Poincare-Bendixson Thm

Theorem 6.32 (Poincaré–Bendixson). *Let D be a connected subset of $\mathbb{R}^2$ and φ be a flow on D. Suppose that the forward orbit of some $p \in D$ is contained in a compact set and that $\omega(p)$ contains no equilibria. Then $\omega(p)$ is a periodic orbit.*

$$\boxed{\frac{dX}{dt} = F(X)} \qquad (*)$$

(1) $X \in \mathbb{R}^2$ (A two dimensional result)

(2) Forward orbit $\Gamma^+(p) = \{\phi_t(p)\}_{t \geq 0} \subseteq$ Compact Set

(closed and bounded in $\mathbb{R}^n$)

($\Longrightarrow$ limit pts $z = \lim_{t_i \to +\infty} \phi_{t_i}(p)$ exists)

(3) $\omega(p) = \{$ all limit pts of $\{\phi_t(p)\}_{t \geq 0} \}$
(Omega limit set)

$F(z) \neq 0$ for $z \in \omega(p)$

Conclusion: $\omega(p) = \gamma$

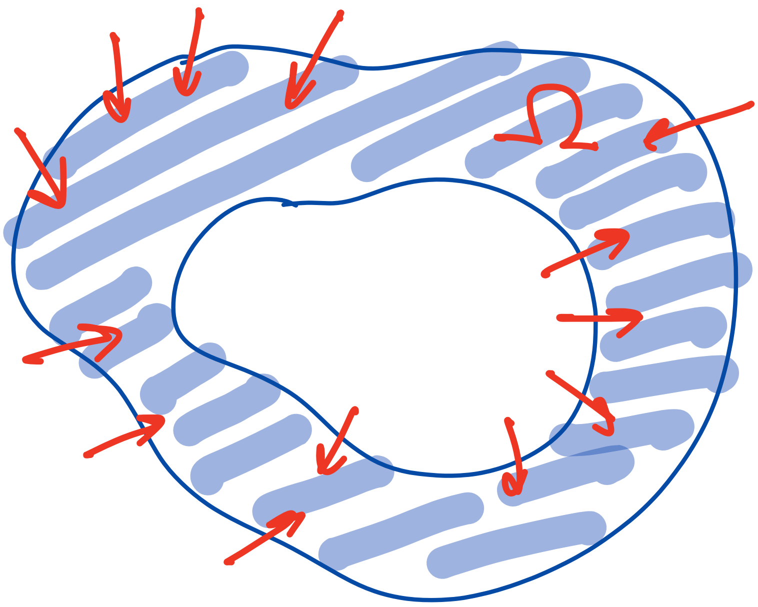
where γ is a periodic orbit of $(*)$: $\gamma(t+T) = \gamma(t)$

Examples

Find an invariant set (1) Ω which resembles an annulus and contains no equilibrium pt. (2)

(1) $x_0 \in \Omega \implies \phi_t(x_0) \in \Omega$ for $t \geq 0$

(2) $z \in \Omega \implies F(z) \neq 0$



$h(r) = 1 - r^2$

Ex 1

$$\begin{cases} \dot{x} = -y + x(1 - x^2 - y^2) \\ \dot{y} = x + y(1 - x^2 - y^2) \end{cases}$$

