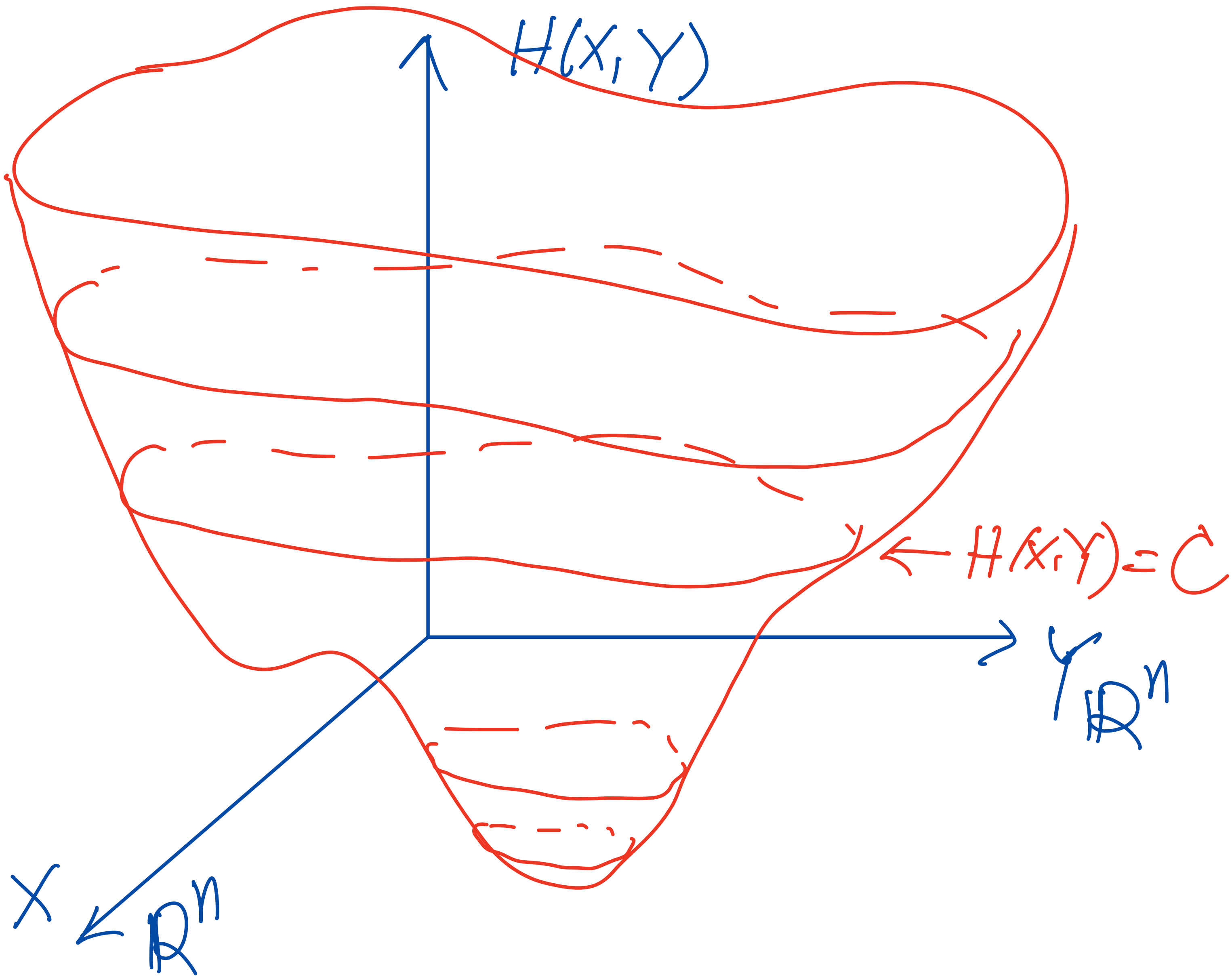
Hamiltonian System is extremely degenerate!

