Lienard System

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

[Meiss, Sec 6.7]
[Perko, Sec 3.8]
[Teschl, Sec 7.2]

Lienard System

$$\ddot{x} + \underbrace{f(x)\,\dot{x}}_{\substack{\| \\ (F(x))^{\cdot}}} + g(x) = 0$$

$$(F(x))^{\cdot} \quad \text{where} \quad F(x) = \int_0^x f(s)\,ds$$

$$(\underbrace{\dot{x} + F(x)}_{})^{\cdot} + g(x) = 0$$

Set $\quad y = \dot{x} + F(x)$

Lienard System

"Hamiltonian"

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases}$$

$\int g(x)\,dx$

$\parallel$

$G'(x) = g(x)$

Consider $\boxed{H(x,y) = \frac{1}{2}y^2 + G(x)}$

$$\frac{d}{dt}H(x,y) = g(x)\,\dot{x} + y\,\dot{y}$$

$$= -g(x)F(x) \quad \Big\} \gtrless 0\,?$$

Lienard System

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases}$$

$$\begin{pmatrix} F(-x) = -F(x) \\ g(-x) = -g(x) \end{pmatrix}$$

Assume

(1) $F(x)$ and $g(x)$ are odd functions

(2) $g(x) > 0$ (< 0) for $x > 0$ (< 0)

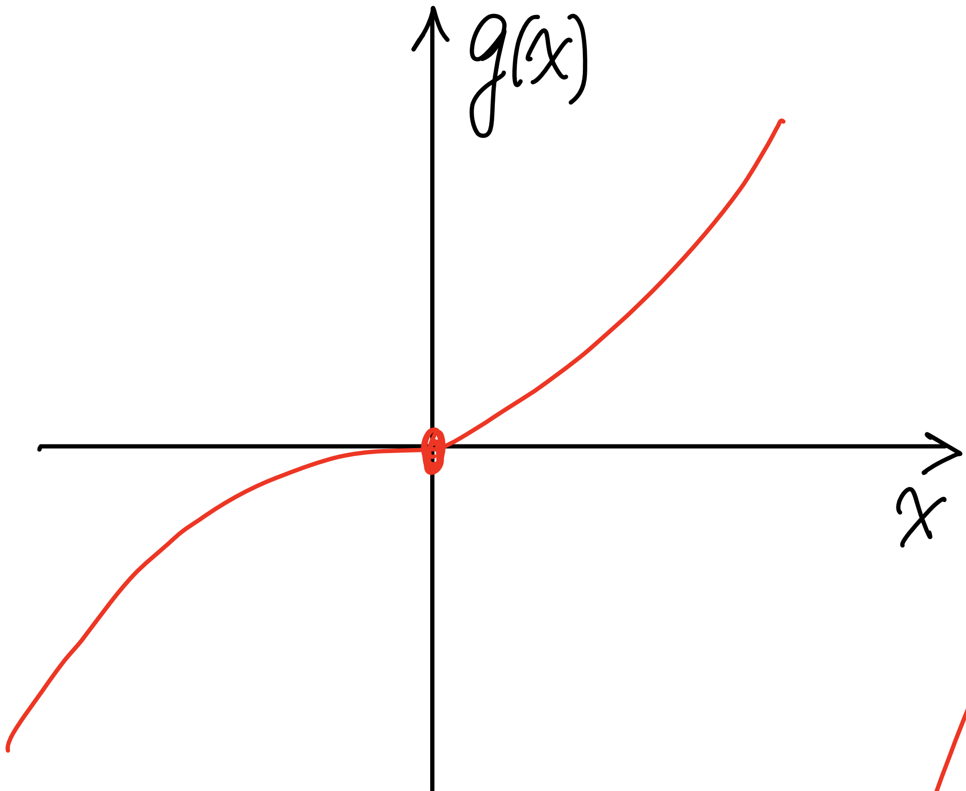
(3) There is a unique $a > 0$ s.t.

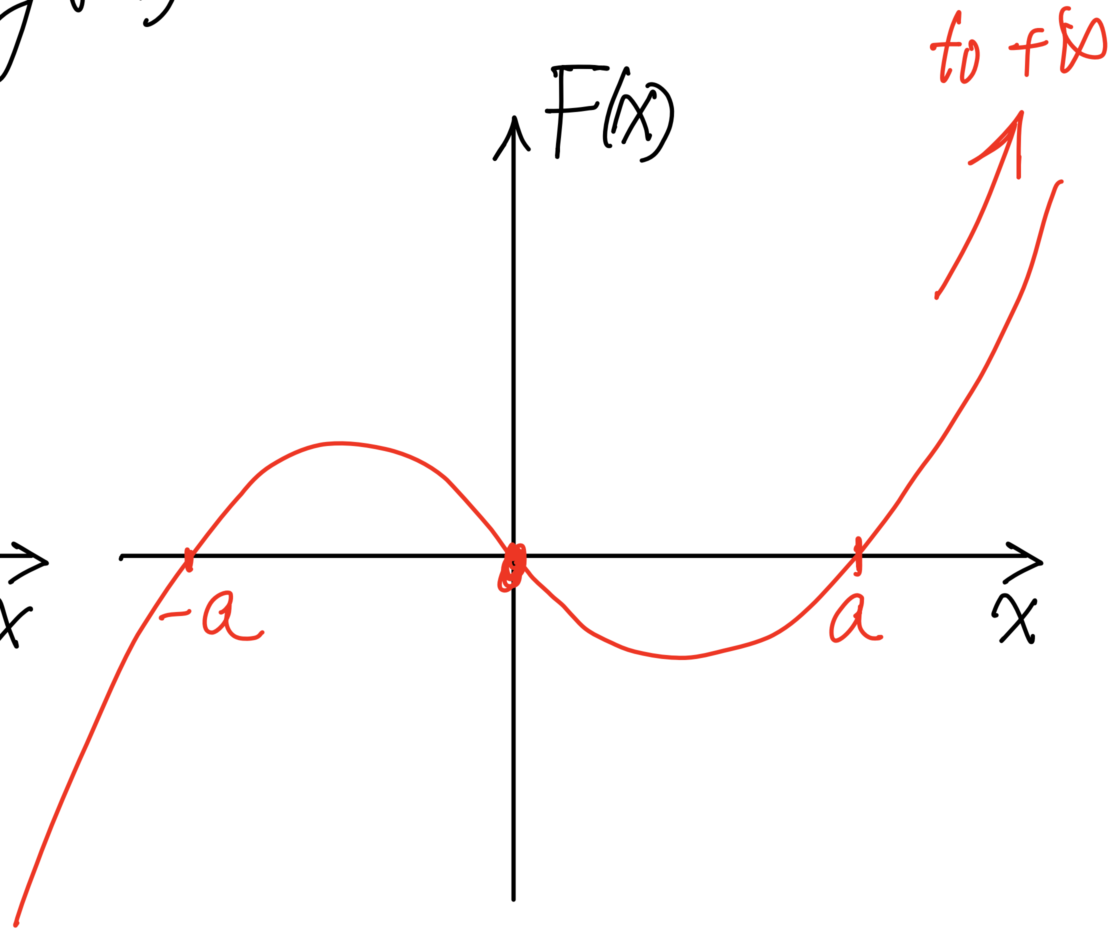
$F(x) < 0$ for $0 < x < a$ and $F(x) > 0$ for $x > a$.

(4) $F(x) \uparrow \rightarrow +\infty$ as $x \rightarrow +\infty$, $x > a$

Lienard System

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases}$$





Lienard System

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases}$$

Then there is a unique limit cycle which is stable.