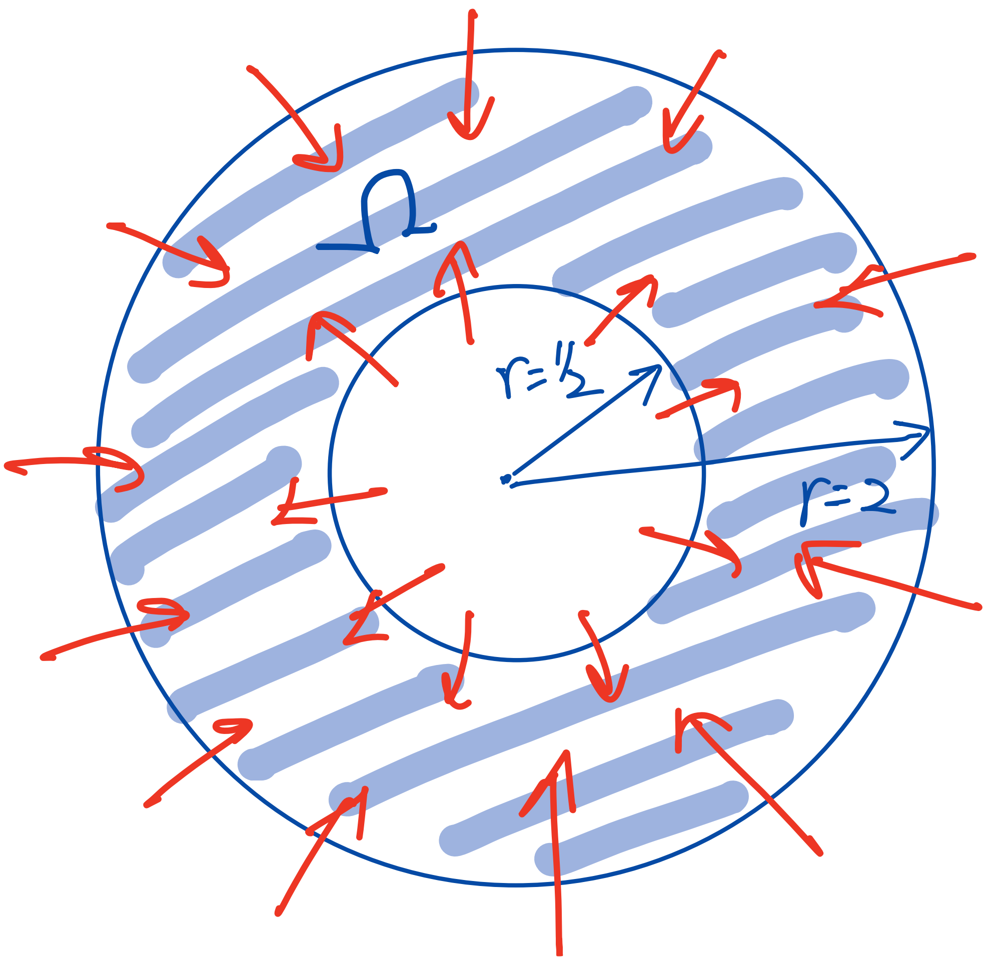
$$\left(\begin{array}{l} \dot{x} = -y + x h(r) \\ \dot{y} = x + y h(r), \end{array} \quad r = \sqrt{x^2 + y^2} \right)$$

$x = r\cos\theta,$
$y = r\sin\theta$ $\implies \dot{r} = r h(r), \quad \dot{\theta} = 1$

$$\dot{r} = r(1-r^2) \qquad (\dot{\theta} = 1)$$

① 0 is the only equil. pt.

② $\dot{r} \begin{cases} > 0 & \text{at } r = 1/2 \\ < 0 & \text{at } r = 2 \end{cases}$



Ex 2

Example 6.39. Let

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + y(1 - x^2 - 2y^2). \end{aligned} \tag{6.35}$$

The only equilibrium is the origin. The rate of change of the polar radius for (6.35) is

$$\dot{r} = \frac{y^2}{r}(1 - x^2 - 2y^2).$$

When $r^2 = 1$, then $\dot{r} = -y^4/r \leq 0$, and when $r^2 = 1/2$, then $\dot{r} = y^2x^2/r \geq 0$. This implies that the annulus $R = \{(x, y) : 2^{-1/2} \leq r \leq 1\}$ is positively invariant—note that even though $\dot{r} = 0$ at some points on the boundary of R, orbits cannot leave the annulus. We conclude there is at least one limit cycle in R. A numerical solution confirms our analysis; see Figure 6.16. ∎

