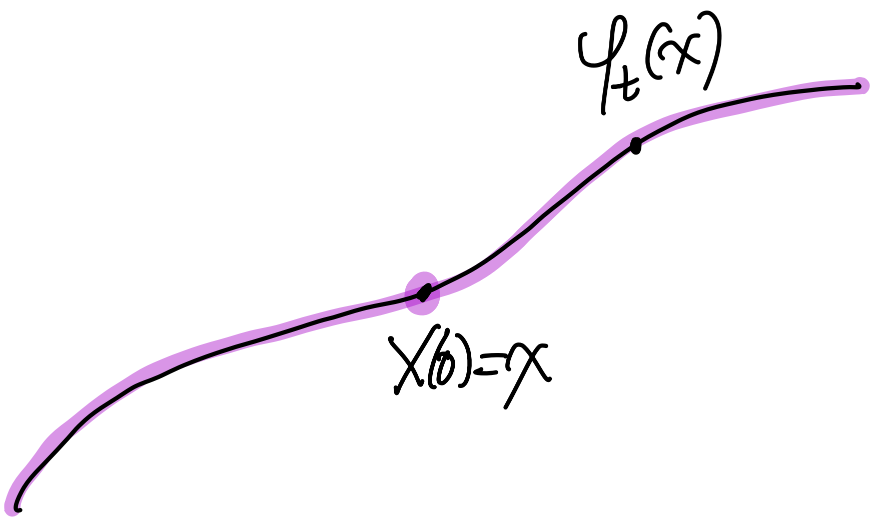
Limit Sets of Dynamical Systems (M, Chapter 4.9, 4.10)

$$\frac{dX}{dt} = F(X), \quad X(0) = x, \quad X(t) = \varphi_t(x)$$

flow/solution map

[Assume global existence of solution.]



- $\Gamma_x = \{ \varphi_t(x) : t \in \mathbb{R} \}$ (orbit, trajectory, solution curve)
- $\Gamma_x^+ = \{ \varphi_t(x) : t \geq 0 \}$ (forward orbit)
- $\Gamma_x^- = \{ \varphi_t(x) : t \leq 0 \}$ (backward orbit)

1 Omega (ω)-limit set $\omega(x)$: limit points of $\varphi_t(x)$ as $t \to +\infty$

$$\boxed{\omega(x) = \{ y : \exists\, t_1, t_2, \cdots t_j \to +\infty,\ \text{s.t. } \varphi_{t_i}(x) \to y \}}$$

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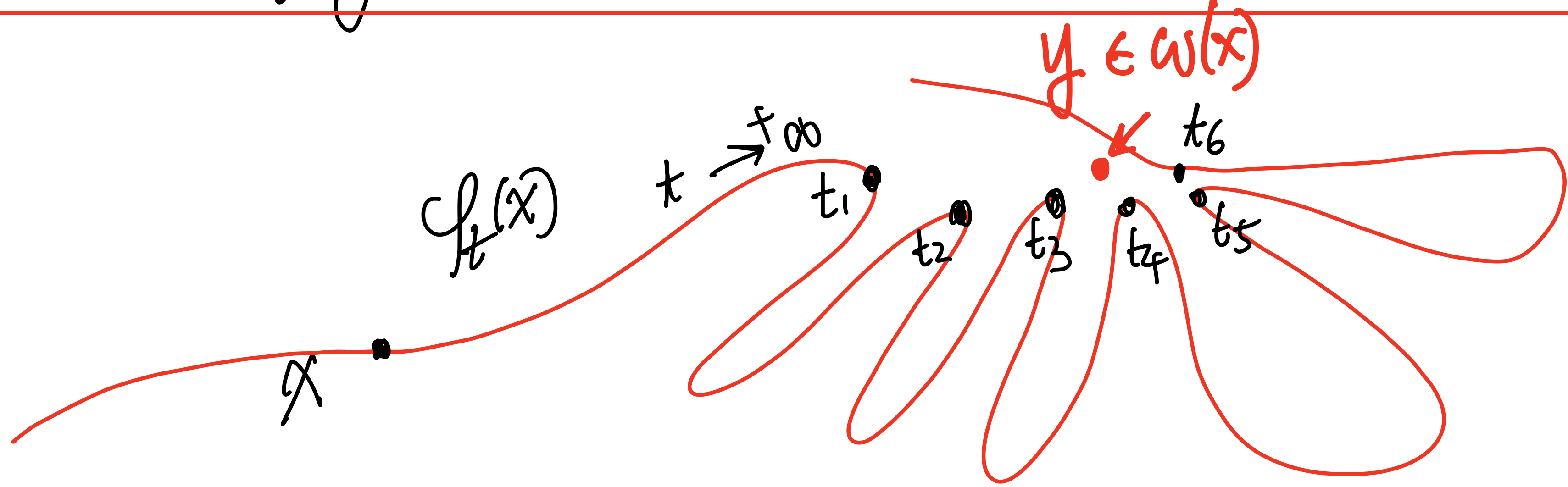


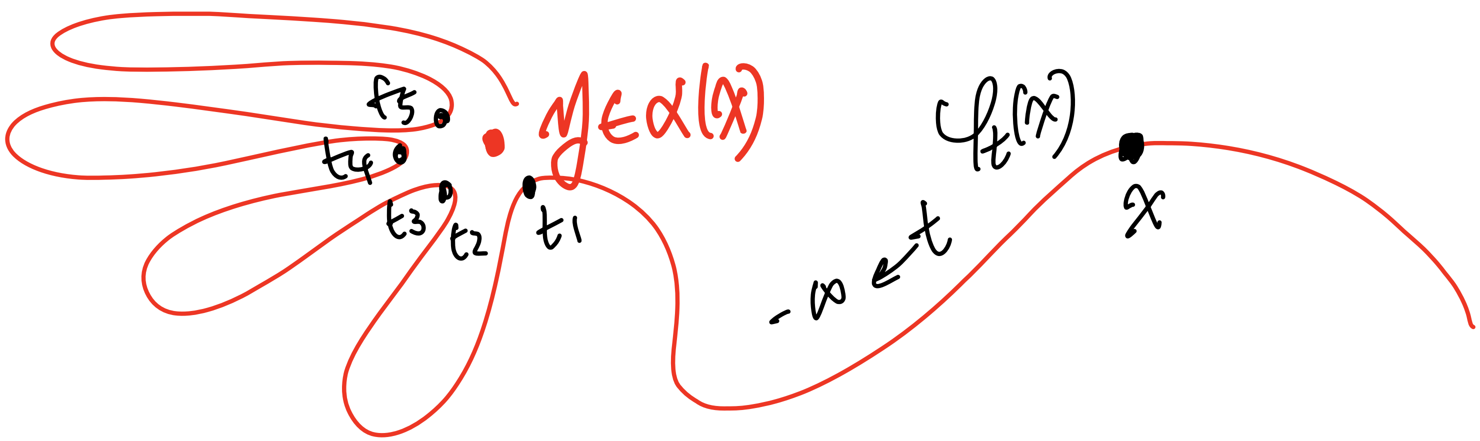
Note: If $z \in \Gamma_x$ then $\omega(z) = \omega(x)$.

Hence, OK to write $\omega(x) = \omega(\Gamma_x)$

② Alpha (α)-limit set $\alpha(x)$: limit points of

$\varphi_t(x)$ as $t \to -\infty$

$$\boxed{\alpha(x) = \{ y : \exists\, t_1, t_2, \cdots t_j \to -\infty, \text{ s.t. } \varphi_{t_i}(x) \to y \}}$$



Note: If $z \in \Gamma_x$, then $\alpha(z) = \alpha(x)$.

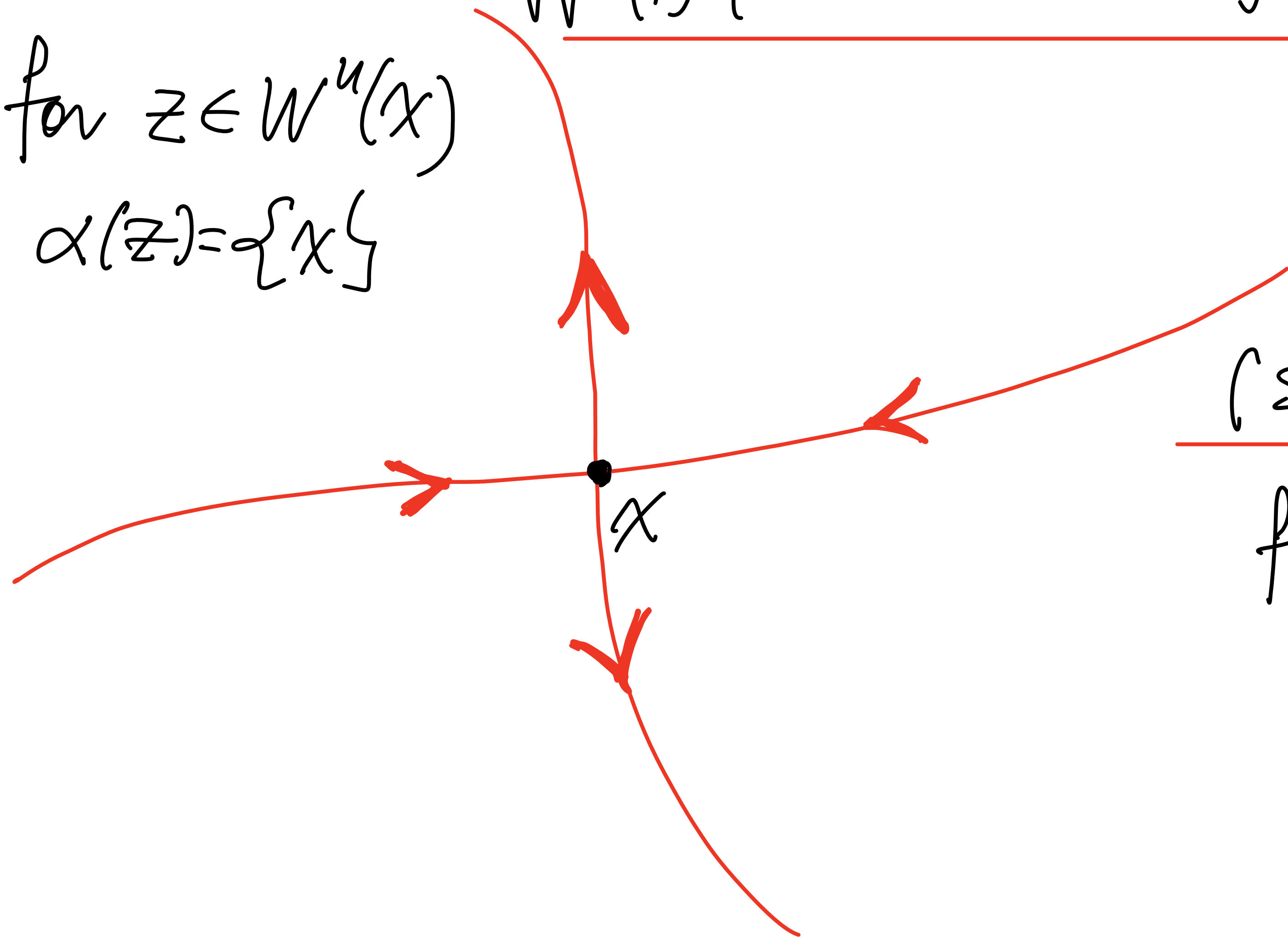
Hence, OK to write $\alpha(x) = \alpha(\Gamma_x)$

Examples: Consider a hyperbolic critical point.