Lienard System

An Example: van der Pol oscillator

$$\ddot{x} = -\underbrace{(x^2-1)}\dot{x} - x \rightarrow \text{force}$$

$\gamma(x) = x^2 - 1$, friction as a function of x

$$\ddot{x} + \underbrace{(x^2-1)}_{f(x)}\dot{x} + x = 0$$

$$\ddot{x} + \left(\frac{x^3}{3} - x\right)^{\cdot} + x = 0 \quad F(x)$$

$$\begin{cases} \dot{x} = y - \underbrace{\left(\frac{x^3}{3} - x\right)}_{F(x)} \\ \dot{y} = -\underbrace{x}_{g(x)} \end{cases}$$

Simple Observations

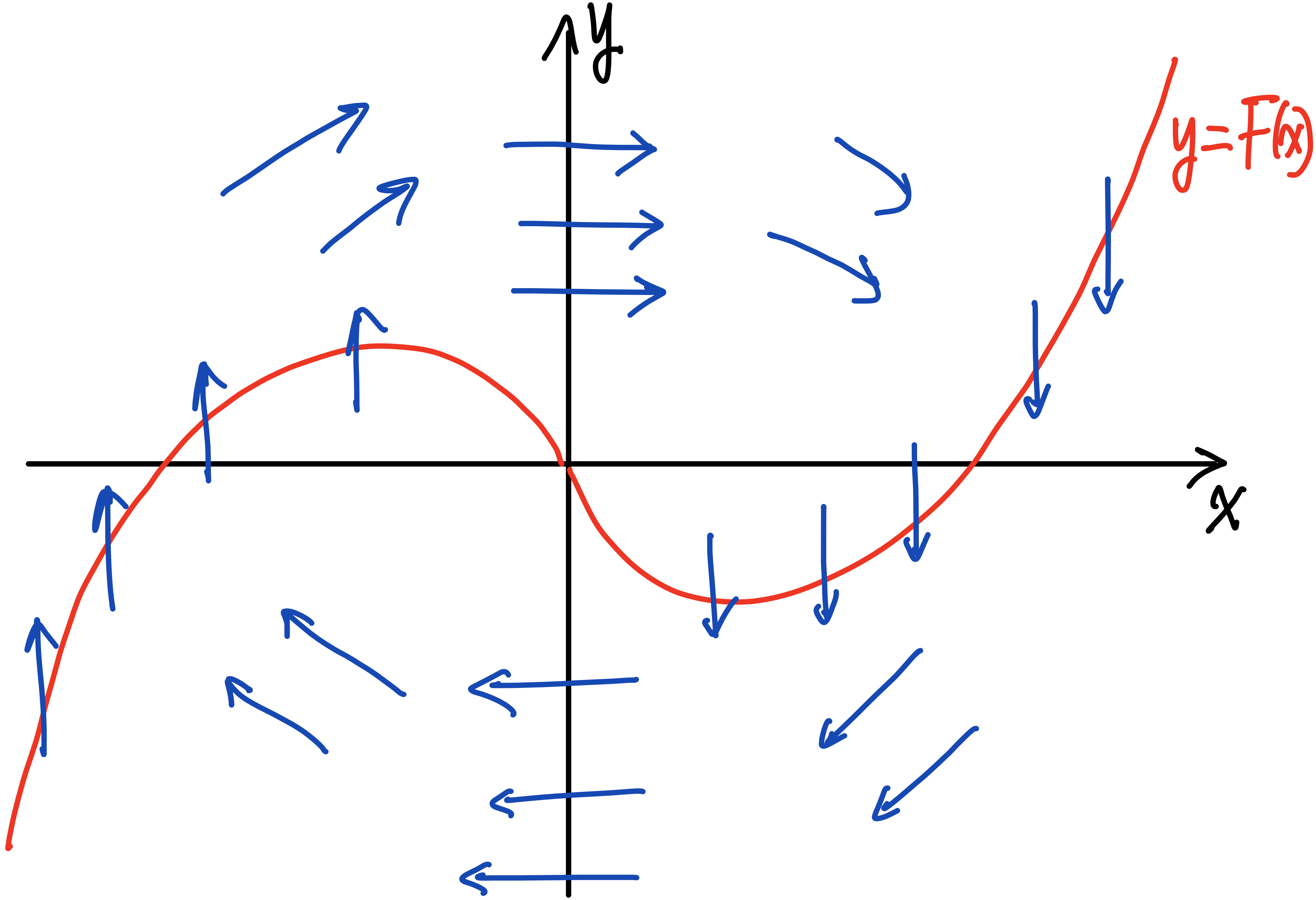
① $(0,0)$ is the only equilibrium point.

$(\, g(0)=0,\ F(0)=0 \,)$

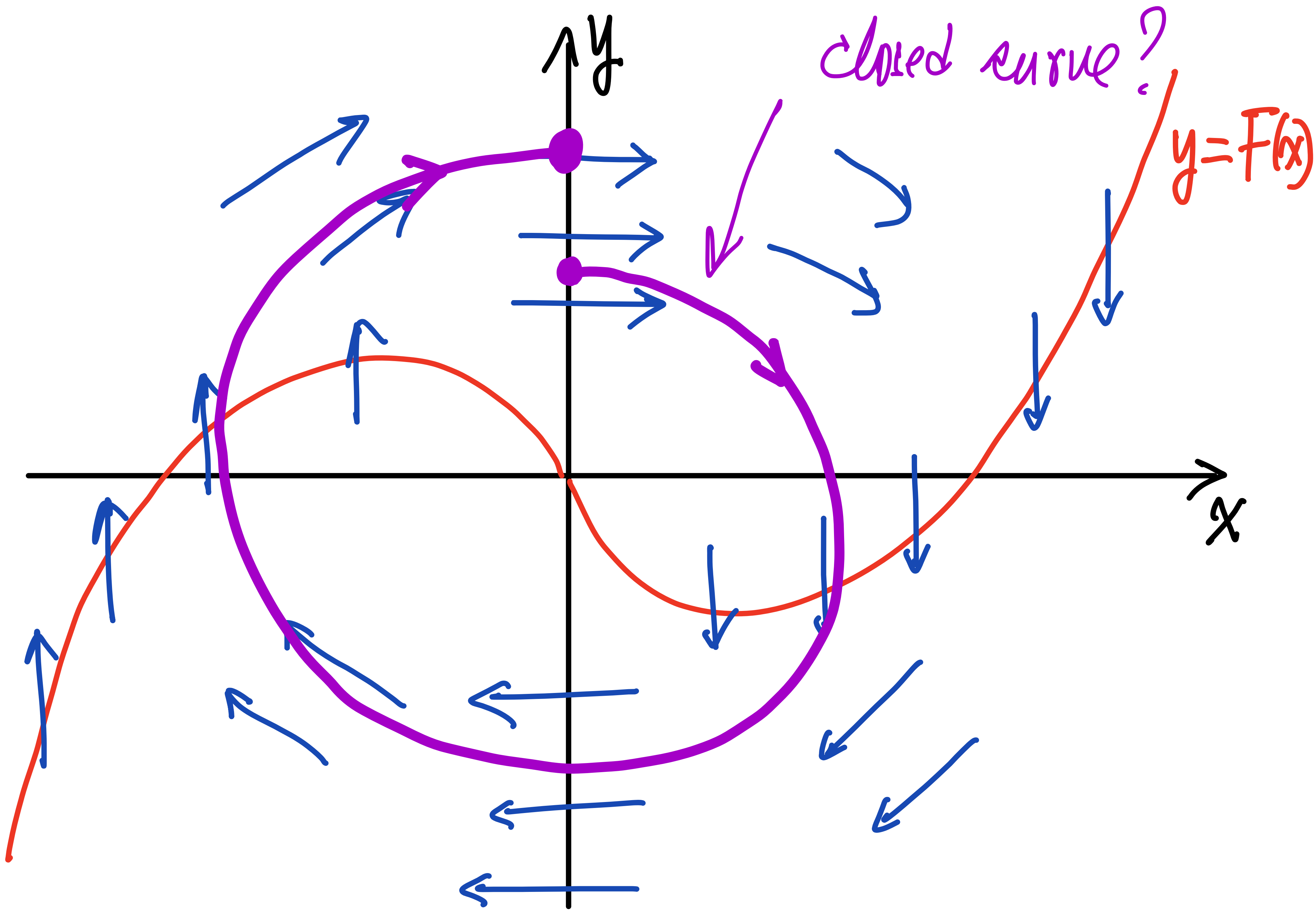
② Look at the nullcines:

$$\dot{x} = y - F(x) \begin{cases} >0 & \text{if } y > F(x) \\ =0 & \text{if } y = F(x) \\ <0 & \text{if } y < F(x) \end{cases}$$

$$\dot{y} = g(x) \begin{cases} >0 & \text{if } x > 0 \\ <0 & \text{if } x < 0 \end{cases}$$



y
x
y=F(x)


y
x
closed curve?
y=F(x)

③ If $(x(t), y(t))$ is a solution, so is $(-x(t), -y(t))$.