"Usually":

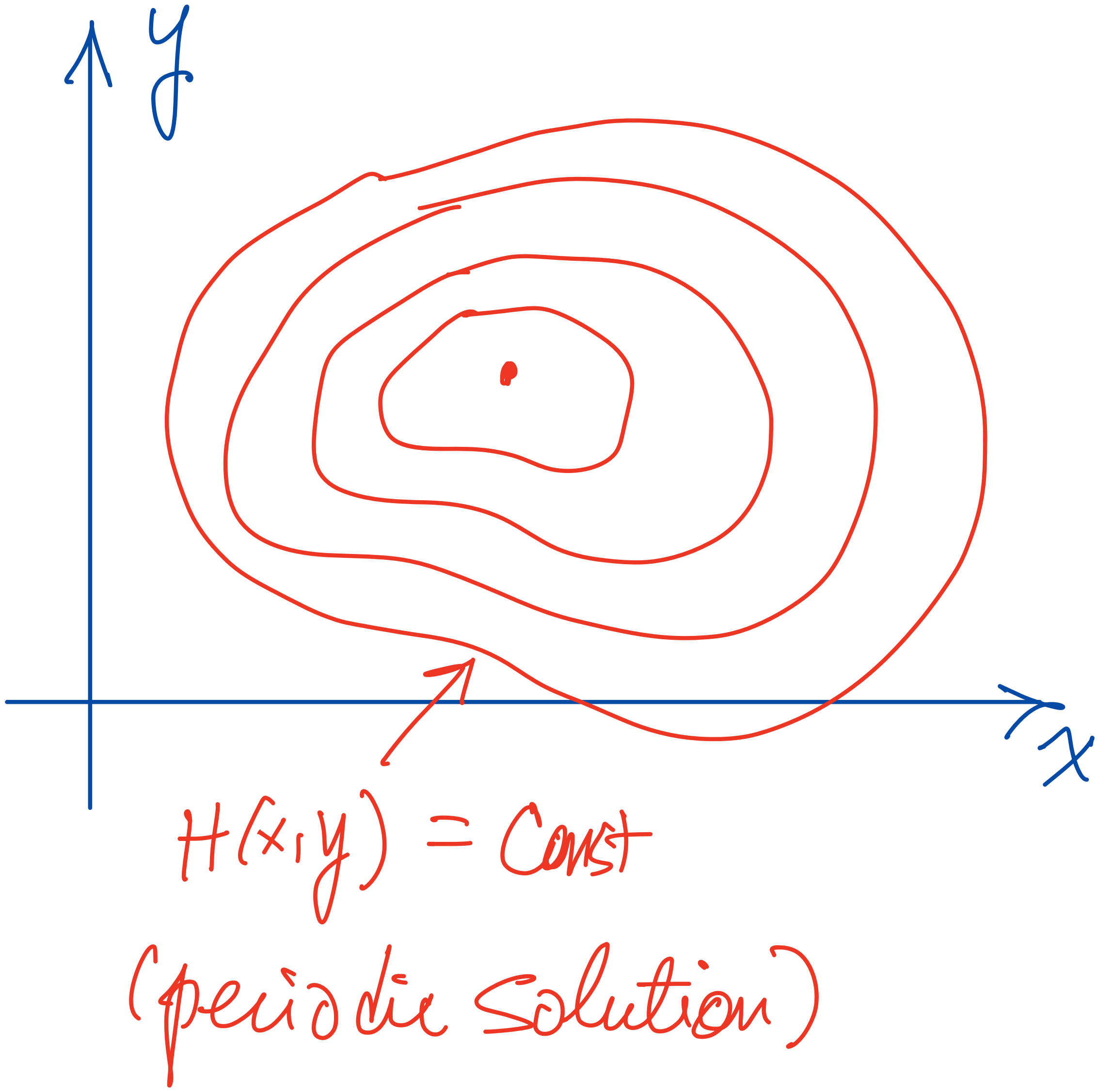
$$H(X,Y) = P.E. + K.E$$

$$= V(X) + \frac{1}{2m}|Y|^2$$

$+\infty$ as $|X| \to \infty$ $\qquad$ $+\infty$ as $|Y| \to \infty$



$n=1$. one degree of freedom



Persistence of Periodic Orbit

$$\frac{dX}{dt} = F(X, \mu) \qquad (*)$$

Suppose $(*)$ has a periodic solution $\gamma_0(t)$ at $\mu = 0$

A sufficient condition for the existence of periodic solution $\gamma_\mu(t)$ for $0 < |\mu| \ll 1$ is

$$\boxed{\mu = 1 \text{ is a simple eigenvalue for } \Phi(T)}$$

[H, Thm 2.4]

where $\Phi(T)$ is the monodromy matrix for the linearized system at $\gamma_0(t)$:

$$\boxed{\frac{dZ}{dt} = A(t)\, Z \qquad A(t) = D_X F(\gamma_0(t), 0)}$$

But this is not true for Hamiltonian system.