$W^u(x)$ (unstable manifold)

for $z \in W^u(x)$

$\alpha(z) = \{x\}$



$W^s(x)$

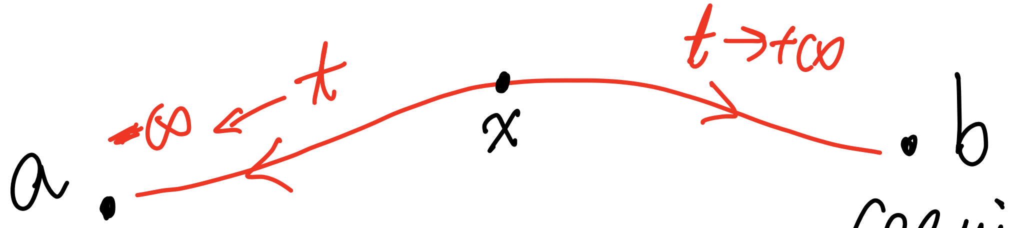
(stable manifold)

for $y \in W^s(x)$,

$\omega(y) = x$

Examples:

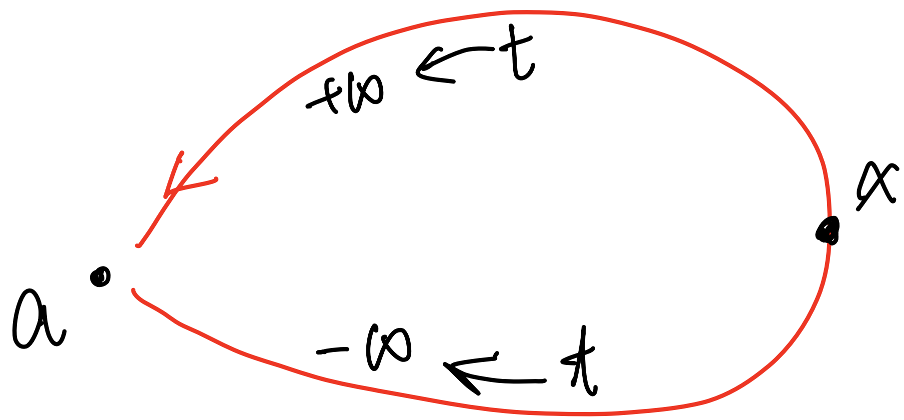
Heteroclinic orbit: $a \xleftarrow{t\to-\infty} \varphi_t(x) \xrightarrow{t\to+\infty} b$



(equilibrium pt) $\alpha(x)=\{a\}$

(equilibrium pt) $\omega(x)=\{b\}$

Homoclinic orbit: $\varphi_t(x) \xrightarrow{t\to\pm\infty} a$



(equilibrium pt.)

$$\omega(x)=\alpha(x)=\{a\}$$

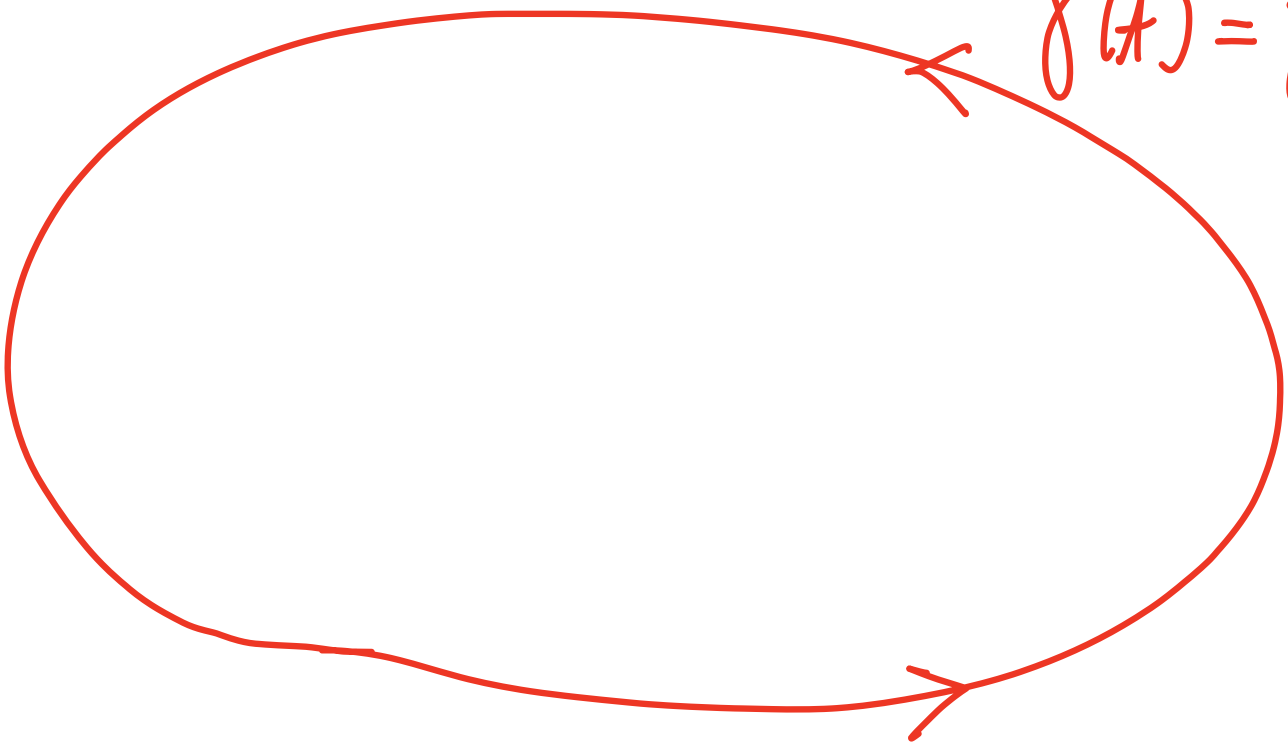
③ Limit Cycle γ $(\exists\, T>0,\ \gamma(t)=\gamma(t+T))$

γ is a periodic orbit that is the ω-limit or α-limit set of some point $x \notin \gamma$

$\gamma = \omega(x)$

$\gamma = \alpha(y)$

$\gamma(t) = \gamma(t+T)$



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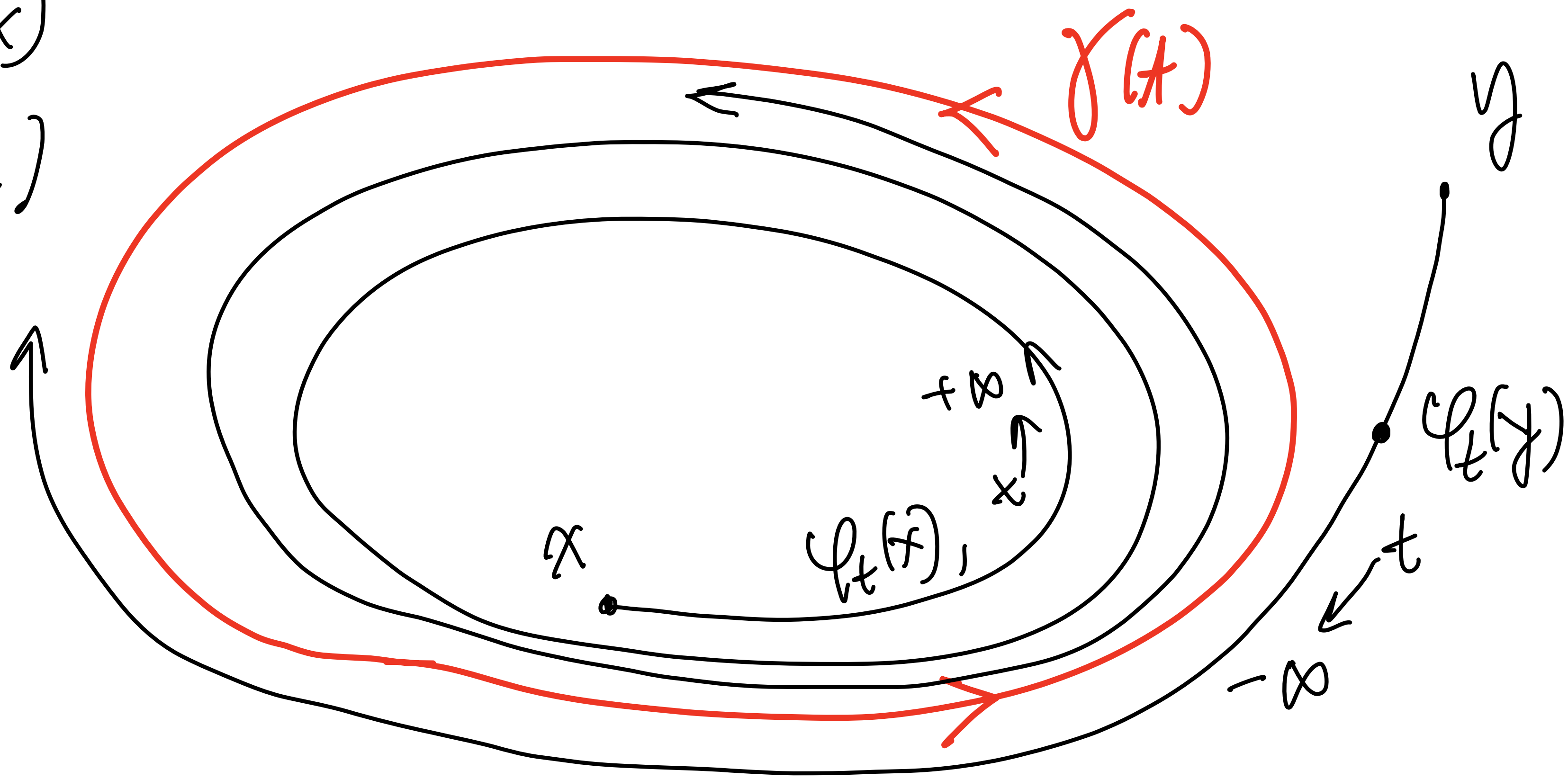