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = g(x) \end{cases} \Longrightarrow \begin{cases} (-x)^{\cdot} = (-y) - F(-x) \\ (-y)^{\cdot} = g(-x) \end{cases}$$

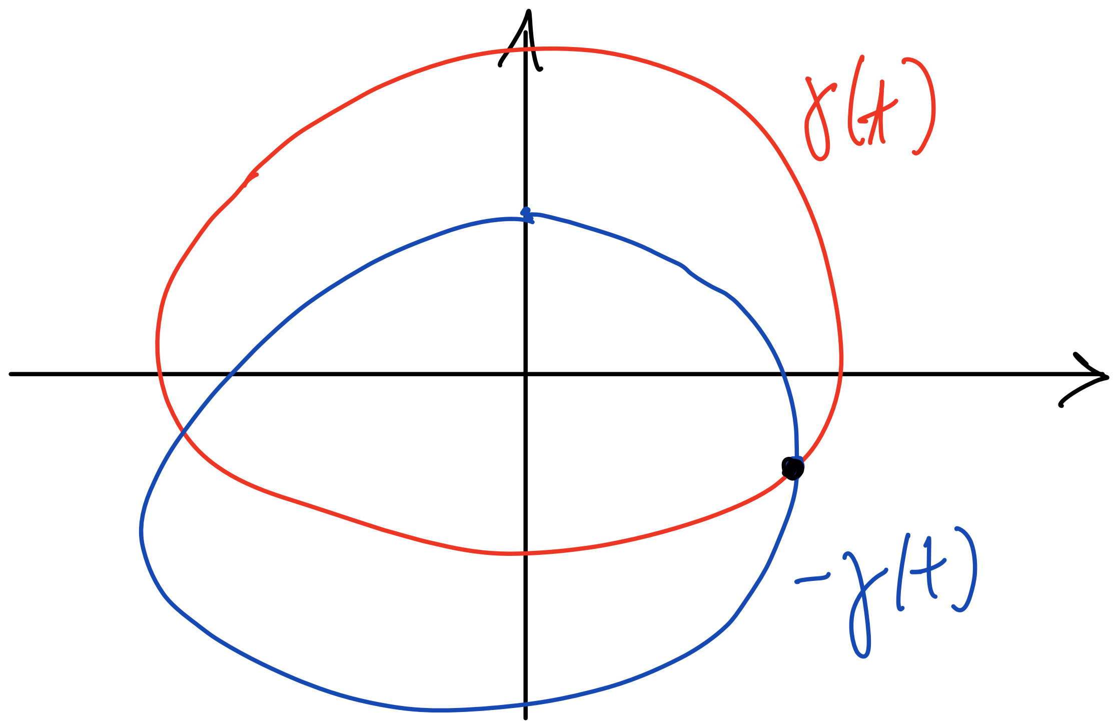
④ Any periodic orbit must enclose the origin (which is the only eq. pt.)

(Any periodic orbit must enclose an eq. pt.)

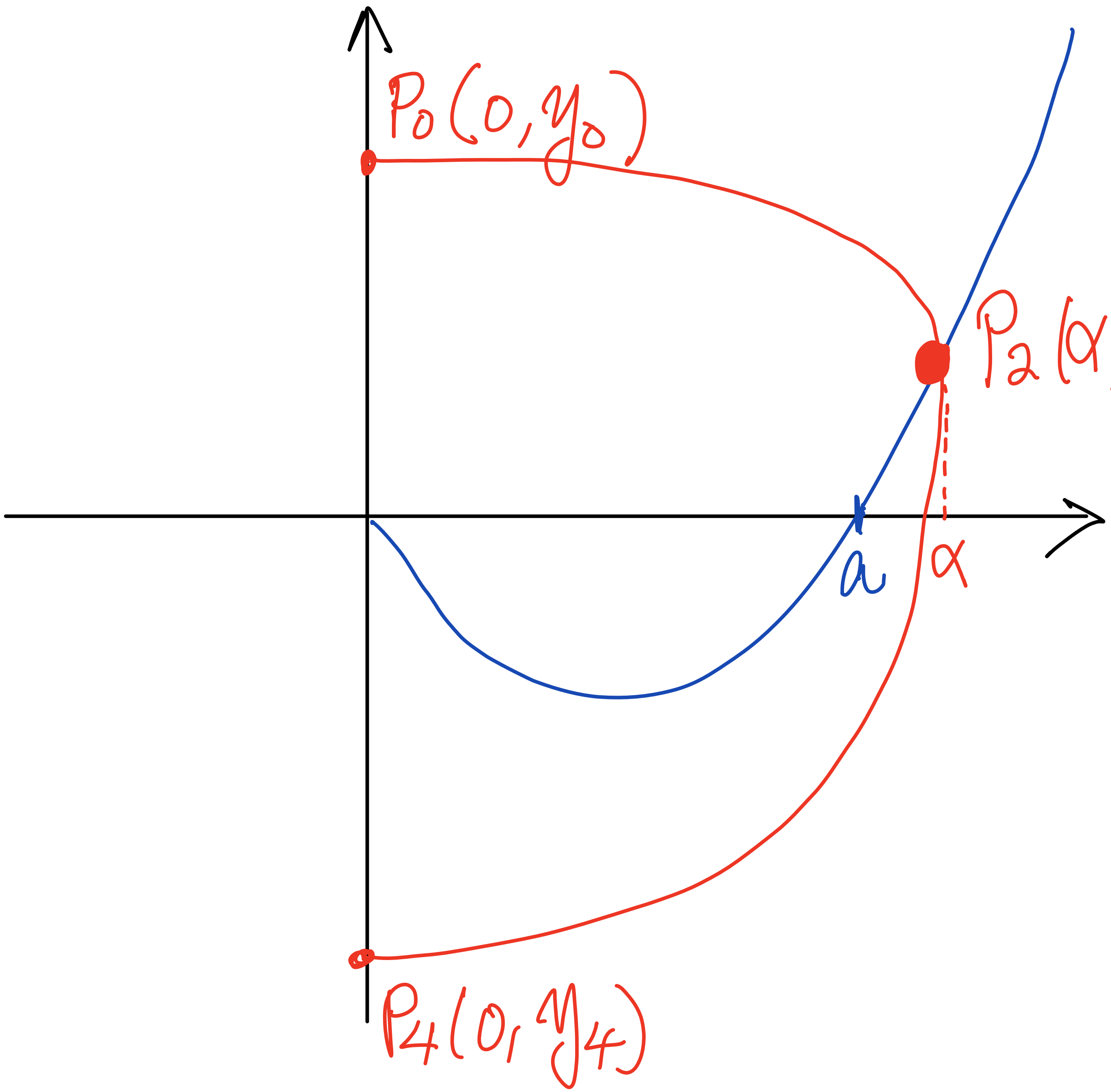
⑤ Any periodic orbit $\{\gamma(t)\}$ must be symmetric w.r.t. the origin

i.e. if $A \in \{\gamma(t)\}$, then $-A \in \{\gamma(t)\}$

Suppose not:



Proof (By shooting method)



Claim

In order for $y_4 = -y_0$, it must be that

$$\alpha > a$$

Proof (By shooting method)

Introducing a "Hamiltonian energy" for the system:

$G(x) = \int_0^x g(s)\,ds$

$$\boxed{H(x,y) = \frac{y^2}{2} + G(x)}$$

Note that at $x = 0$,

$$H(0,y_0) \overset{>}{\underset{<}{=}} H(0,y_4) \text{ iff } |y_0| \overset{>}{\underset{<}{=}} |y_4|$$

Proof (By shooting method)