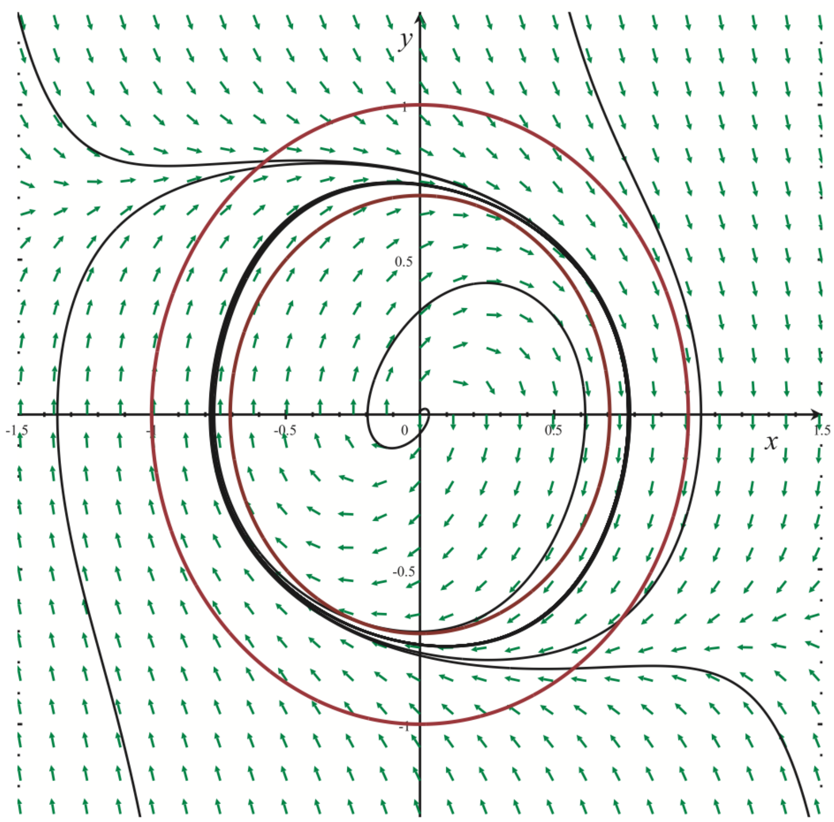


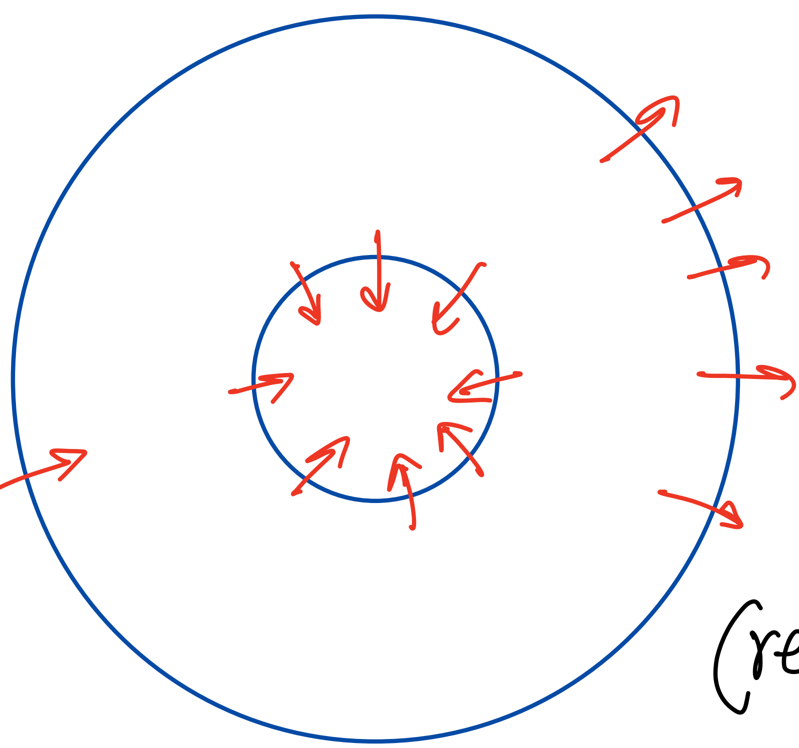
Figure 6.16. *Phase portrait of* (6.35) *showing the limit cycle and the boundaries of* R.

Ex3 [Verhulst, p.48]

$$\dot{x} = -y + x(x^2+y^2-2x-3)$$
$$\dot{y} = x + y\underbrace{(x^2+y^2-2x-3)}_{h(x,y)}$$

$$\dot{r} = r\,h(x,y) = r\underbrace{(r^2-2r\cos\theta-3)}$$
$$\dot{\theta} = 1$$

$$\begin{cases} r^2-2r\cos\theta-3 < 0 & \text{if } r \ll 1 \quad \text{eg. } r<1 \\ r^2-2r\cos\theta-3 = r(r-2\cos\theta)-3 > 0 & \text{if } r>3 \end{cases}$$



no equil. pt.