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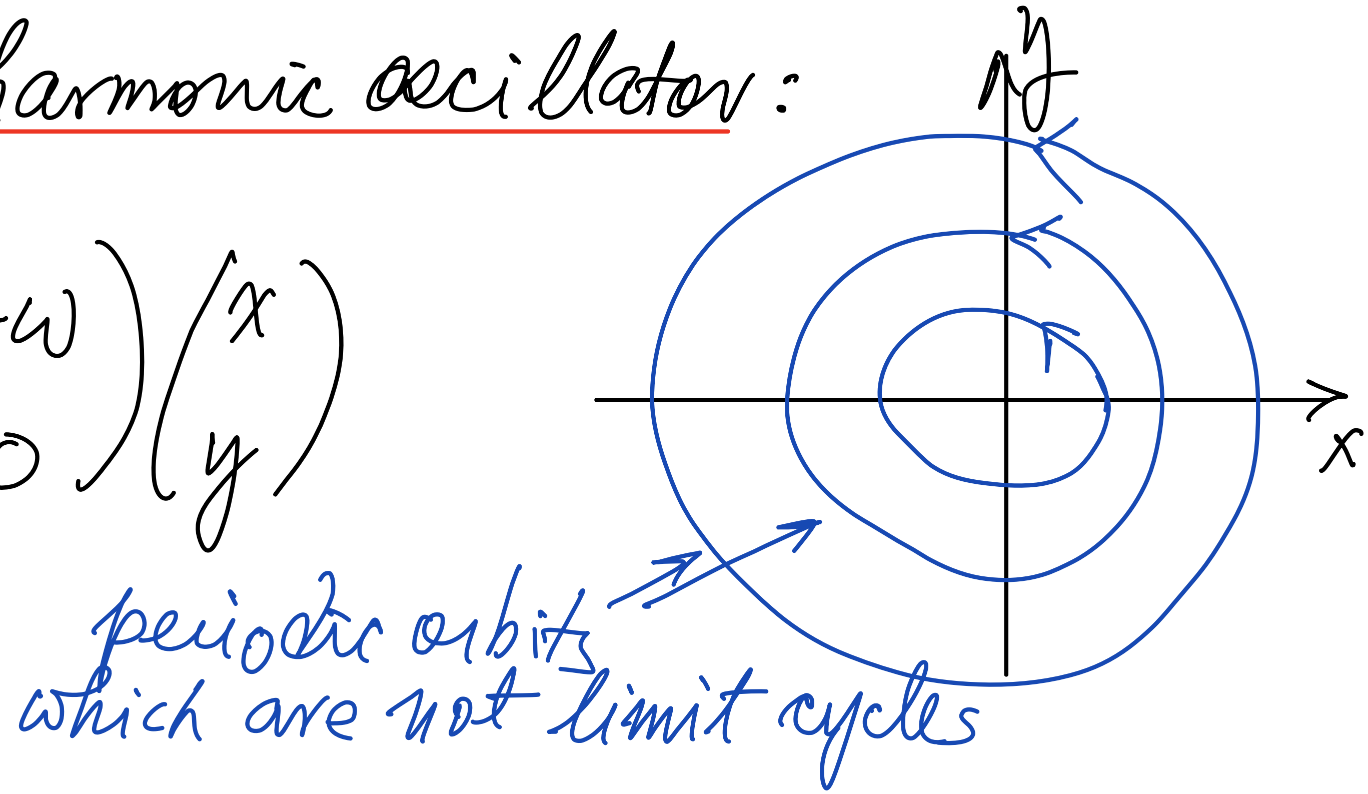
Note: not all periodic orbits are limit cycles

e.g. a center/harmonic oscillator:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

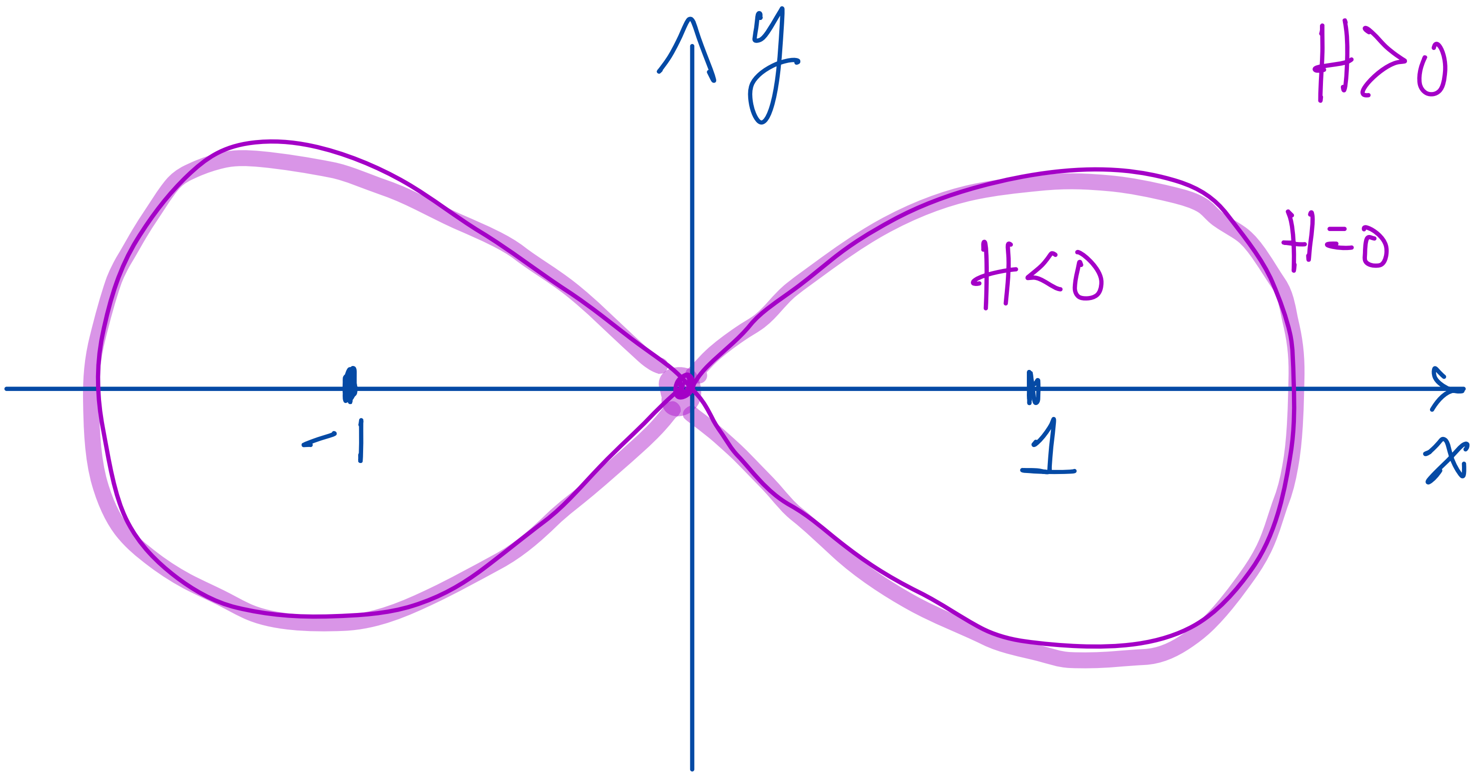


Example 4.43. Consider the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^3 - \mu y\left(y^2 - x^2 + \tfrac{1}{2}x^4\right). \end{aligned} \tag{4.46}$$

$\mu = 0$ Hamiltonian function:

$$H(x,y) = \frac{y^2}{2} + \frac{1}{2}\left(\frac{x^4}{2} - x^2\right)$$



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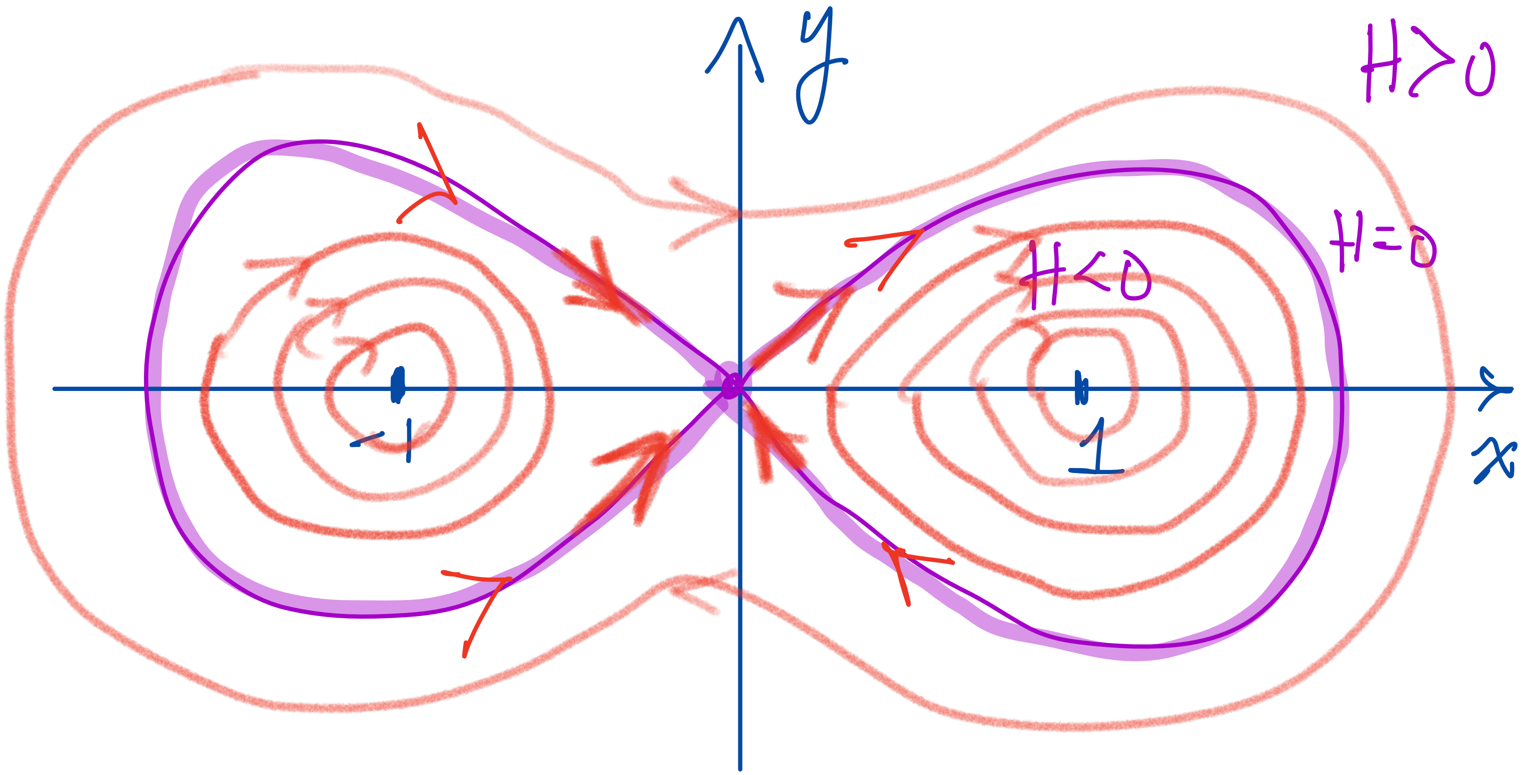

Example 4.43. Consider the system

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= x - x^3 - \mu y \overbrace{\left(y^2 - x^2 + \tfrac{1}{2}x^4\right)}^{H}.\end{aligned} \tag{4.46}$$