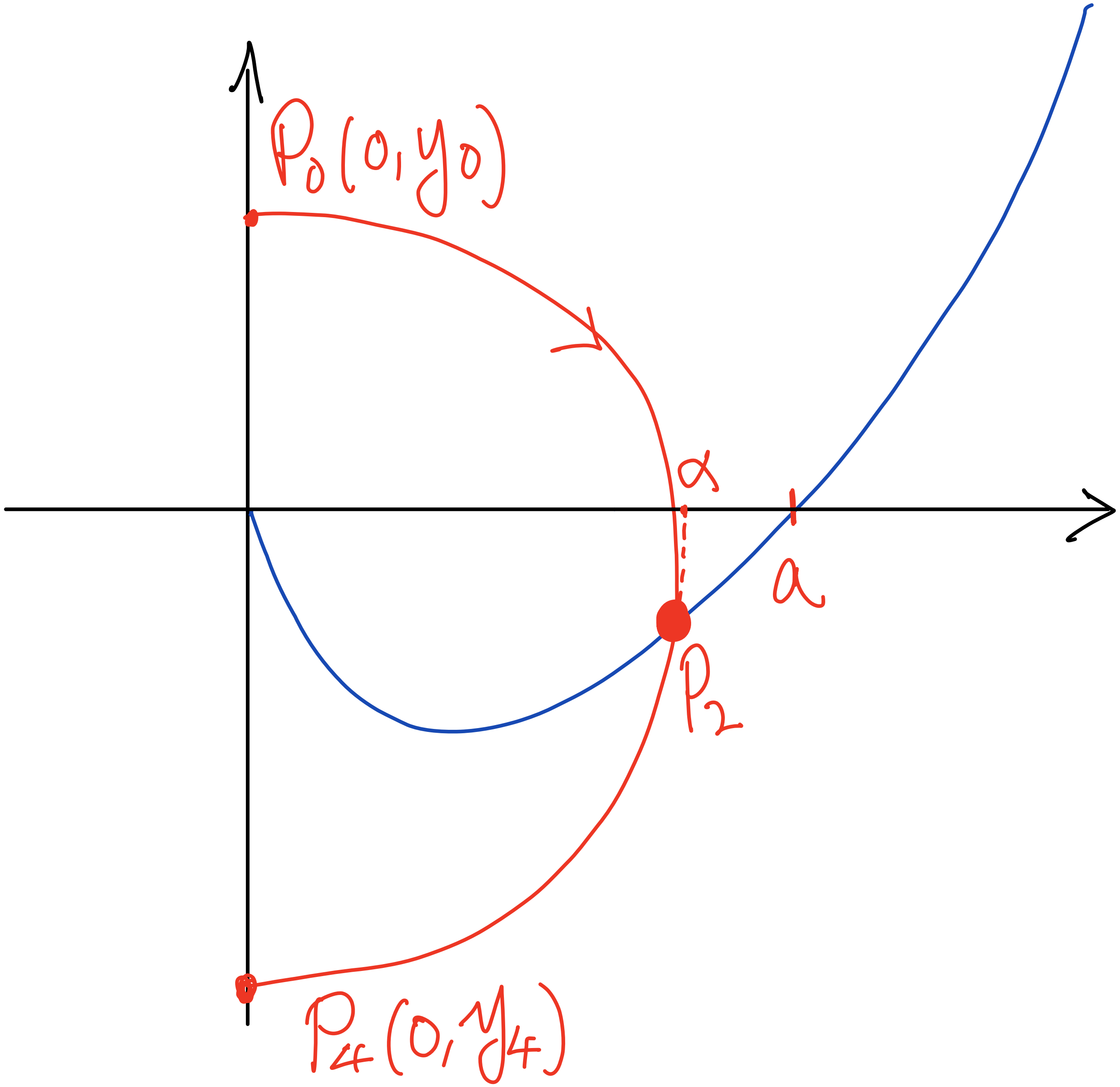
Differentiate $H(x,y)$ along solution:

$$\frac{d}{dt}H(x(t),y(t)) = y\dot{y} + g(x)\dot{x}$$

$$= y(-g(x)) + g(x)(y - F(x))$$

$$= -\underbrace{g(x)}_{\substack{>0 \\ (x>0)}}\,\underbrace{F(x)}_{\substack{<0 \\ (\text{for } 0<x<a)}} > 0 \quad \text{for } 0<x<a$$

Proof (By shooting method)



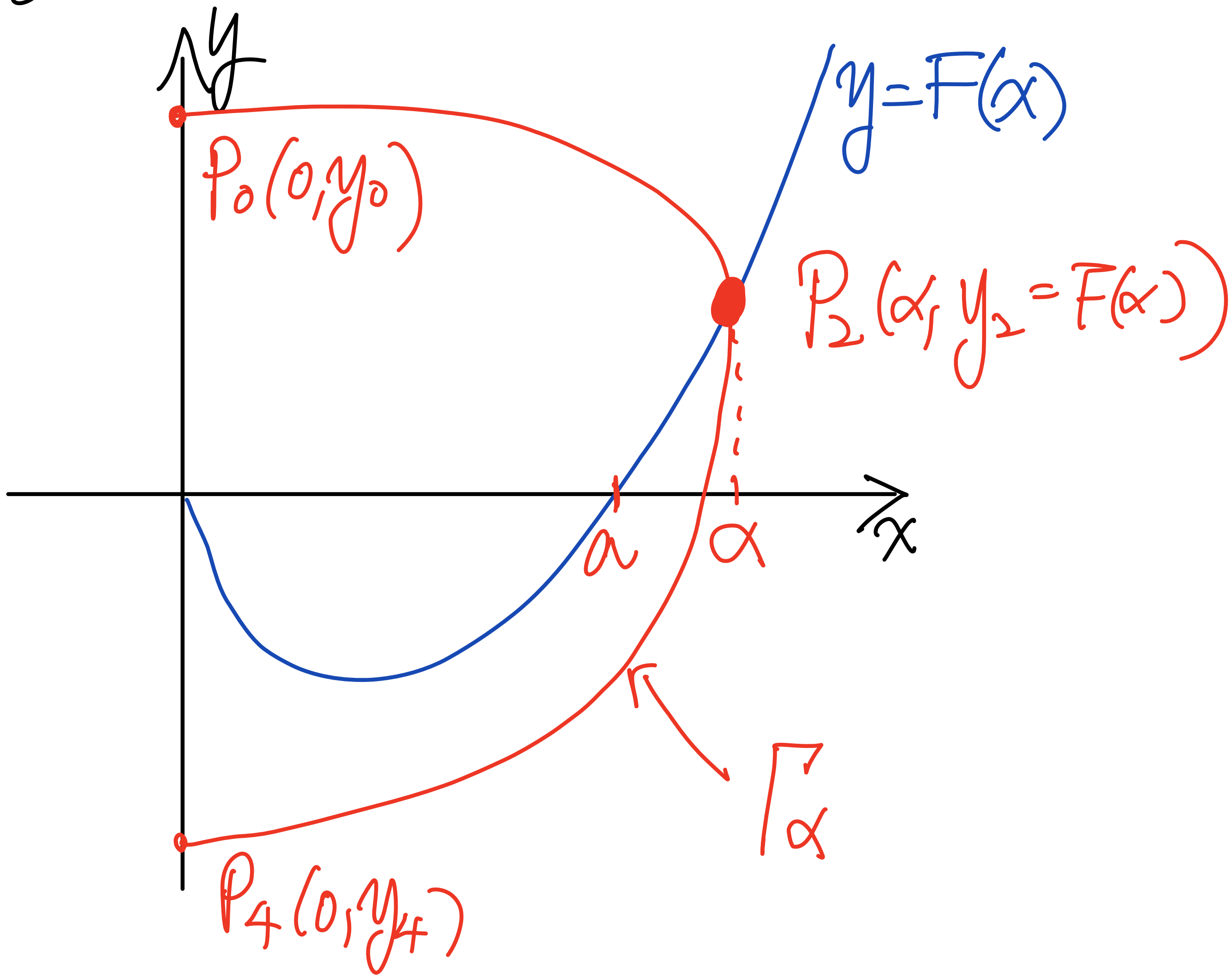
If $\alpha < a$,
H increases along γ.

$H(0, y_4) > H(0, y_0)$

$|y_4| > |y_0|$

Proof (By shooting method)

Shooting from $P_2(\alpha, F(\alpha))$ on $y = F(x)$



Proof (By shooting method)

Shooting from $P_2(\alpha, F(\alpha))$ on $y = F(x)$

Define $\boxed{\phi(\alpha) = \int_{\Gamma_\alpha} dH}$ (line integral along Γ_α)

$$= H(P_4) - H(P_0)$$

$$= \frac{y_4^2}{2} - \frac{y_0^2}{2} \qquad (x = 0)$$

Hence $\phi(\alpha) >, =, < 0$ iff $|y_4| >, =, < |y_0|$

Proof (By shooting method)

Shooting from $P_2(\alpha, F(\alpha))$ on $y = F(x)$

Define $\boxed{\phi(\alpha) = \int_{\Gamma_\alpha} dH}$ (line integral along Γ_α)

Claim:

① $\phi(\alpha) > 0$ if $0 < \alpha < a$

② $\phi(\alpha)$ is decreasing in α if $\alpha > a$.