(reverse time)

Ex 4 [Glendinning, p.136]

$$\dot{x} = y + \frac{1}{4}x\underline{(1-2r^2)}$$

$$\dot{y} = -x + \frac{1}{2}y\underline{(1-r^2)}$$

2 diff. fcts

$$\dot{r} = \underbrace{\frac{1}{4}r(1+\sin^2\theta) - \frac{1}{2}r^3}_{\begin{cases} >0 & \text{if } r \ll 1 \\ <0 & \text{if } r \gg 1 \end{cases}}$$

$$\frac{1}{4}r\left(1+\sin^2\theta - 2r^2\right)$$

($\Rightarrow$ no equil. pt. in $\{a < r < b\}$ for a small, b big)

In general, $\dot{x} = -y + x\,h(x,y)$
$\dot{y} = x + y\,g(x,y)$

$r^2 = x^2 + y^2, \quad \theta = \tan^{-1}\frac{y}{x}$

$$r\dot{r} = x\dot{x} + y\dot{y}$$
$$= x[-y + xh] + y[x + yg]$$
$$= x^2 h + y^2 g$$

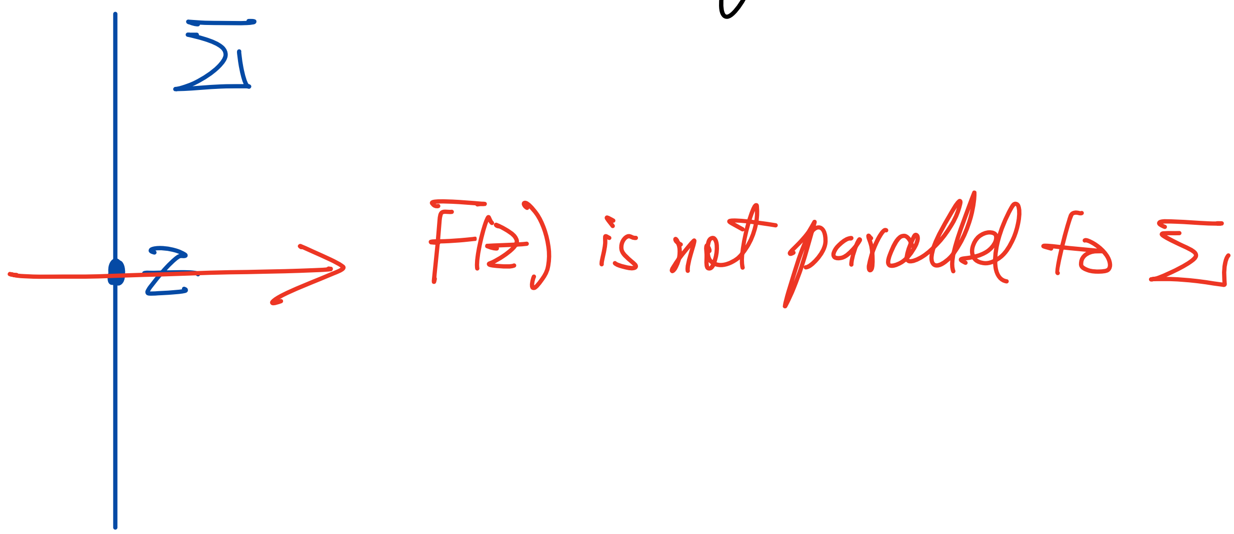
$$\dot{r} = \frac{x^2 h(x,y) + y^2 g(x,y)}{r^2} \quad \left(\begin{array}{l} = h(x,y) \\ \text{if } h = g \end{array} \right)$$

$$\dot{\theta} = \frac{1}{1 + \frac{y^2}{x^2}} \left[\frac{x\dot{y} - y\dot{x}}{x^2} \right]$$
$$= \frac{1}{r^2}\left[x(x + yg) - y(-y + xh) \right]$$
$$= \frac{1}{r^2}\left[x^2 + y^2 + xy(g - h) \right] \quad (= 1 \text{ if } h = g)$$

Proof of Poincare-Bendixson (Outline)

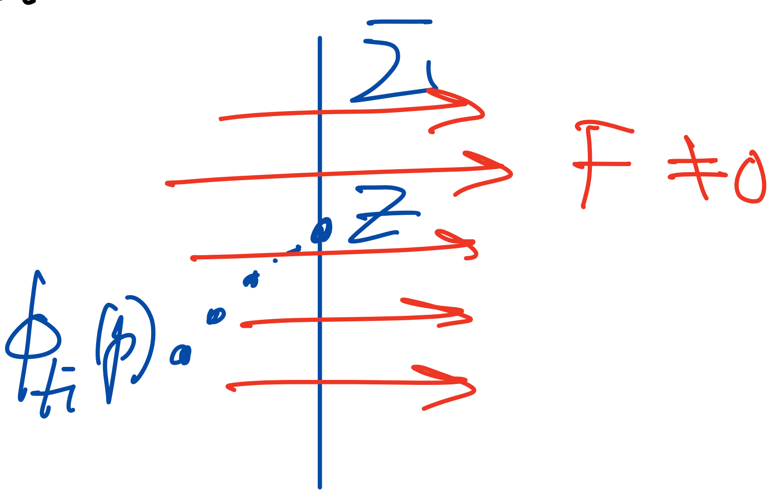
① $p \longrightarrow \omega(p) \neq \phi$