$\mu \neq 0$, $\mu > 0$ $\quad H(x,y) = \dfrac{y^2}{2} + \dfrac{1}{2}\left(\dfrac{x^4}{2} - x^2\right)$

$$\frac{dH}{dt}(x,y) = H_x \dot{x} + H_y \dot{y}$$

$$= \cancel{(x^3 - x)y} + y(\cancel{x - x^3} - \mu y H)$$

$$= -\mu y^2 H$$

$$= \begin{cases} < 0 & \text{if } H > 0 \\ > 0 & \text{if } H < 0 \end{cases}$$

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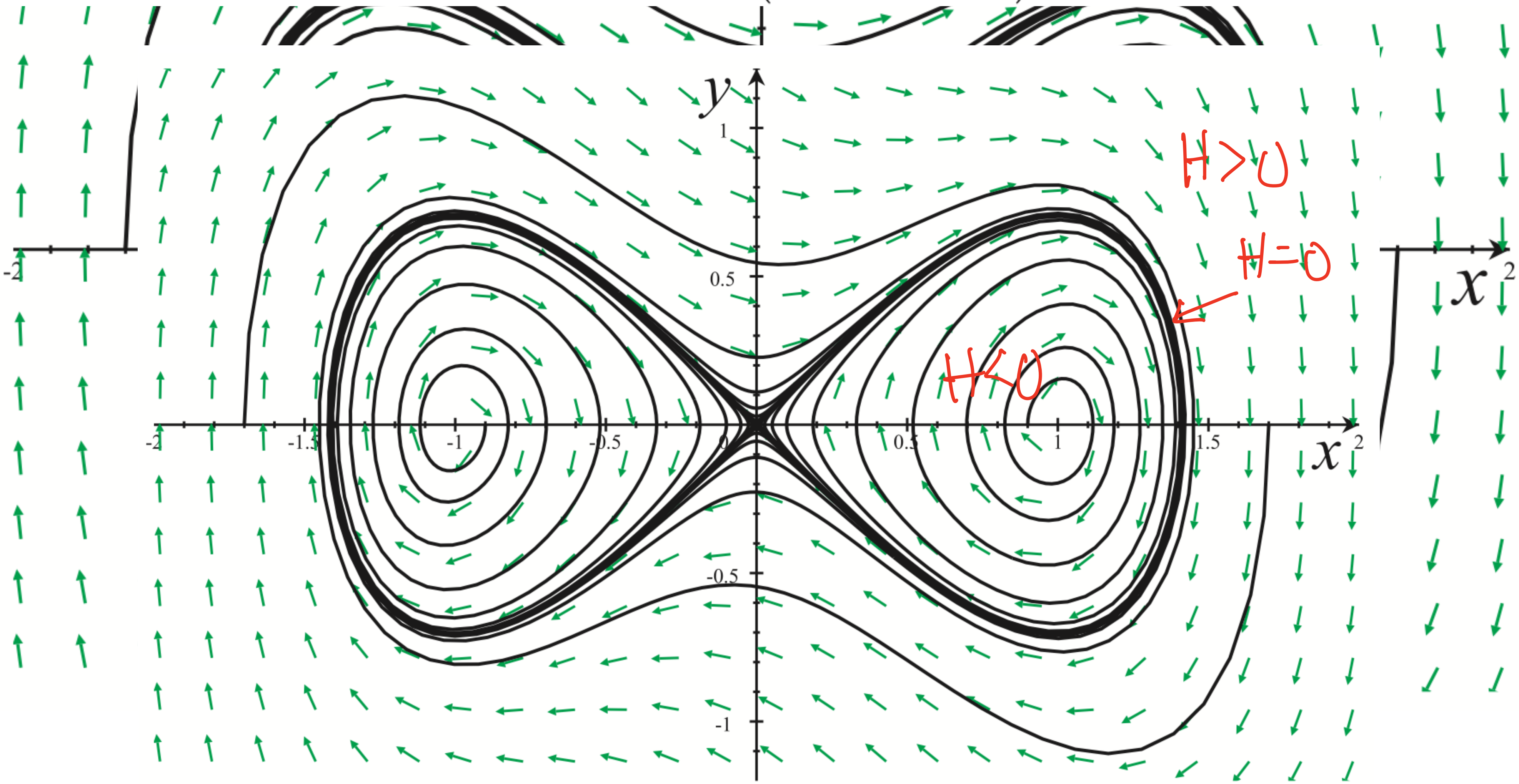

Figure 4.18. *Attracting figure-eight orbit of* (4.46) *for* $\mu = 0.5$.

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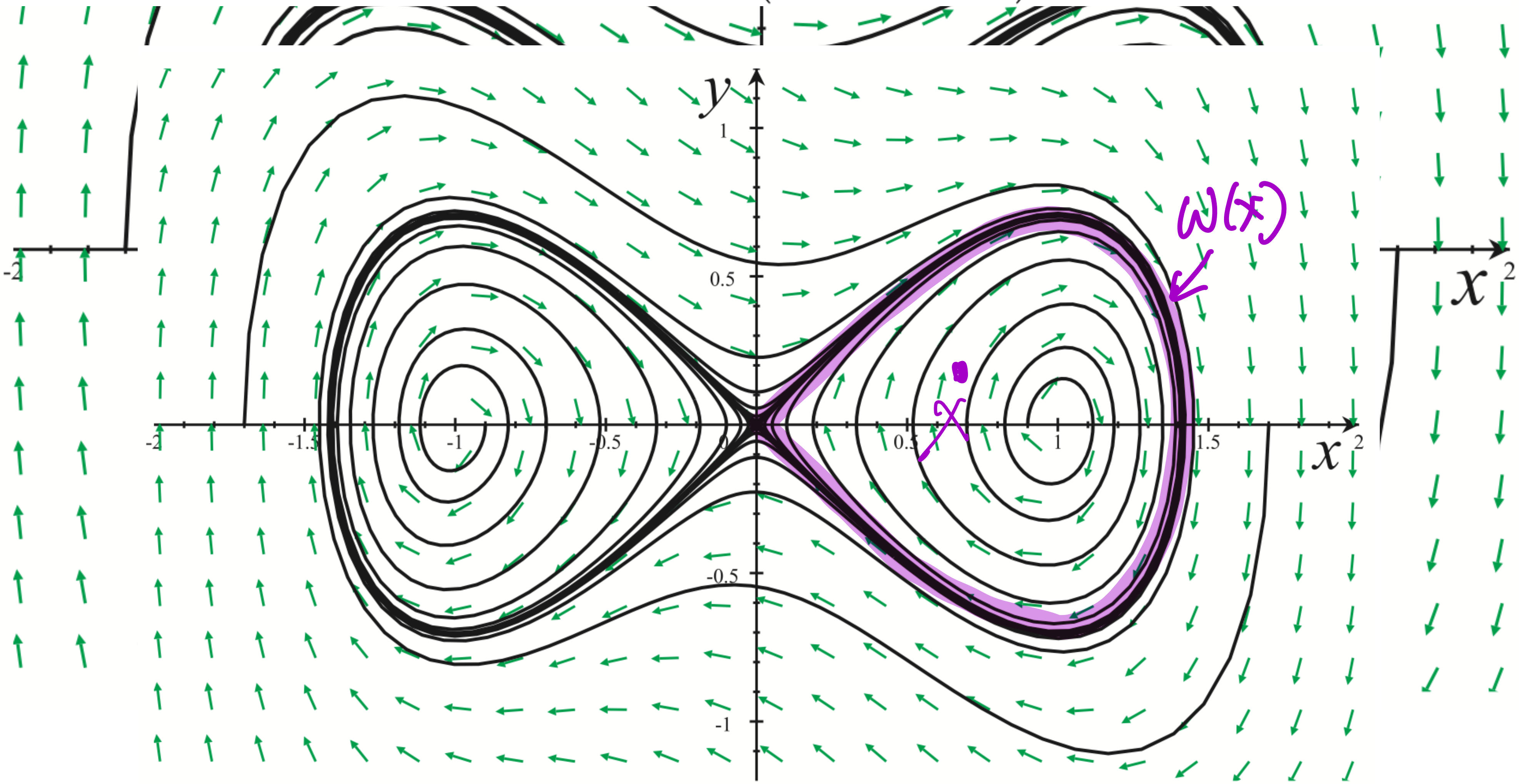

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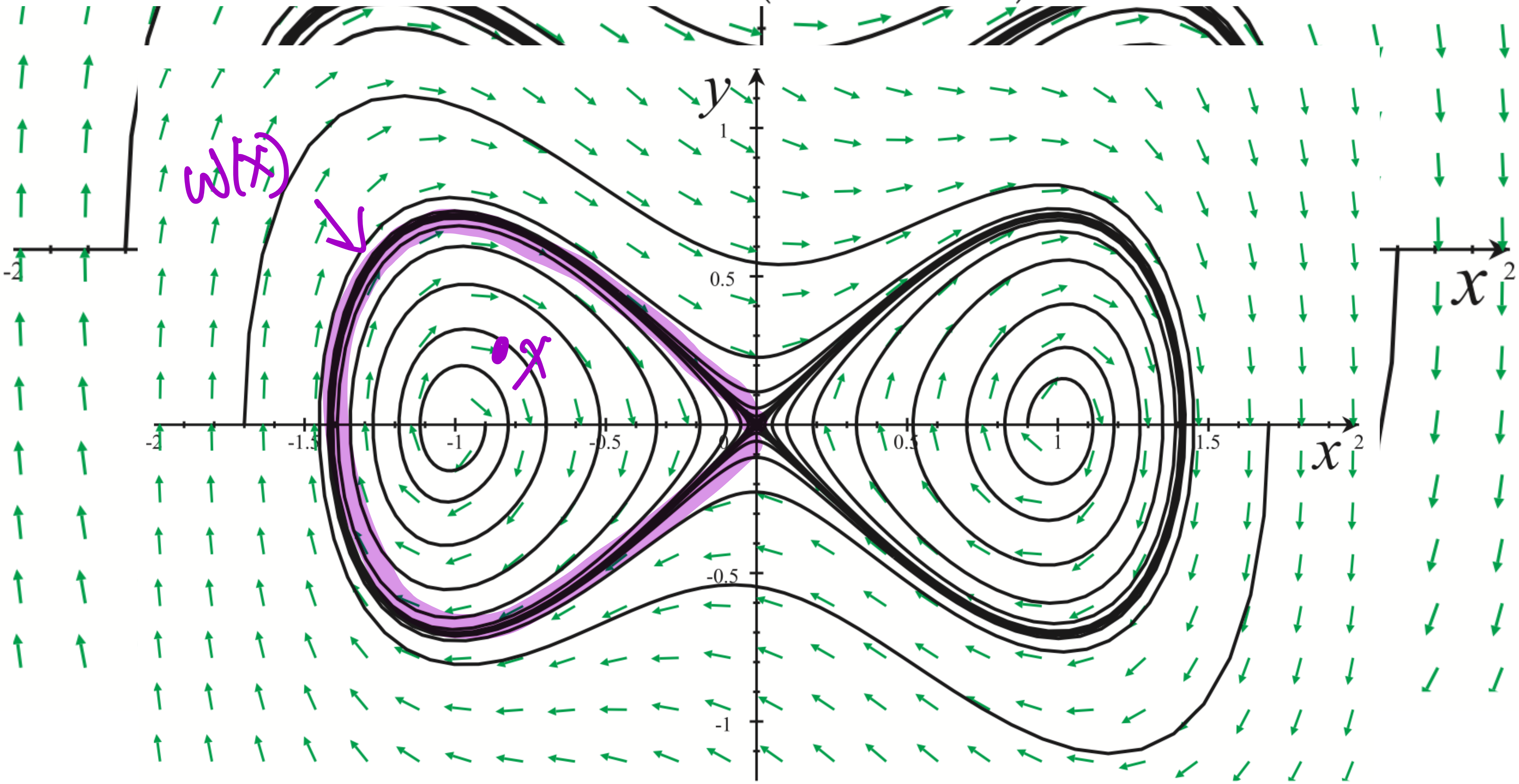