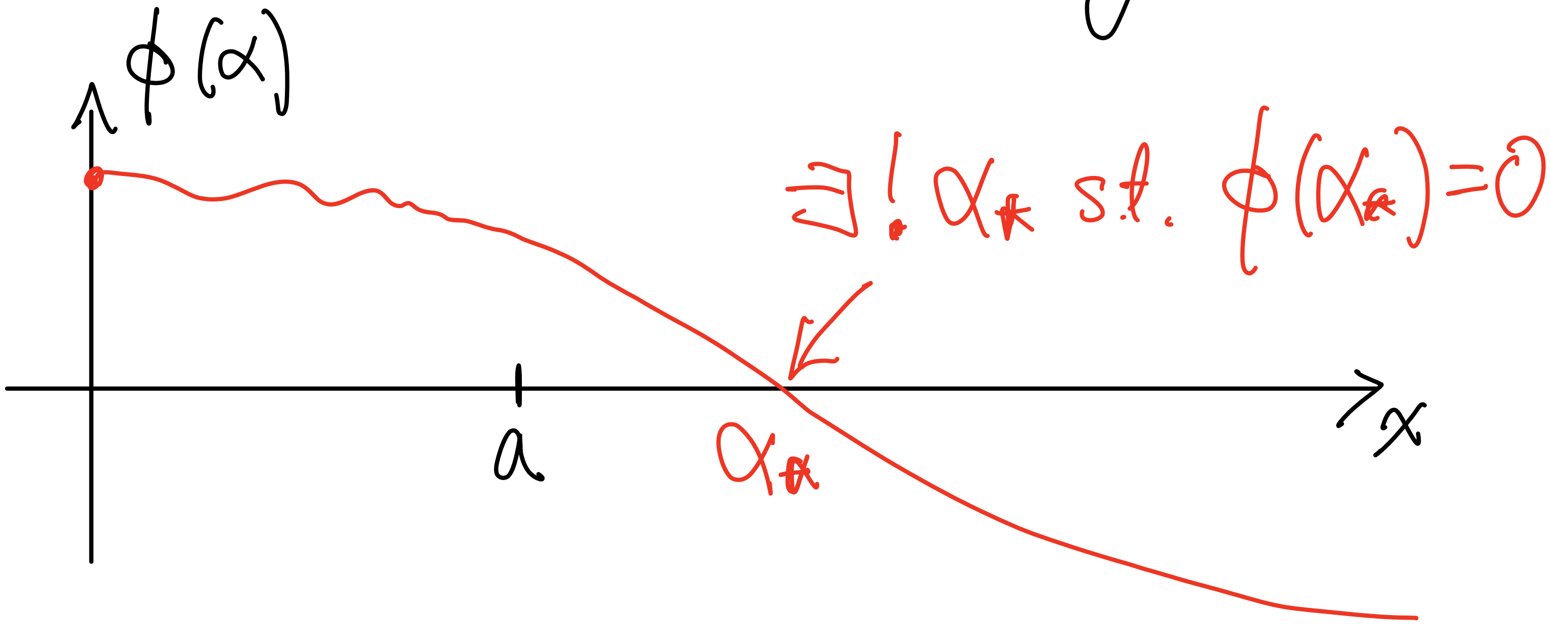
③ $\phi(\alpha) \longrightarrow -\infty$ as $\alpha \longrightarrow +\infty$

Proof (By shooting method)

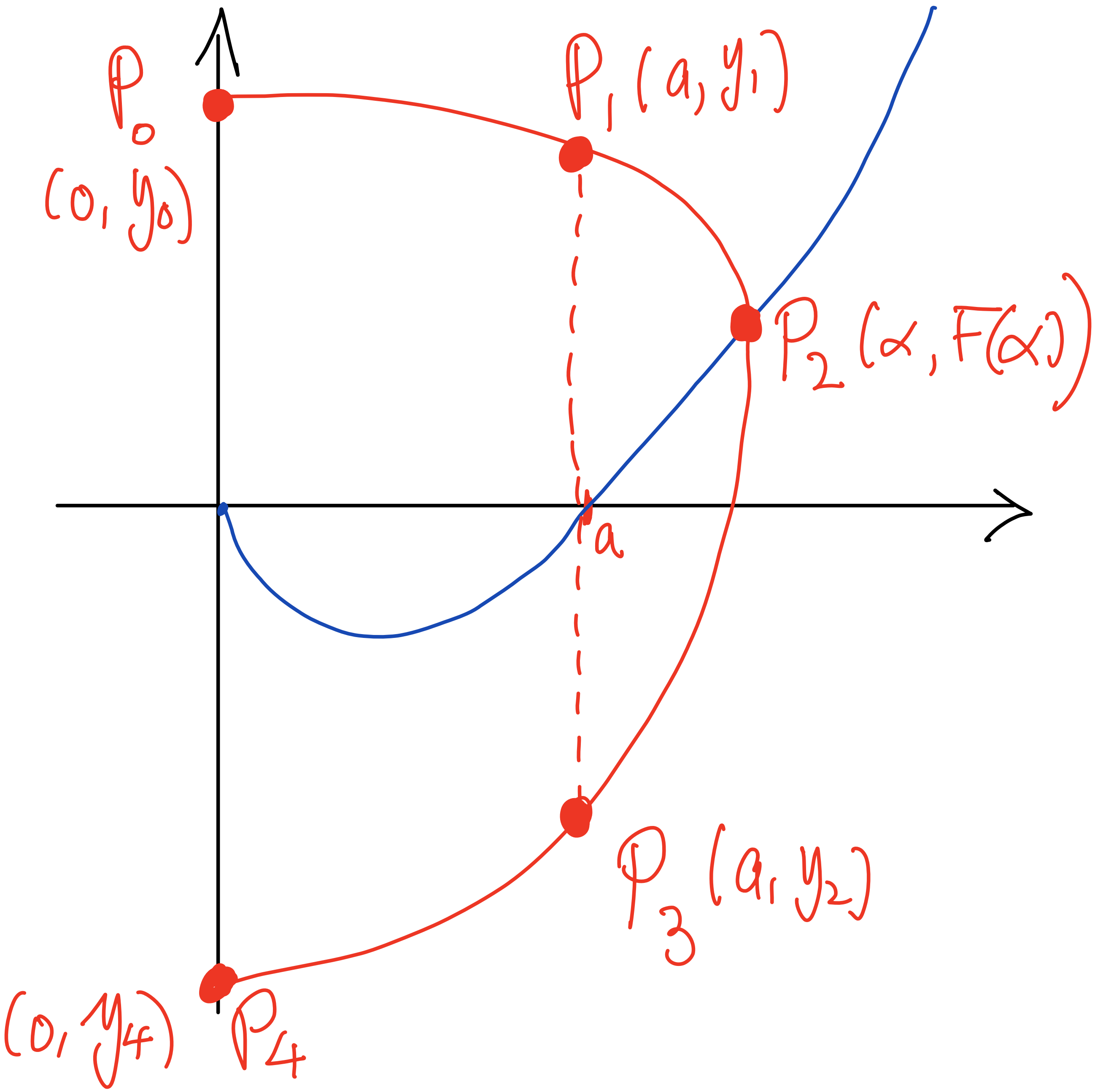
Shooting from $P_2(\alpha, F(\alpha))$ on $y = F(x)$

Define $\boxed{\phi(\alpha) = \int_{\Gamma_\alpha} dH}$ (line integral along Γ_α)

Claim:



Shooting from $P_2(\alpha, F(\alpha))$



$$\phi(\alpha) = \int_{\overline{\Gamma_\alpha}} dH$$

$$= \int_{P_0P_1} dH$$

$$+ \int_{P_1P_3} dH$$

$$+ \int_{P_3P_4} dH$$

Shooting from $P_2(\alpha, F(\alpha))$

$$\int_{P_0P_1} dH = \int_{P_0P_1} \frac{dH}{dt}\,dt = \int_{P_0P_1} \frac{dH}{dt}\frac{dt}{dx}\,dx$$

$$= \int_0^a -g(x)F(x)\,\frac{1}{\dot{x}}\,dx$$

$$\quad\ \ -\qquad +\qquad -$$

$$= \int_0^a -\frac{g(x)F(x)}{y-F(x)}\,dx \qquad > 0,$$

$y>0$ $\nearrow$ $\qquad +$

$\downarrow$ as $y\uparrow$

Shooting from $P_2(\alpha, F(\alpha))$

$$\int_{P_3P_4} dH = \int_{P_3P_4} \frac{dH}{dt}\, dt = \int_a^0 \frac{dH}{dt}\frac{dt}{dx}\, dx$$

$$= \int_a^0 -g(x)F(x)\frac{1}{\dot{x}}\, dx$$

$$= \int_a^0 -\frac{g(x)F(x)}{y-F(x)}\, dx > 0,$$

$y<0$

$- \quad - \quad + \quad - \quad -$

$\downarrow$ as $|y| \uparrow$

Shooting from $P_2(\alpha, F(\alpha))$

$$\int_{P_1P_3} dH = \int_{P_1P_3} \frac{dH}{dt}\, dt = \int_{P_1P_3} \frac{dH}{dt}\frac{dt}{dy}\, dy$$