Let $z \in \omega(p)$. Since $F(z) \neq 0$, there exists a tranversal segment Σ at z

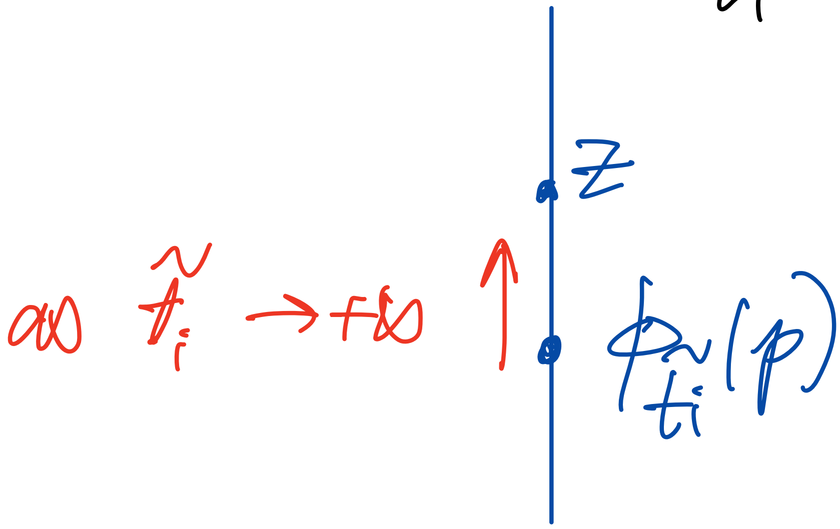


② as $\exists\, t_1, t_2, \cdots \to +\infty$ s.t.