(1) Consider $n=1$ (Hamiltonian system with 1 deg. of freedom)

$$\begin{cases} \dot{x} = \partial_y H(x,y) \\ \dot{y} = -\partial_x H(x,y) \end{cases}$$ ← suppose it has a periodic solution $\gamma(t)$

$$\overset{2\times 2}{\Phi(T)} = \overset{2\times 2}{M} = \left[\begin{array}{c|c} P'(0) & 0 \\ \hline \cancel{\quad} & 1 \end{array}\right]$$

from time shift

$$P'(0) = e^{\int_0^T \operatorname{div} F(\gamma(t))dt}$$

$$F(x,y) = (\partial_y H(x,y), -\partial_x H(x,y))$$

$$\operatorname{div} F = \partial_x \partial_y H - \partial_y \partial_x H = 0$$

Hence $P'(0) = e^0 = 1$

i.e. $\mu = 1$ is a repeated eigenvalue of M

[H, Thm 2.4] cannot be "easily" utilized

In fact $\gamma(t)$ can be easily "destroyed."

$$\dot{x} = \partial_y H(x,y) + \varepsilon g_1(x,y)$$
$$\dot{y} = -\partial_x H(x,y) + \varepsilon g_2(x,y) \qquad \varepsilon \ll 1$$

Consider $\frac{d}{dt} H(x(t), y(t))$

$$= (\partial_x H)\,\dot{x} + (\partial_y H)\,\dot{y}$$
$$= (\partial_x H)(\partial_y H + \varepsilon g_1) + (\partial_y H)(-\partial_x H + \varepsilon g_2)$$
$$= \varepsilon\Big[\underbrace{(\partial_x H) g_1 + (\partial_y H) g_2}_{\text{choose } g_1, g_2 \text{ s.t. it is } < 0}\Big]$$

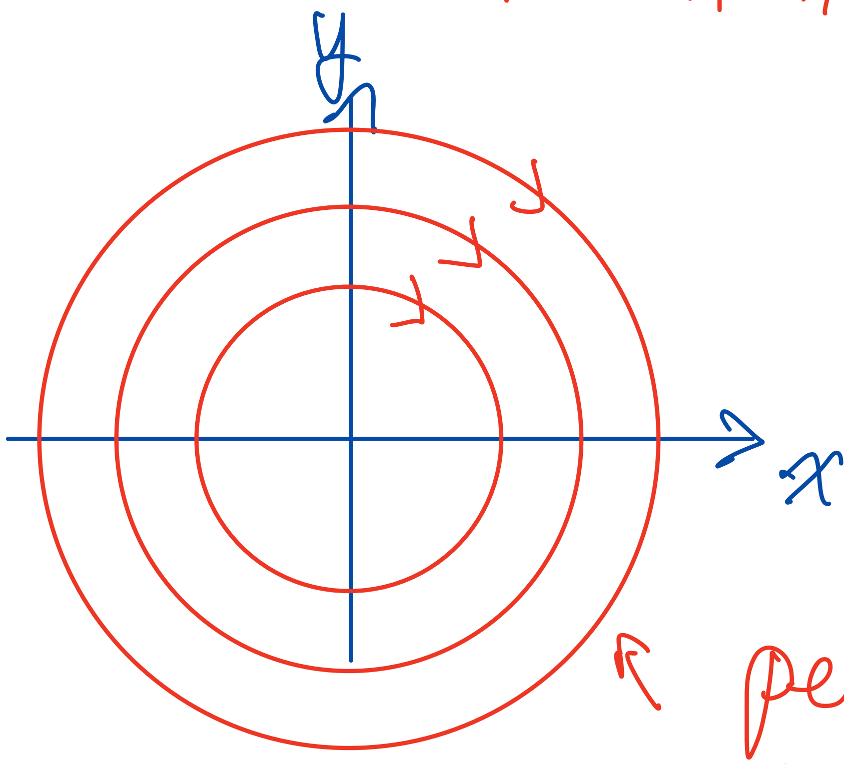
eg $g_1 \equiv 0$, $g_2 \equiv -\partial_y H$

$$\dot{x} = \partial_y H(x,y)$$
$$\dot{y} = -\partial_x H(x,y) \underbrace{- \nu \dot{x}}_{\text{friction}}$$

$-\nu\,\partial_y H(x,y)$

② $\frac{dX}{dt} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\text{Harmonic oscillator}} X = \begin{pmatrix} y \\ -x \end{pmatrix}$

$\lambda = \pm i$, non-hyperbolic



periodic solutions

"degenerate"

$$F(x,y) = (y, -x)$$

$$\operatorname{div} F = \partial_x y + \partial_y(-x) = 0 = P'(0)$$