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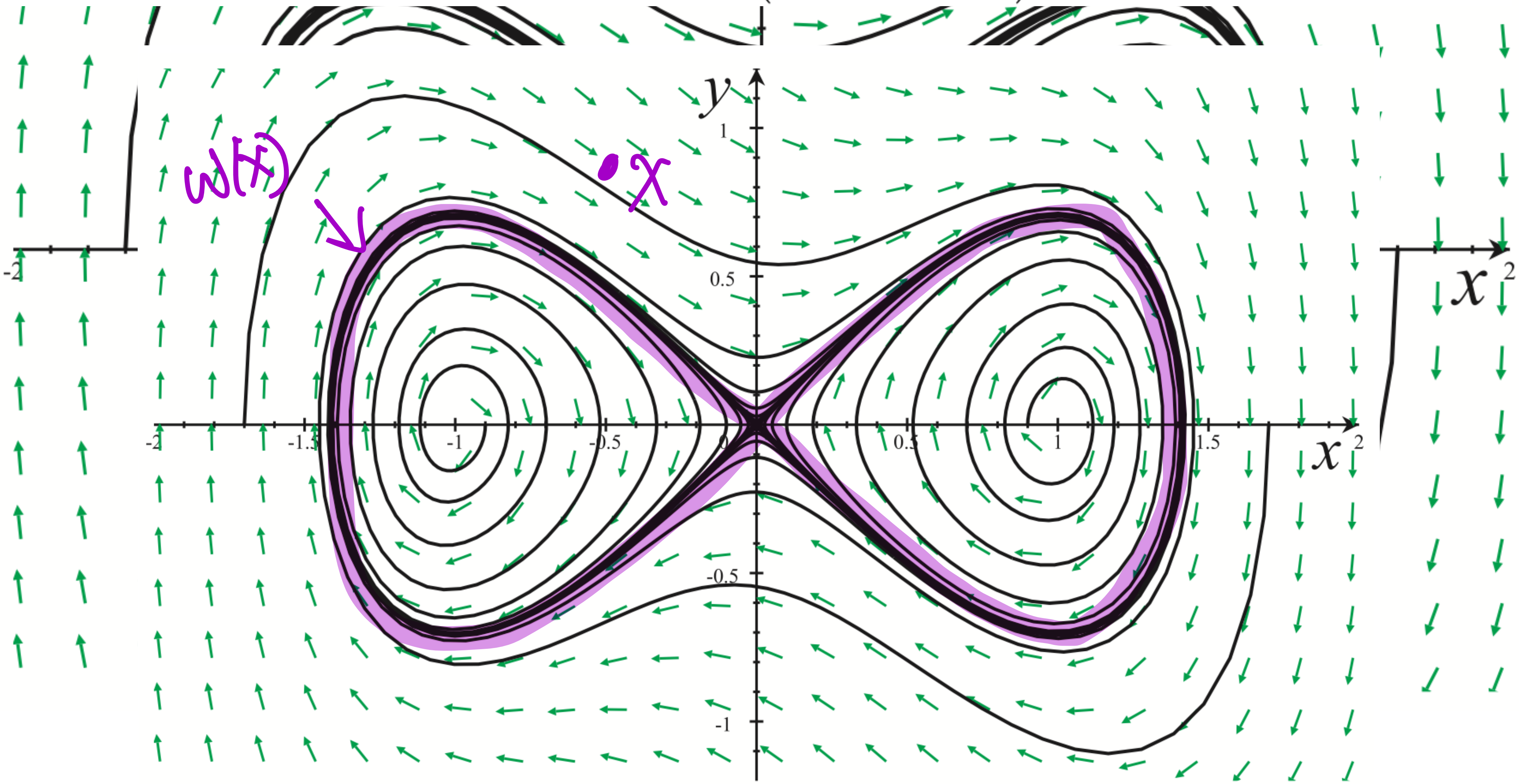


Figure 4.18. *Attracting figure-eight orbit of* (4.46) *for* $\mu = 0.5$.

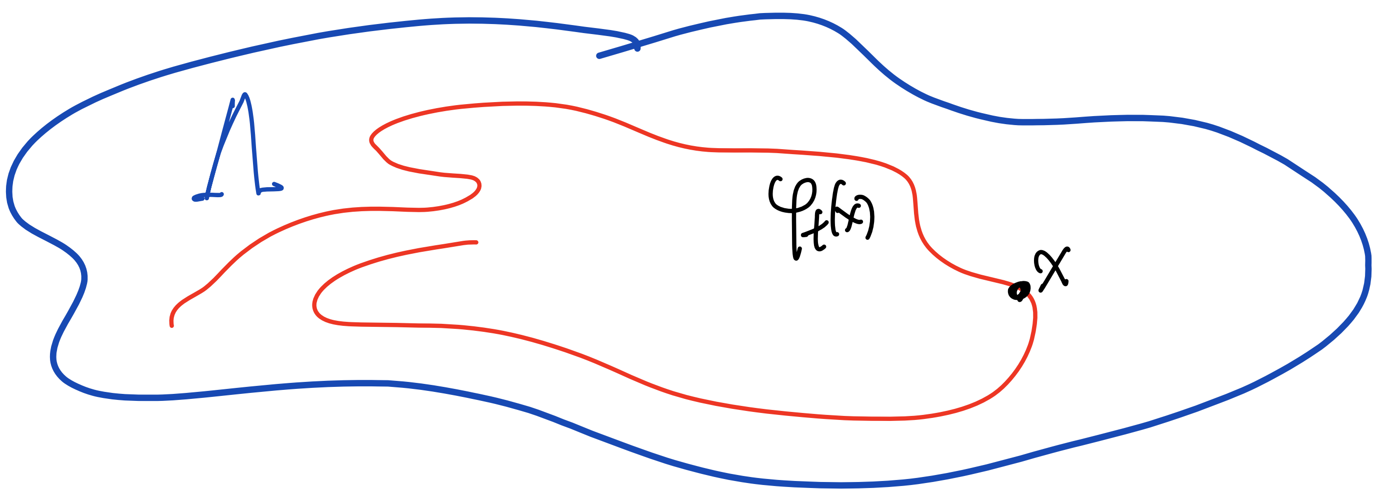
4 Invariant Set Λ

Λ is invariant if $\varphi_t(\Lambda) \subseteq \Lambda$,

i.e. $\forall x \in \Lambda$, $\varphi_t(x) \in \Lambda$.

Λ is forward invariant, if $\varphi_t(\Lambda) \subseteq \Lambda$, $t \geq 0$

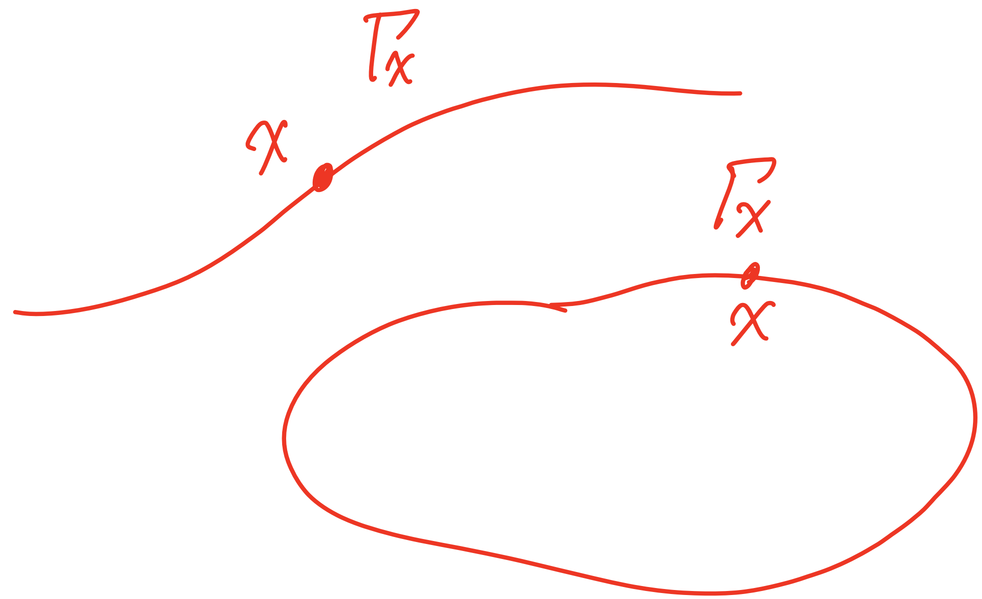
Λ is backward invariant if $\varphi_t(\Lambda) \subseteq \Lambda$, $t \leq 0$

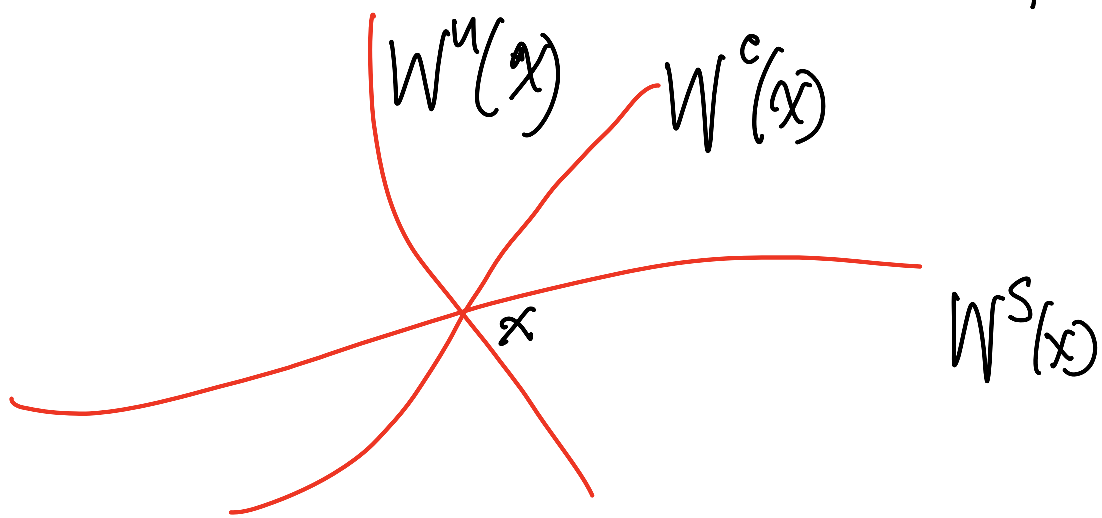


④ Invariant Set Λ

Examples

- Any orbit Γ_x,
- Periodic orbit
- Stable, unstable, center manifolds





4 Invariant Set Λ

Note that for any invariant set Λ,

$\{\varphi_t(\Lambda)\}_{t>0}$ is nested,

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pf for $t, s > 0$, $\Lambda \xrightarrow{\varphi_t} \varphi_t(\Lambda) \subseteq \Lambda$

$$\varphi_s(\Lambda) \subseteq \Lambda \xrightarrow{\varphi_t} \underbrace{\varphi_t(\varphi_s(\Lambda)) \subseteq \varphi_t(\Lambda)}$$

i.e. $\varphi_{t+s}(\Lambda) \subseteq \varphi_t(\Lambda)$

Lemma (M, 4.40) $\omega(x)$ & $\alpha(x)$ are closed sets.

[A set is closed if it contains all of its limit pts.]