$$= \int_{y_1}^{y_3} -g(x)\, F(x) \frac{1}{(-g(x))}\, dy$$

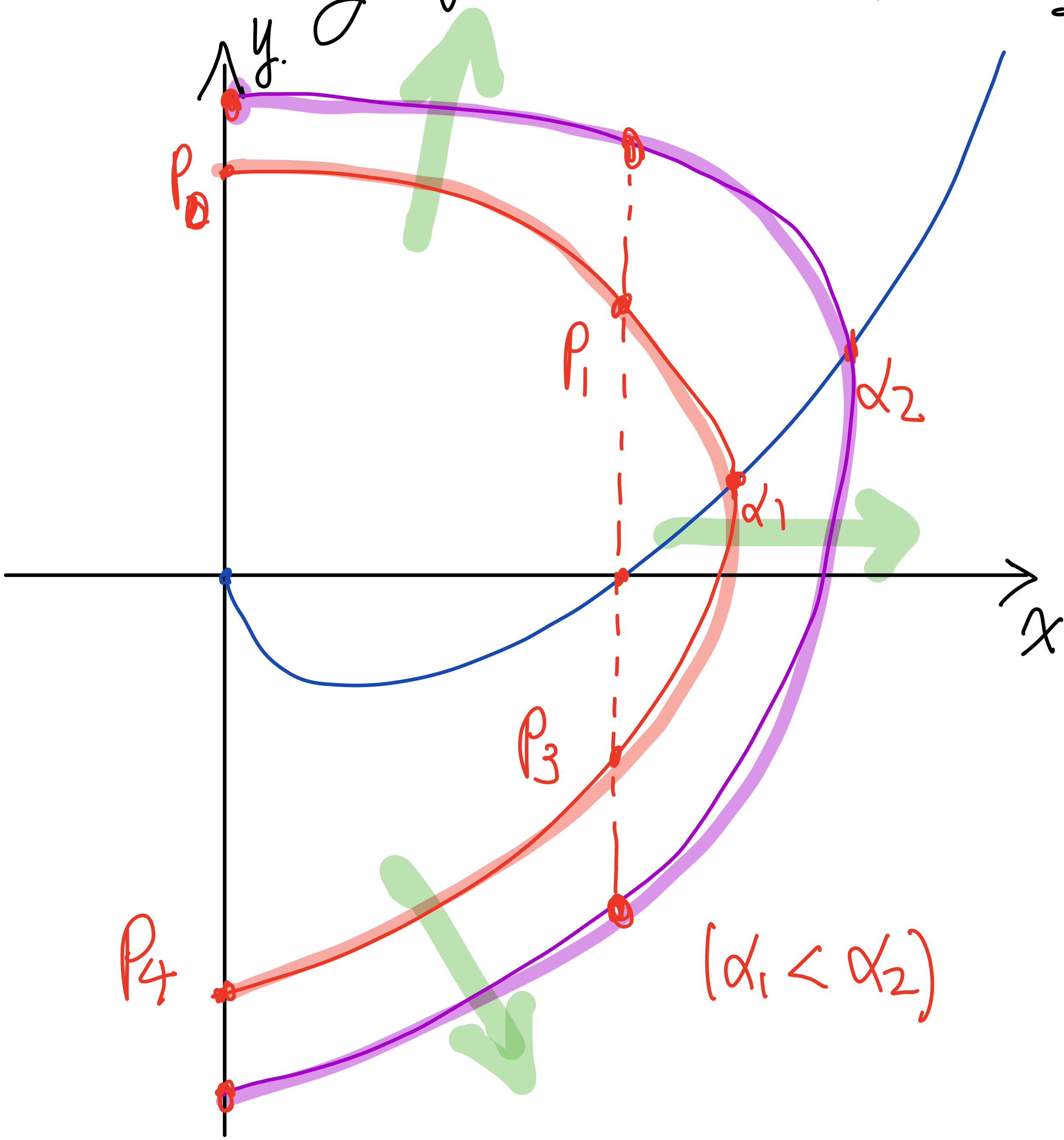
$$= \int_{y_1}^{y_3} F(x)\, dy < 0,$$

dy: <0

$F(x)$: >0

$$\boxed{\text{Very negative as } x \longrightarrow +\infty.}$$

Shooting from $P_2(\alpha, F(\alpha))$



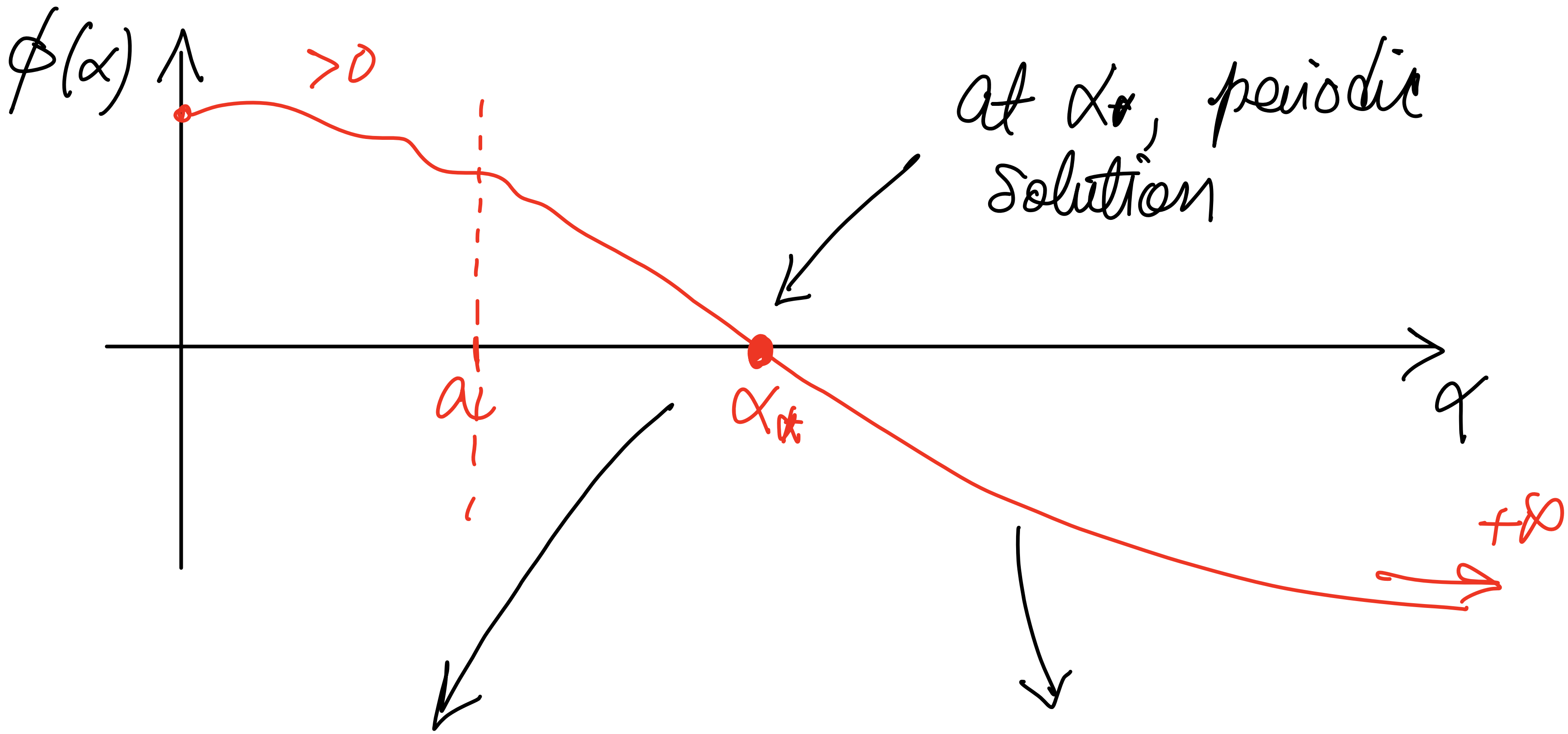
$\Gamma(\alpha_1) \,:\, \Gamma(\alpha_2)$