$$\phi_{t_i}(p) \longrightarrow z$$

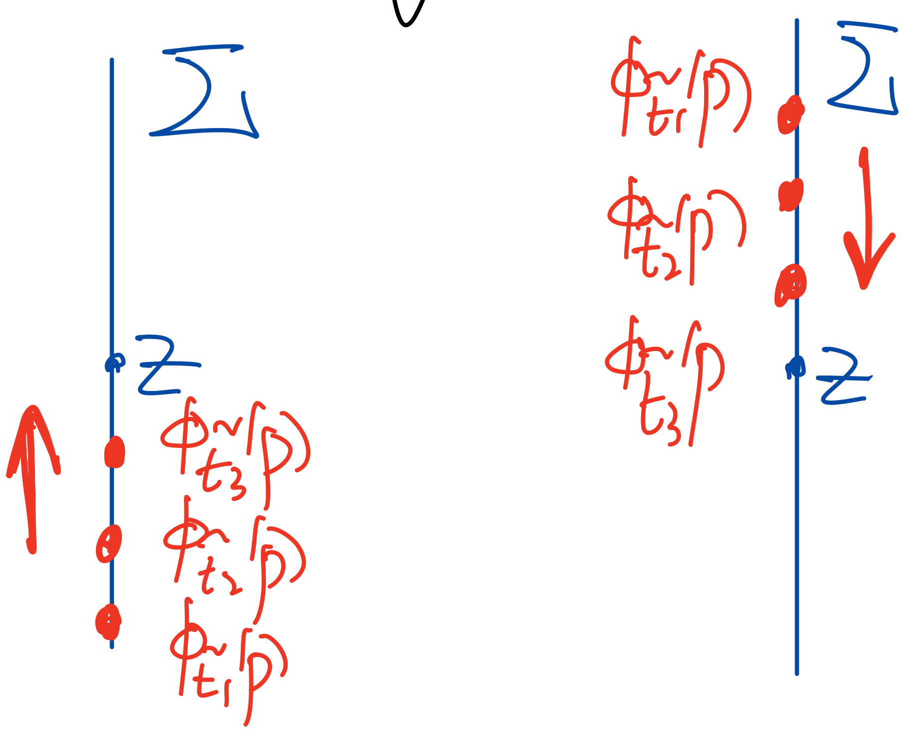


③ $\exists\ \tilde{t}_i \to +\infty$ s.t. $\phi_{\tilde{t}_i}(p)$ lies on Σ

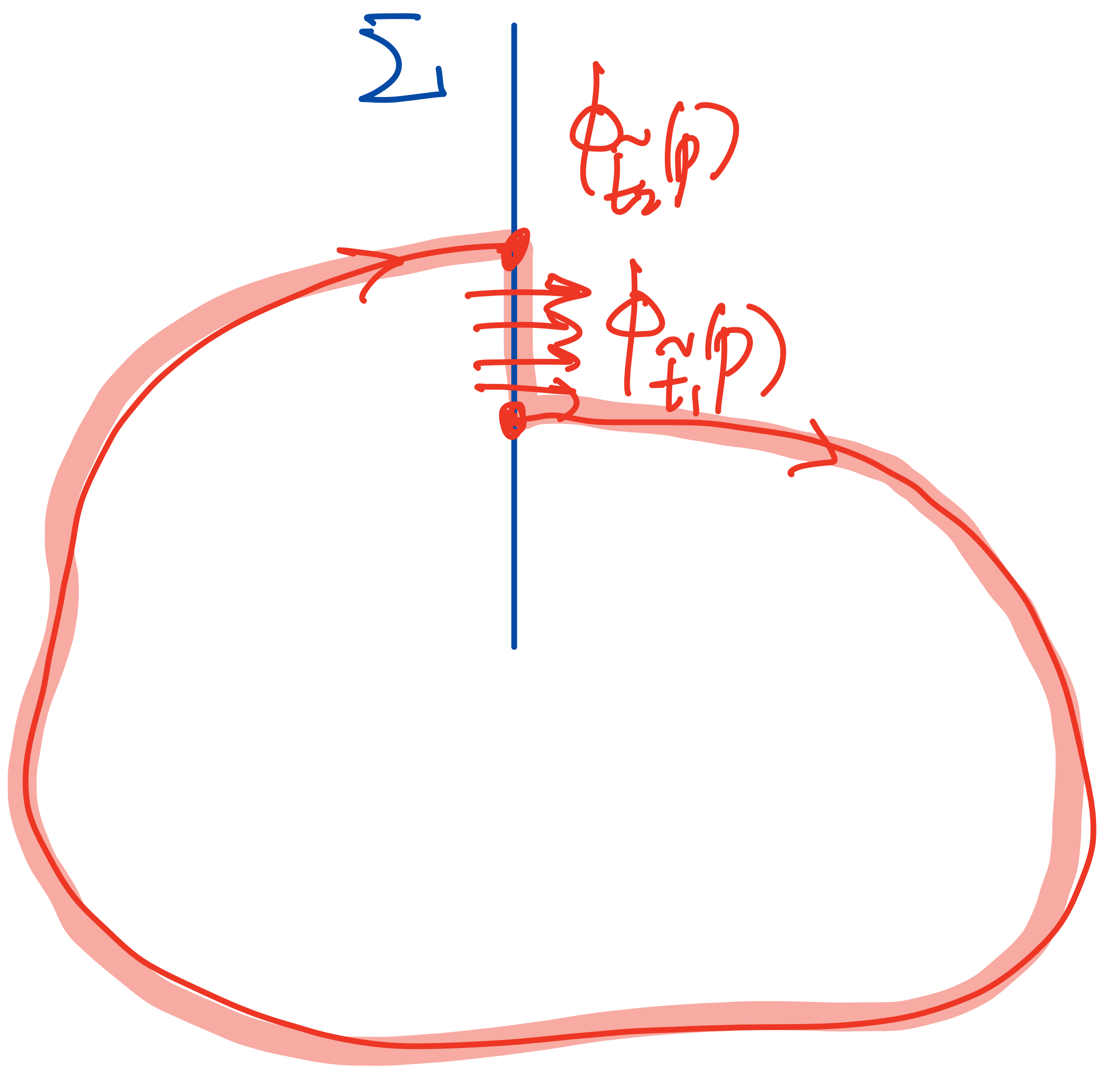


$$\lim_{i\to\infty} \phi_{\tilde{t}_i}(p) = z$$

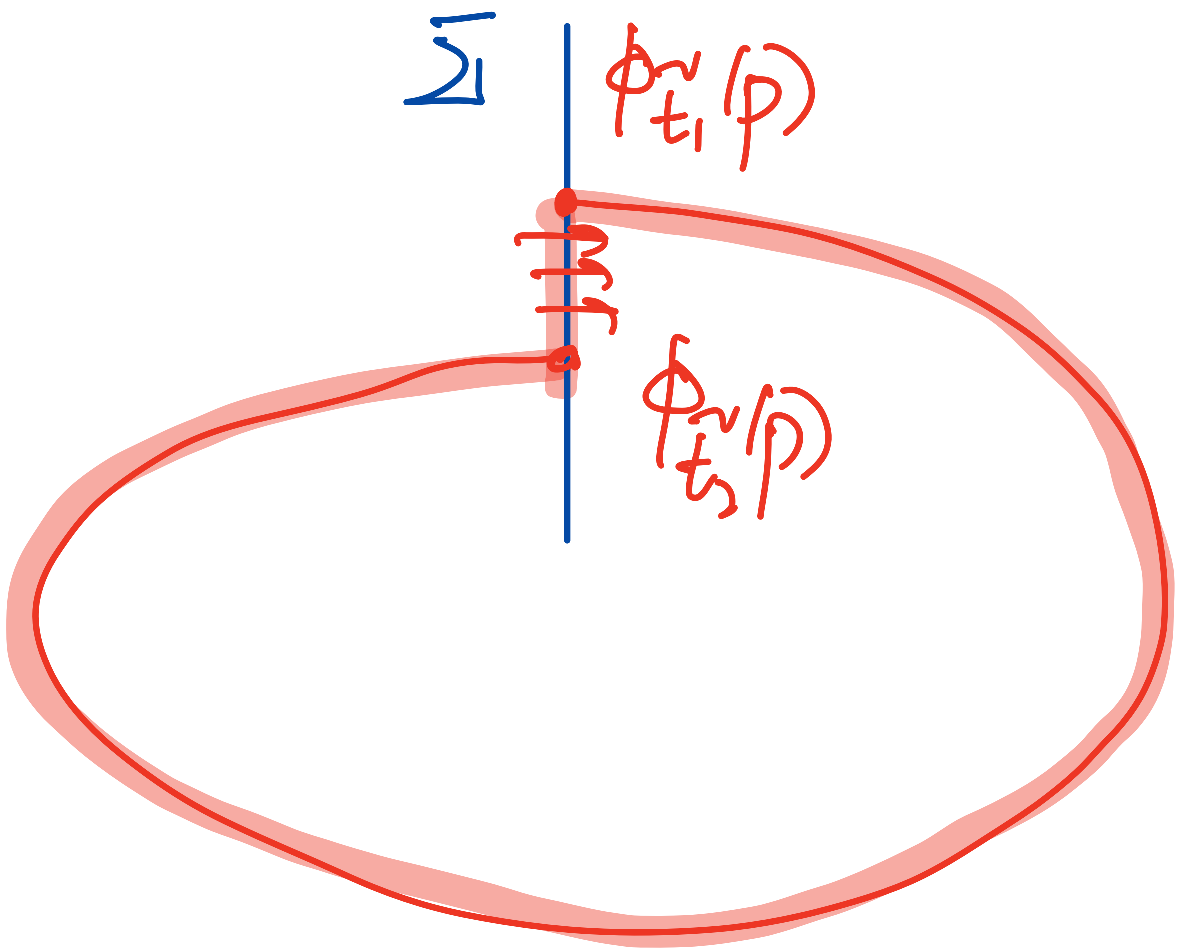
④ $\phi_{\tilde{t}_i}(p)$ converges to z monotonically



Theorem 6.34 (Jordan curve). *A simple closed curve (a set that is homeomorphic to $\mathbb{S}^1$) in $\mathbb{R}^2$ separates the plane into two connected components: one bounded, called the interior, and one unbounded, called the exterior.*