Proof Let $y_1, y_2, y_3, \ldots \in \omega(x)$, & $y_i \longrightarrow y_*$

Claim: $y_* \in \omega(x)$

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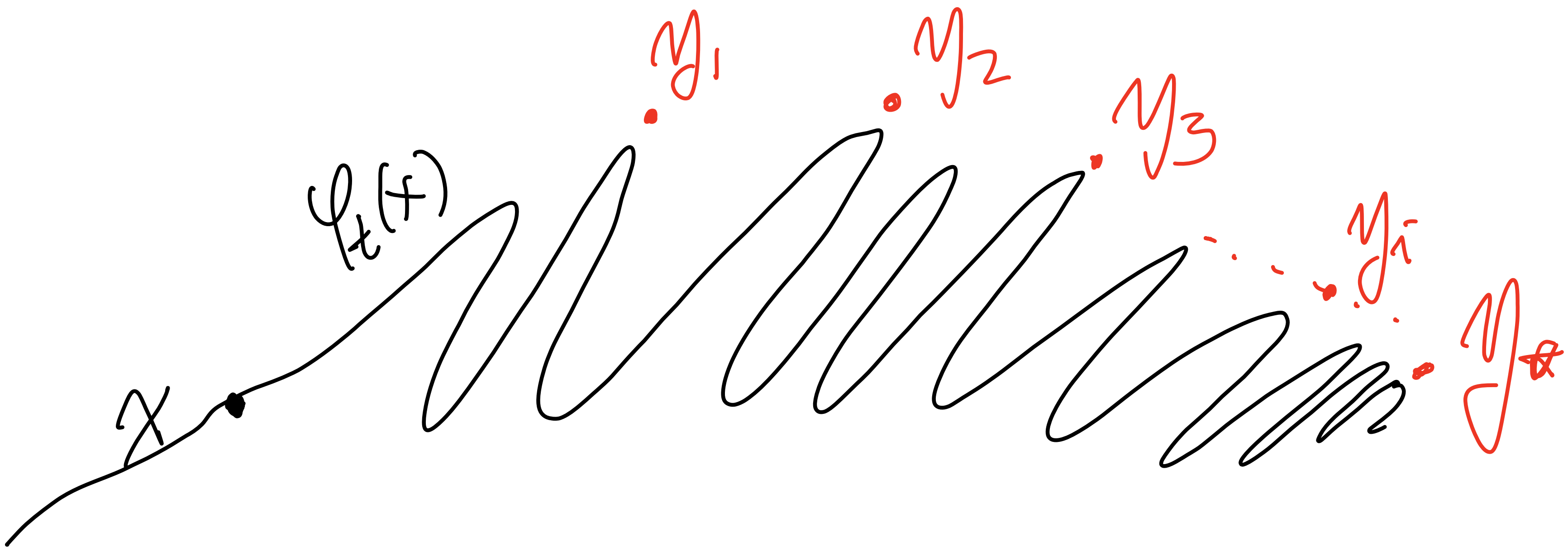

Lemma 4.41. $\omega(x)$, $\alpha(x)$ are invariant.

i.e. if $y \in \omega(x)$ $(\alpha(x))$, then $\varphi_t(y) \in \omega(x)$ $(\alpha(x))$

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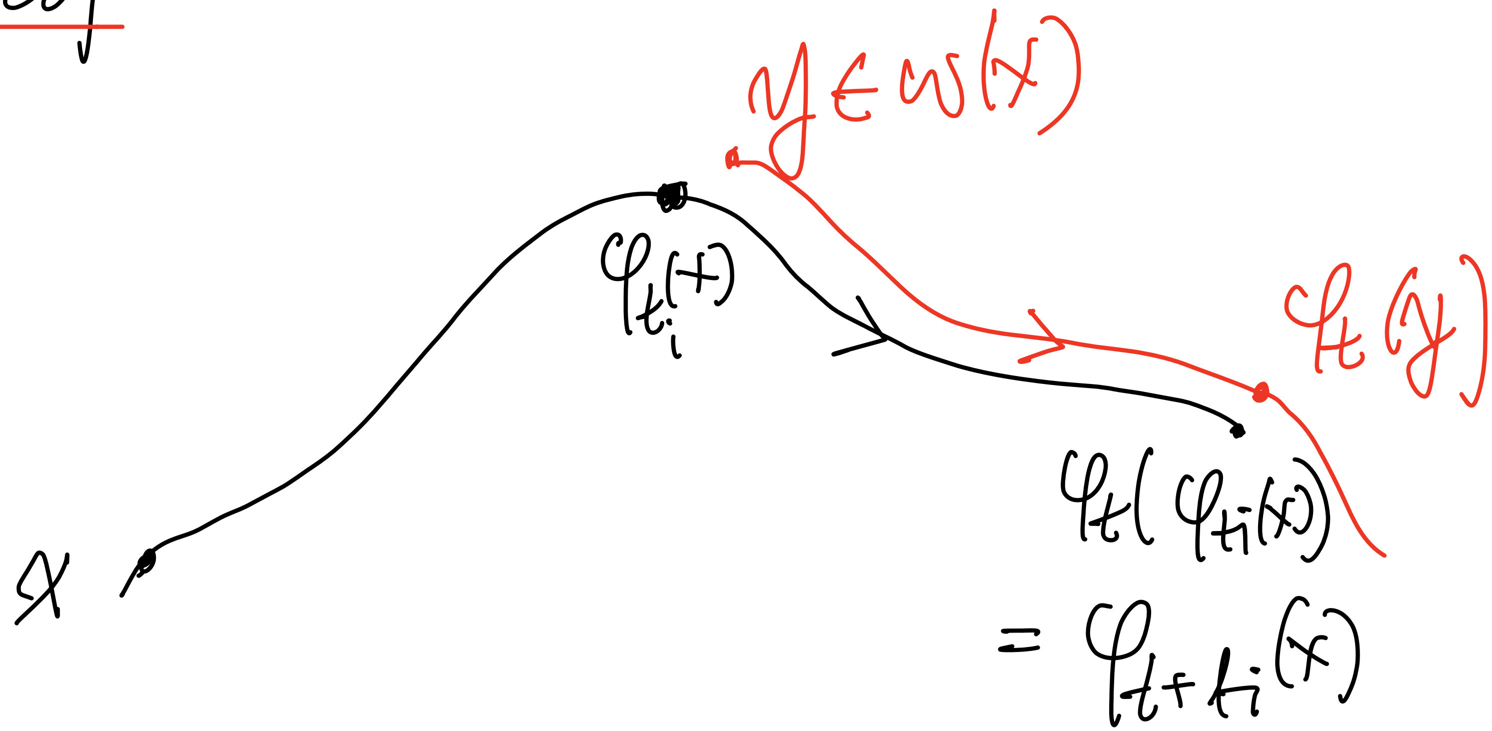

Lemma 4.42 (Compact, Connected & Convergence)

If $\Gamma_x^+ \subseteq K$ (a compact set), then

$\omega(x)$ is non-empty, compact, and connected.

Furthermore, $\varphi_t(x) \longrightarrow \omega(x)$

[Note: • A set K is compact if any sequence $\{x_i\}$ has a limit point in K.

• A set S is connected if it cannot be written as $S = A \cup B$ for two non-empty closed, disjoint subsets.]

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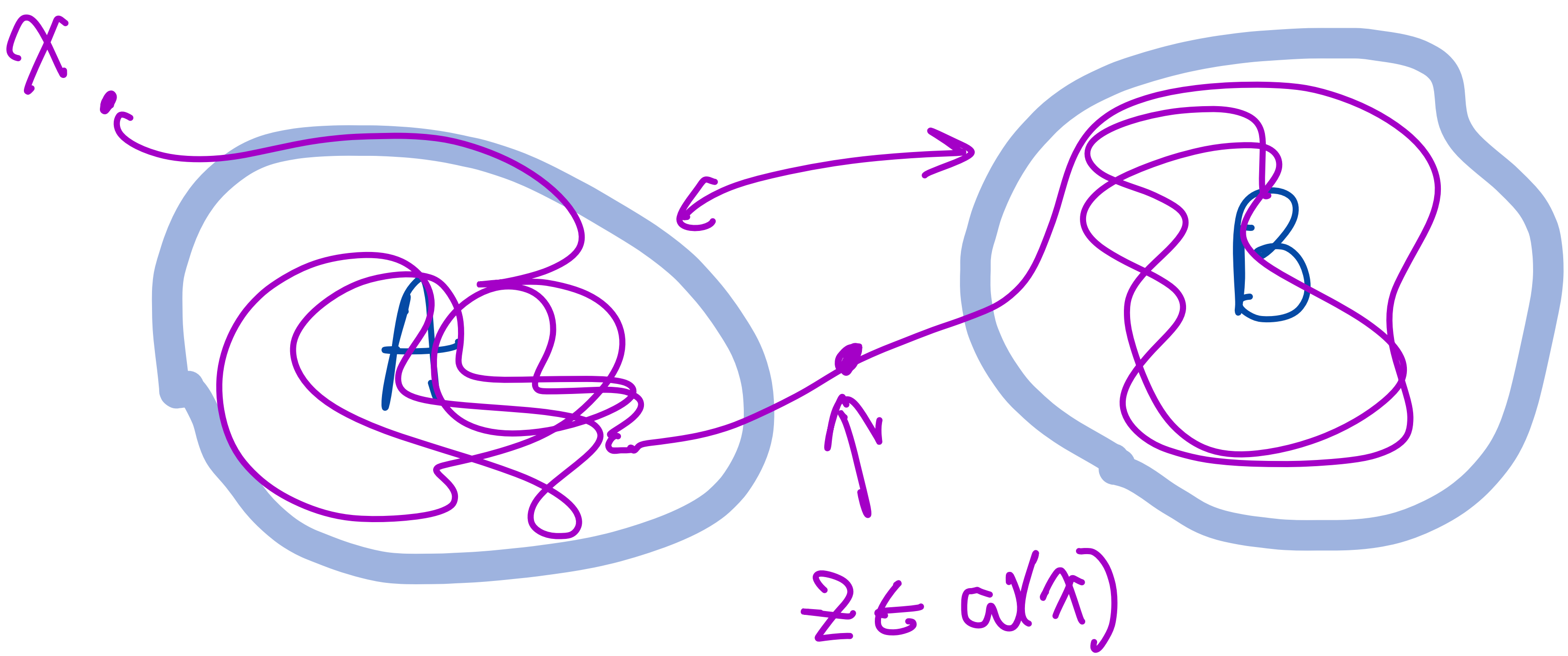
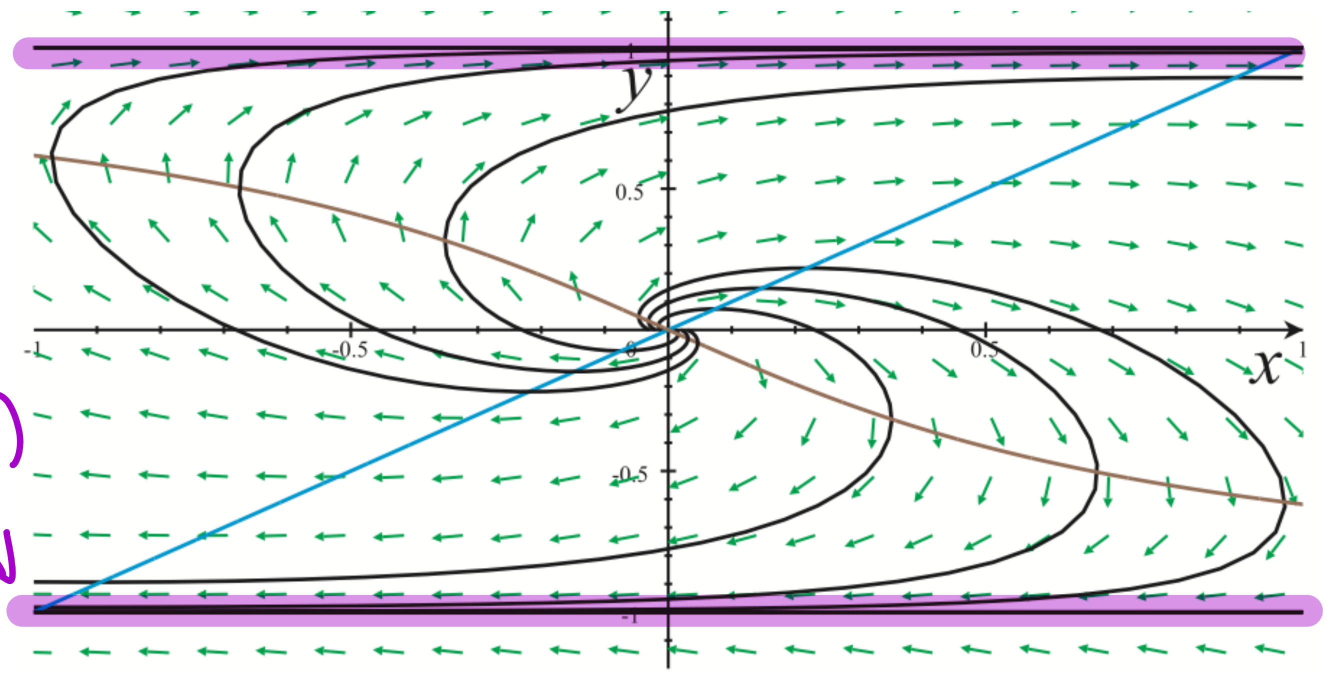