$$\int_{P_0P_1(\alpha_1)} dH > \int_{P_0P_1(\alpha_2)} dH$$

$$\int_{P_1P_3(\alpha_1)} dH > \int_{P_1P_3(\alpha_2)} dH$$

$$\int_{P_3P_4(\alpha_1)} dH > \int_{P_3P_4(\alpha_2)} dH$$

Stability of Periodic Orbit



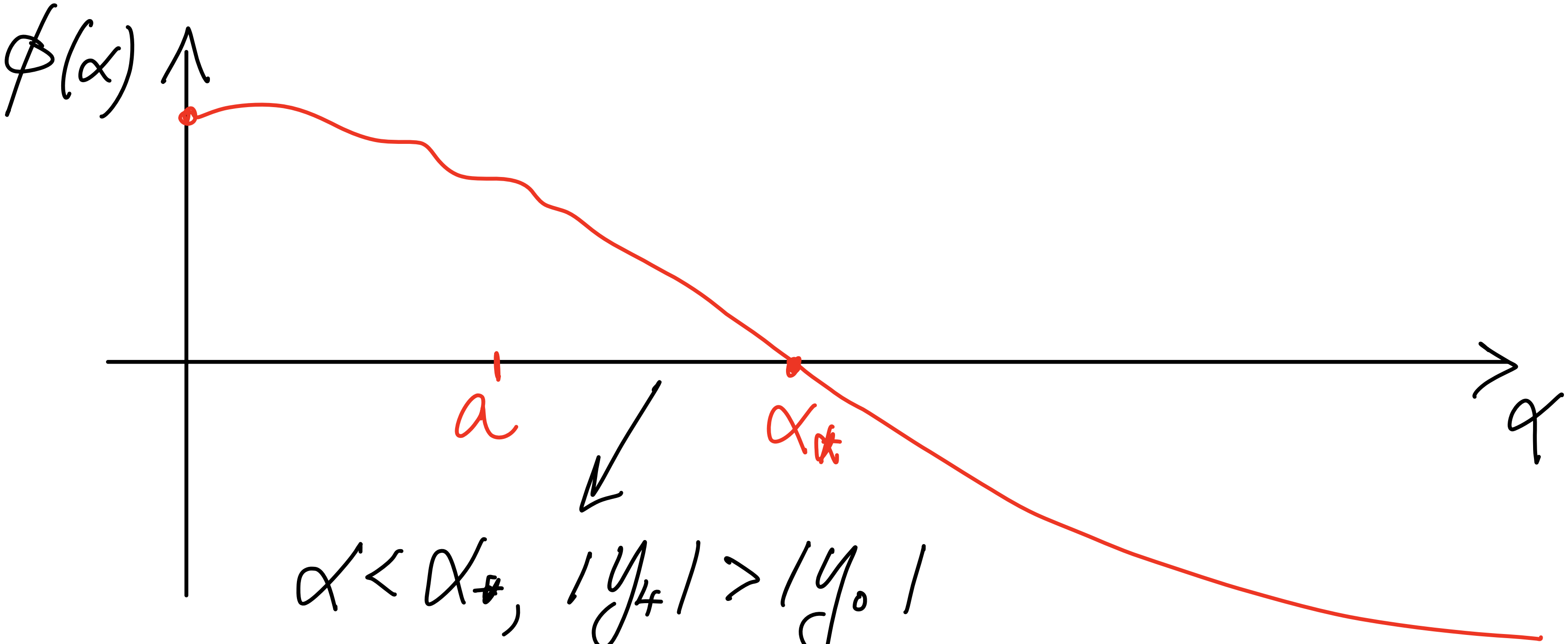
$|y_4| > |y_0|$

$(\alpha < \alpha_*)$

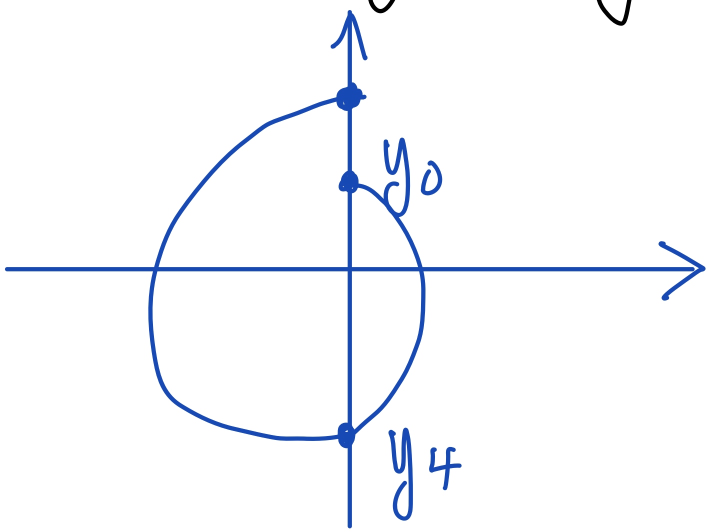
$|y_4| < |y_0|$

$(\alpha > \alpha_*)$

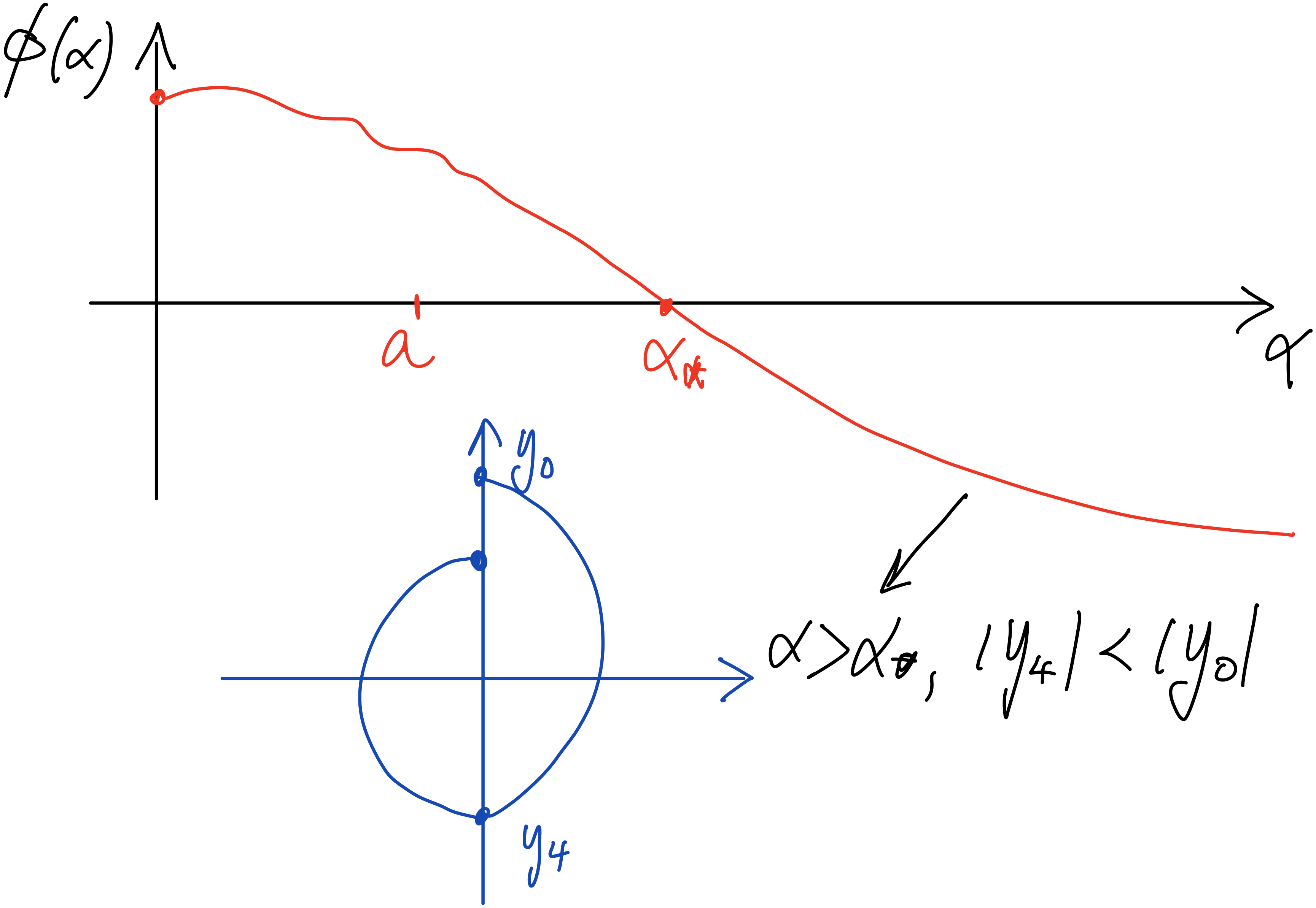
Stability of Periodic Orbit



$\alpha < \alpha_*, \ |y_4| > |y_0|$

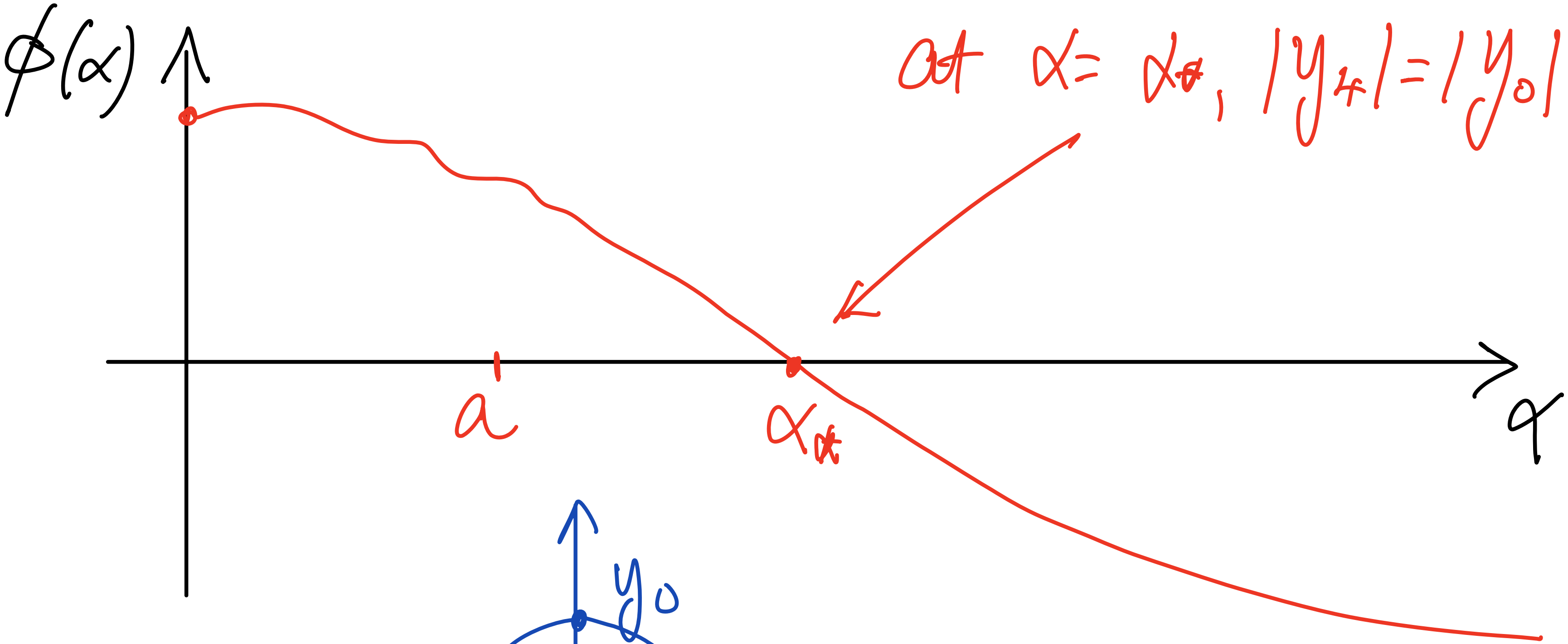


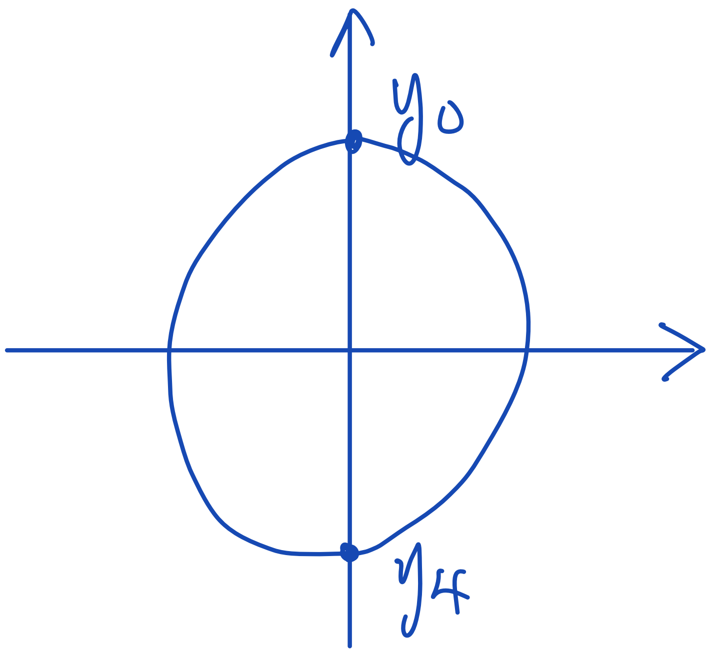
Stability of Periodic Orbit



$\alpha > \alpha_*$, $|y_4| < |y_0|$

Stability of Periodic Orbit





Example 6.45. Consider the system [M] van der Pol

$$\begin{aligned} \dot{x} &= y - x(x^2 - 1), \\ \dot{y} &= -x. \end{aligned} \tag{6.40}$$

These functions satisfy the conditions of Theorem 6.42 and so there is a unique limit cycle. The phase portrait is shown in Figure 6.19. ∎