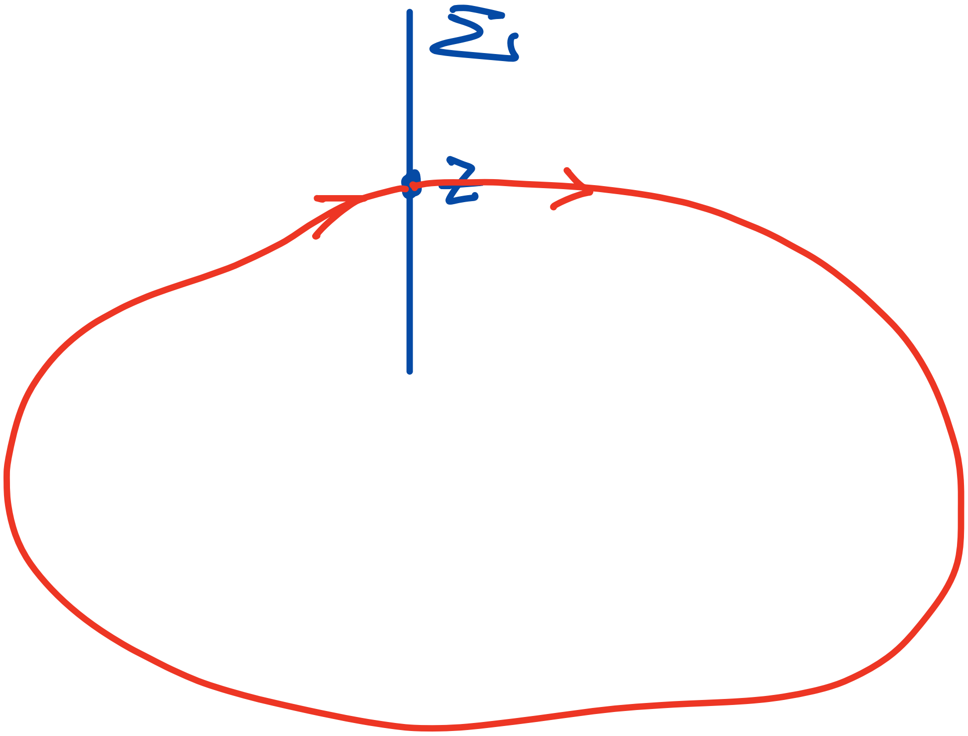


Σ
$\phi_{\tilde{t}_2}(p)$
$\phi_{\tilde{t}_1}(p)$

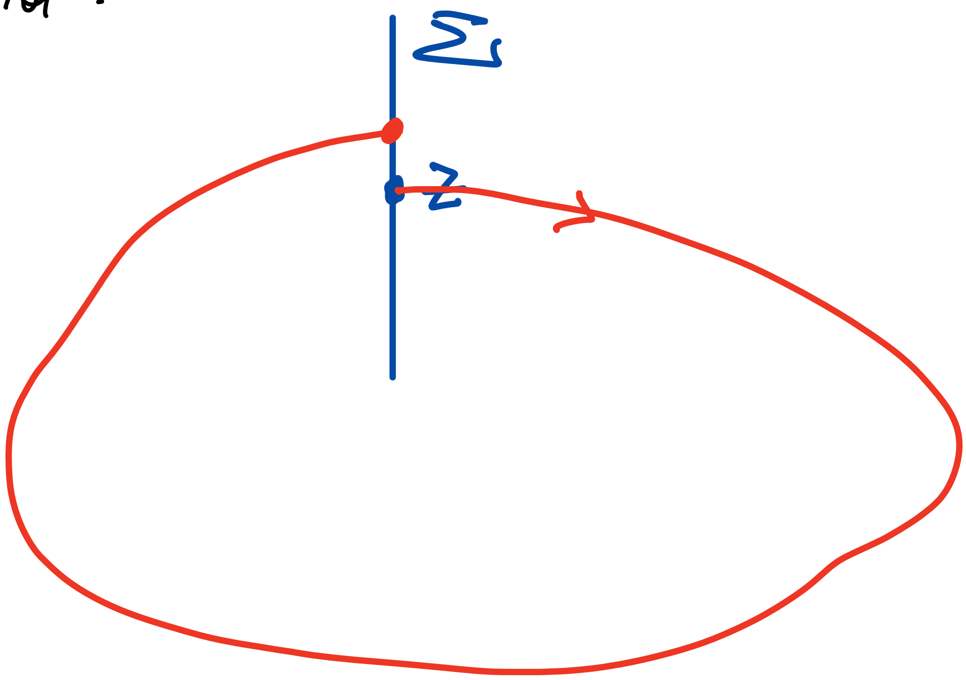


Σ
$\phi_{\tilde{t}_1}(p)$
$\phi_{\tilde{t}_3}(p)$

⑤ $\Gamma^{+}(z) = \{\phi_t(z)\}_{t \geq 0}$ must be a periodic orbit.



If not:



then $z \notin \omega(p)$!!