Example 4.44. Consider the system

$$\begin{aligned} \dot{x} &= y + x(1-y^2), \\ \dot{y} &= (1-y^2)(y-x). \end{aligned} \tag{4.47}$$



$W(x)$ (unbounded, disconnected)

$y=1$

$y=-1$

Figure 4.19. *Phase portrait of the system* (4.47), *showing the nullclines (blue and brown).*

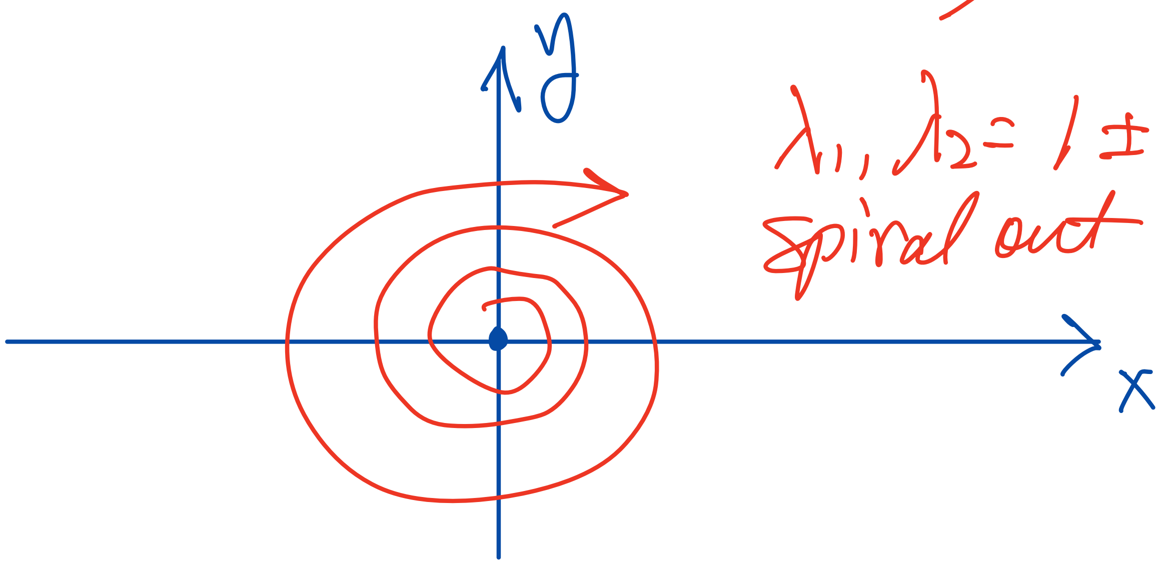
Example 4.44. Consider the system

$$\begin{aligned} \dot{x} &= y + x(1-y^2), \\ \dot{y} &= (1-y^2)(y-x). \end{aligned} \tag{4.47}$$

- $(0,0)$ is an eq. pt.
- Linearize: $\begin{cases} \dot{x} = x+y \\ \dot{y} = -x+y \end{cases} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$\lambda_1, \lambda_2 = 1 \pm i$

spiral out



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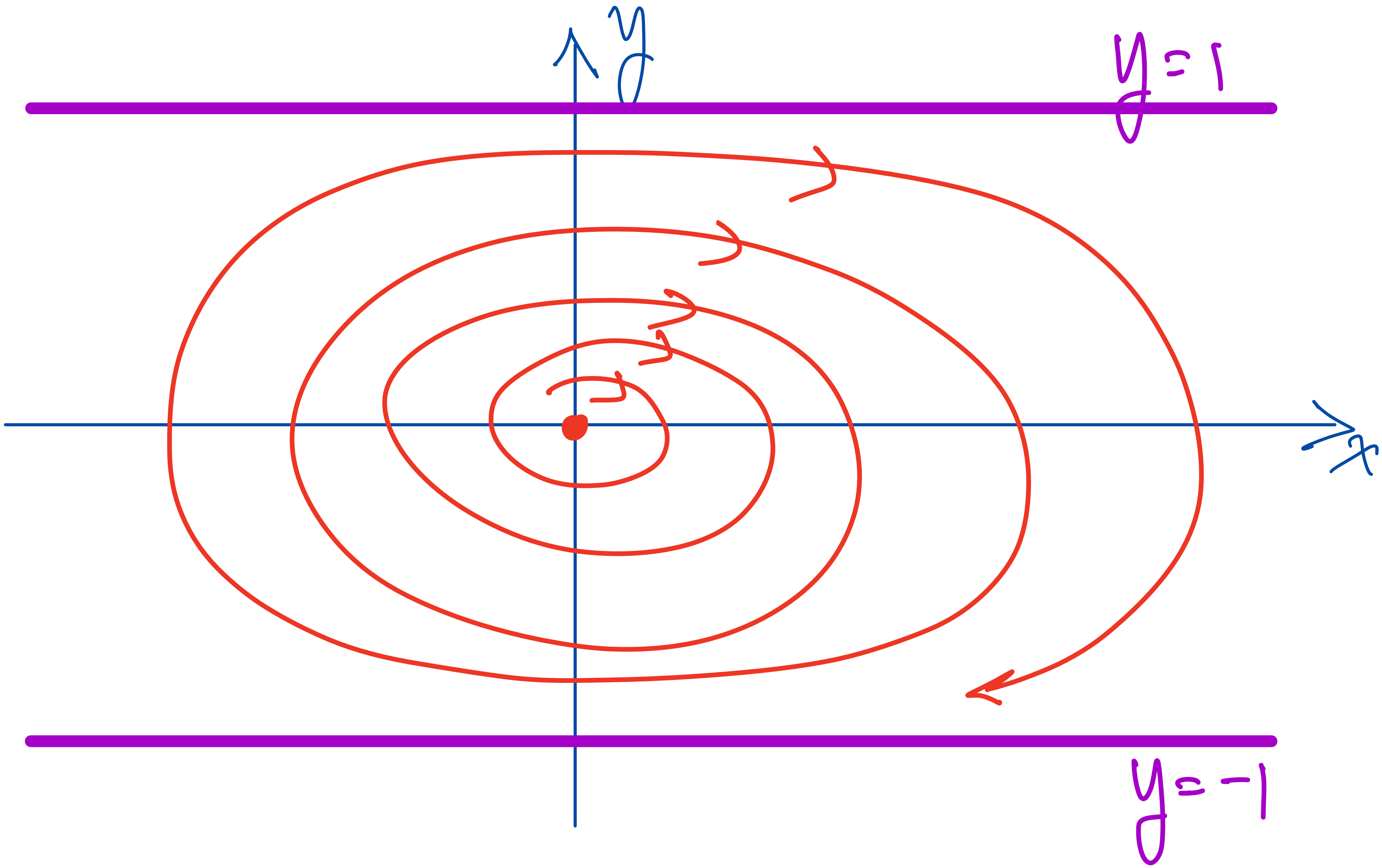
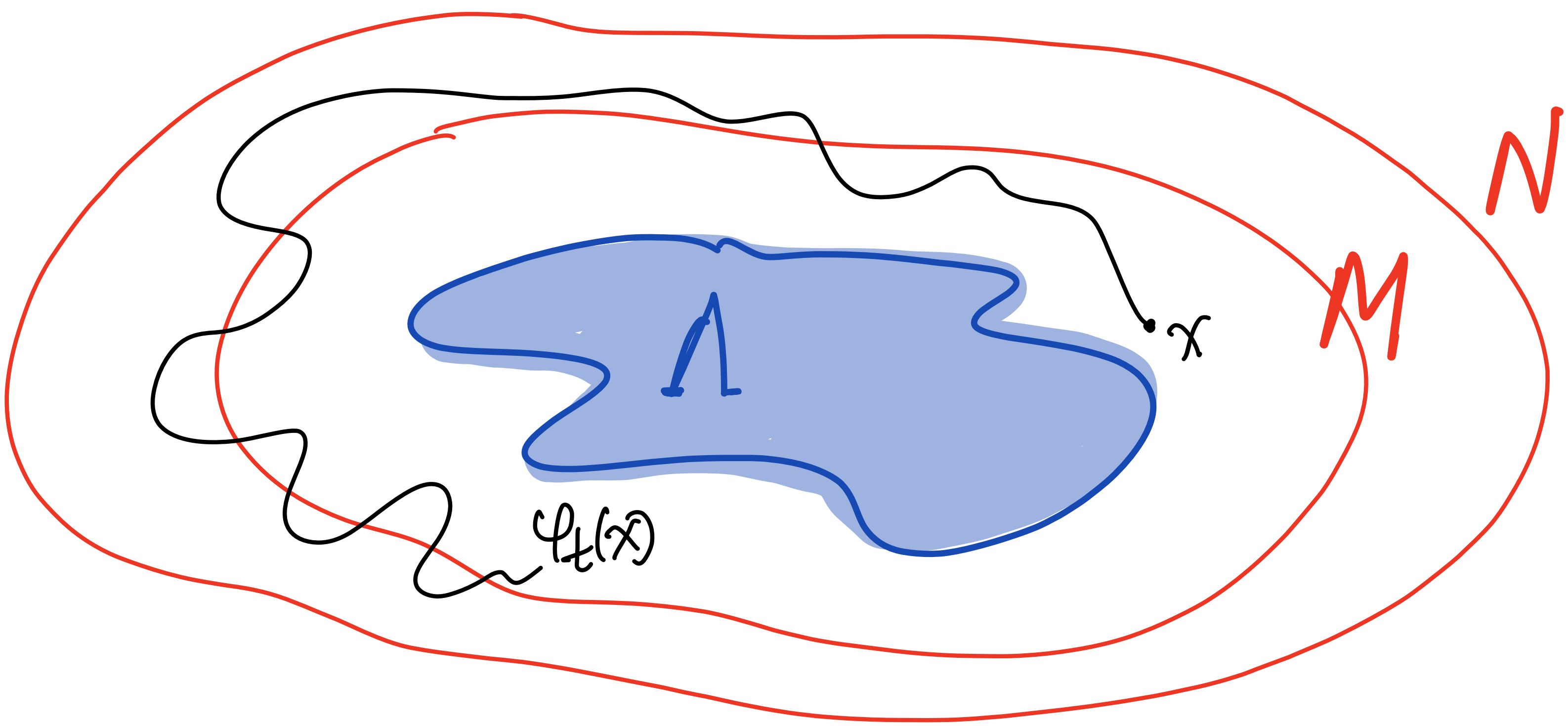


Attractors and Basins (M, 4.10)