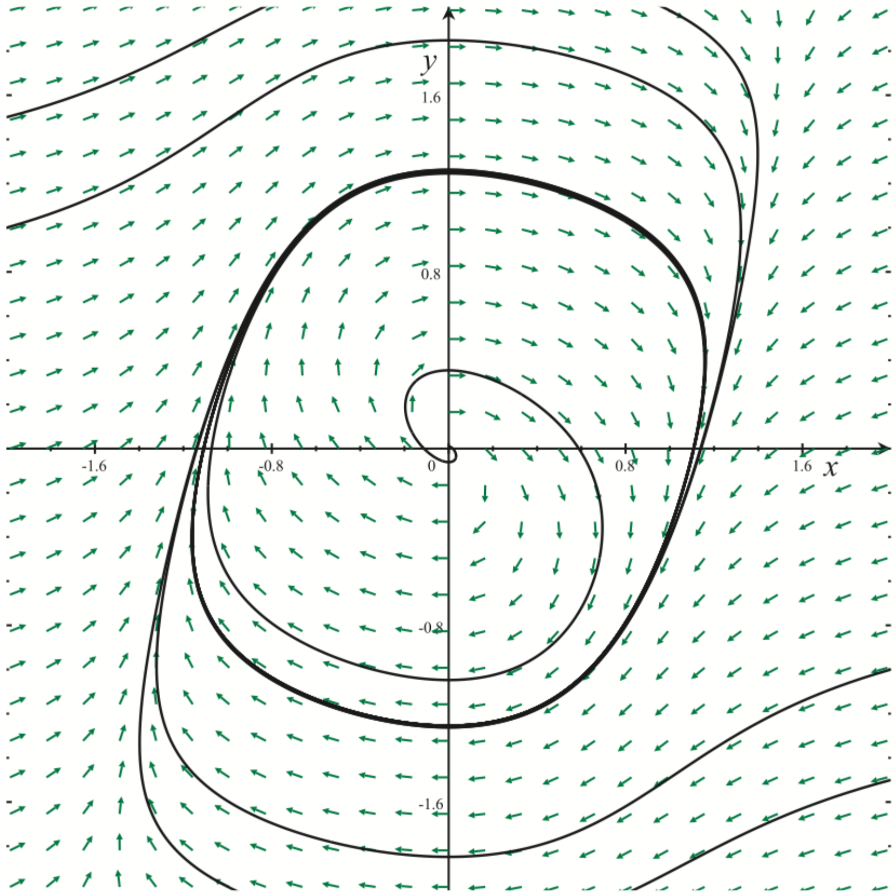


Figure 6.19. *Limit cycle for* (6.40).

Some Extensions [Glendinning, Stab., Instab, Chaos]

(5.14) THEOREM

Consider $\ddot{x} + f(x)\dot{x} + g(x) = 0$ *and suppose that*

i) $xg(x) > 0$ *for* $x \neq 0$, $g(0) = 0$, $f(0) < 0$ *and* $\frac{dg}{dx}(0) > 0$;
ii) $sign(x)F(x) > k > 0$ *for sufficiently large* $|x|$;
iii) $G(x) \to \infty$ *as* $|x| \to \infty$; *where* $G(x) = \int_0^x g(\xi)d\xi$. *Then the system has at least one periodic orbit.*

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12. Suppose $\ddot{x} + f(x)\dot{x} + g(x) = 0$ and let $F(x) = \int_0^x f(u)du$ and $G(x) = \int_0^x g(u)du$. Prove that if

$$g(0) = 0, \quad xg(x) > 0 \text{ if } x \neq 0, \quad f(0) < 0, \quad g'(0) > 0$$

and

$$\operatorname{sign}(x)F(x) \to \infty \quad \text{as} \quad |x| \to \infty$$

then there is at least one periodic orbit.