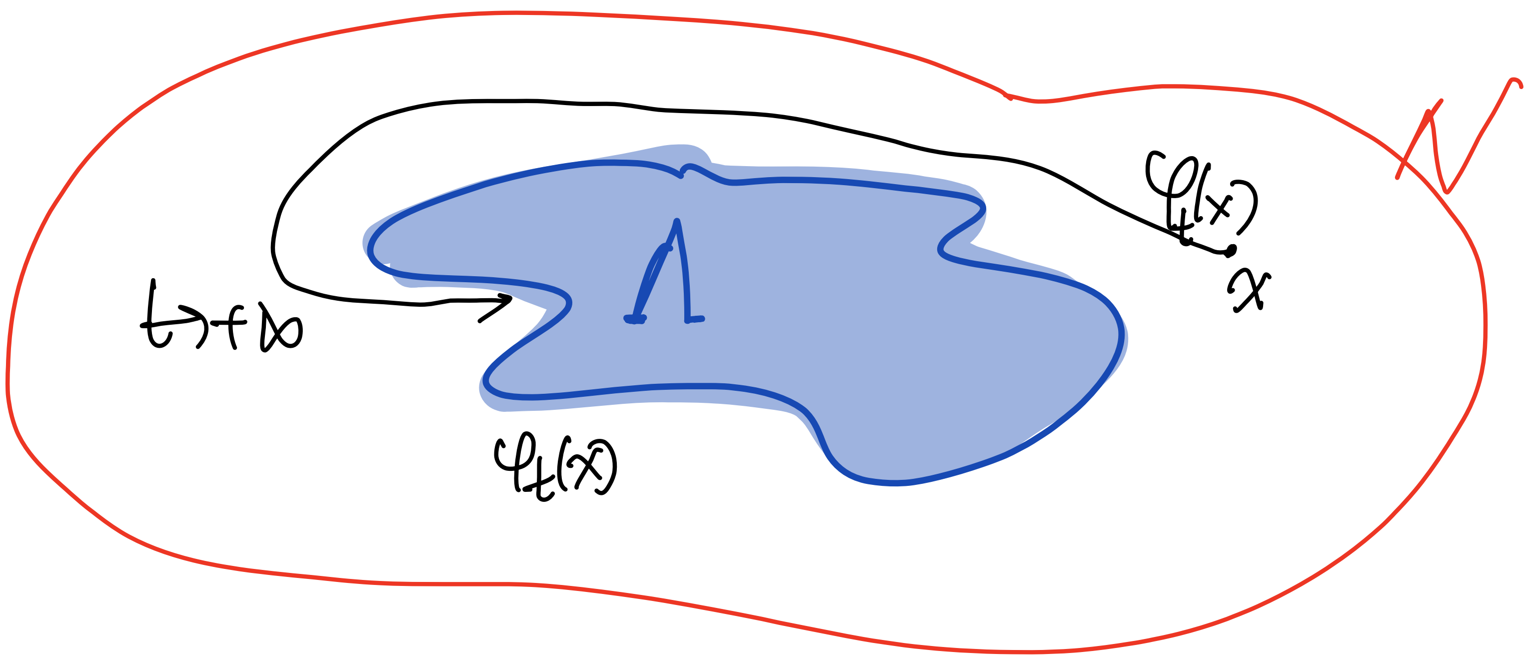
- An invariant set Λ is stable if for any a neighborhood N of Λ, there is a subset M s.t. $\Lambda \subseteq M \subseteq N$ & for $\forall x \in M$, $\varphi_t(x) \in N$, $t > 0$



Attractors and Basins $(M, 4.0)$

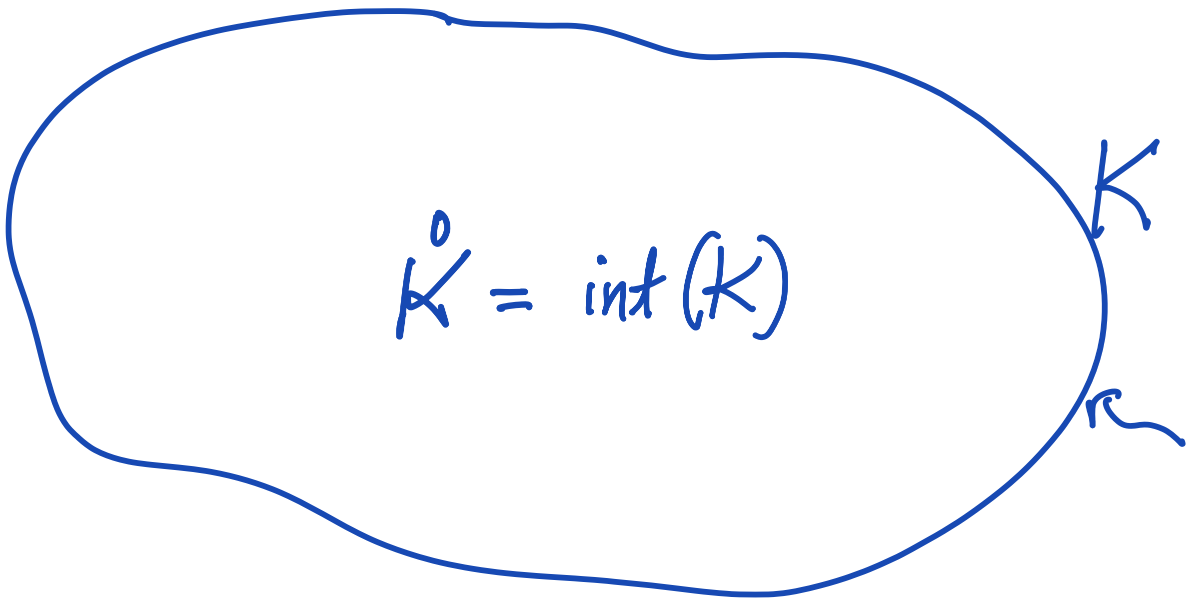
An invariant set Λ is asymptotically stable if there is a neighborhood N of Λ s.t.

$$\forall x \in N, \quad \varphi_t(x) \longrightarrow \Lambda \quad \text{as } t \longrightarrow +\infty$$



Attractors and Basins (M, 4.10)

A trapping region K is a compact set s.t. $\varphi_t(K) \subseteq \text{int}(K)$ $(\mathring{K})$ (interior of K)



$$\boxed{K = \partial K \cup \mathring{K}}$$

$\partial K = \text{bdry}(K)$

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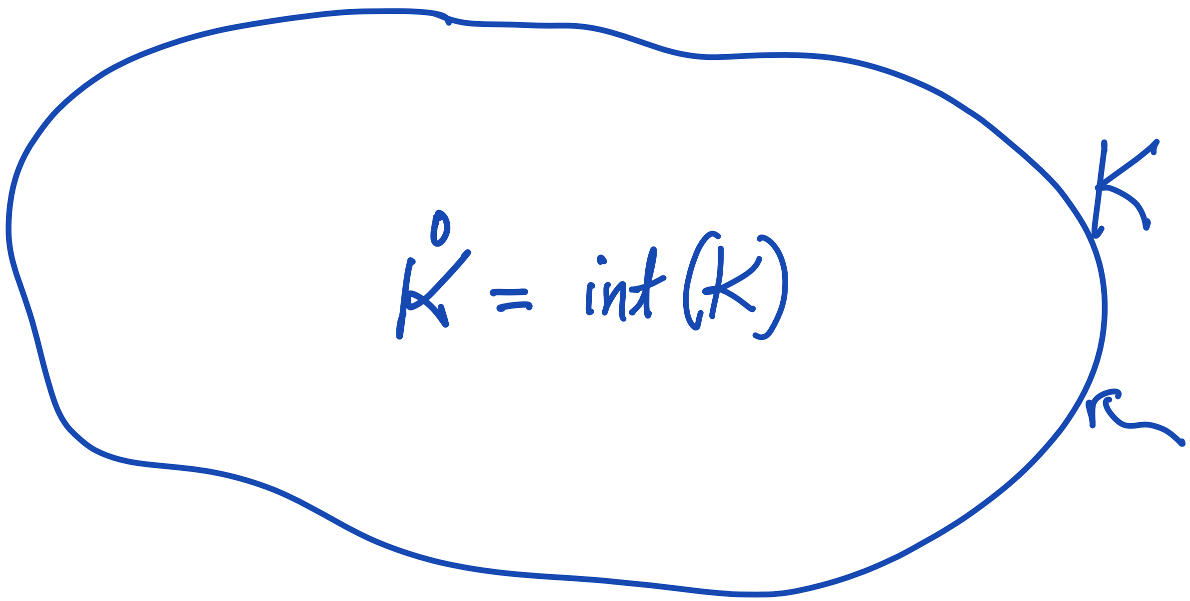


$$\boxed{K = \partial K \cup \mathring{K}}$$

$\partial K = \text{bdry}(K)$

Recall that $\{\varphi_t(K)\}$ is nested as K is invariant:
$\varphi_{t_2}(K) \subseteq \varphi_{t_1}(K)$ for $t_2 > t_1$.

Attractors and Basins (M, 4.10)

Attracting set: Λ is attracting set if there is a trapping region K s.t. $\Lambda = \bigcap_{t>0} \varphi_t(K)$

Basin of attraction. Λ – invariant set.

Basin of Λ $(W^s(\Lambda)) = \{x : \varphi_t(x) \longrightarrow \Lambda\}$

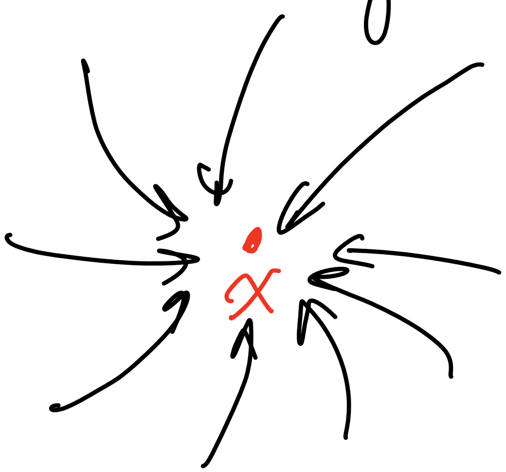
Lemma 4.46. An attracting set is asymptotically stable. Conversely, if a compact set K is asymptotically stable, then it is an attracting set.

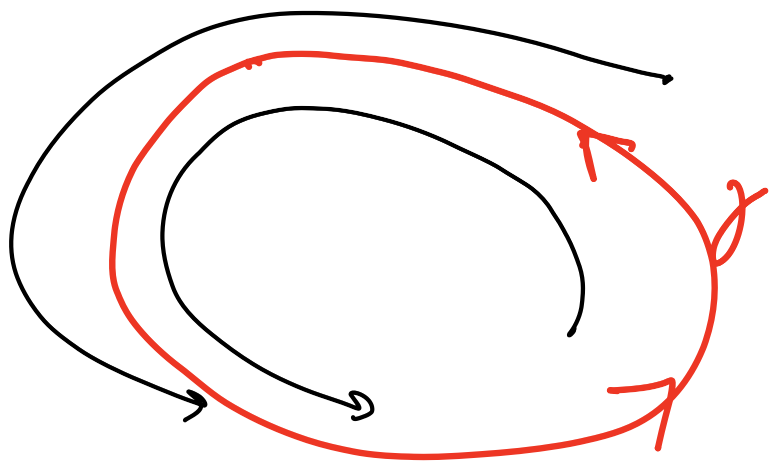
Attractors and Basins (M, 4.10)

Attractor: Λ is an attractor if it is an attracting set, and $\exists x$ s.t. $\Lambda = \omega(x)$

Examples

A stable equilibrium pt., A stable periodic orbit





Example 4.24. The Lorenz system, (1.33), is

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= r x - y - x z, \\ \dot{z} &= x y - b z, \end{aligned} \tag{4.26}$$

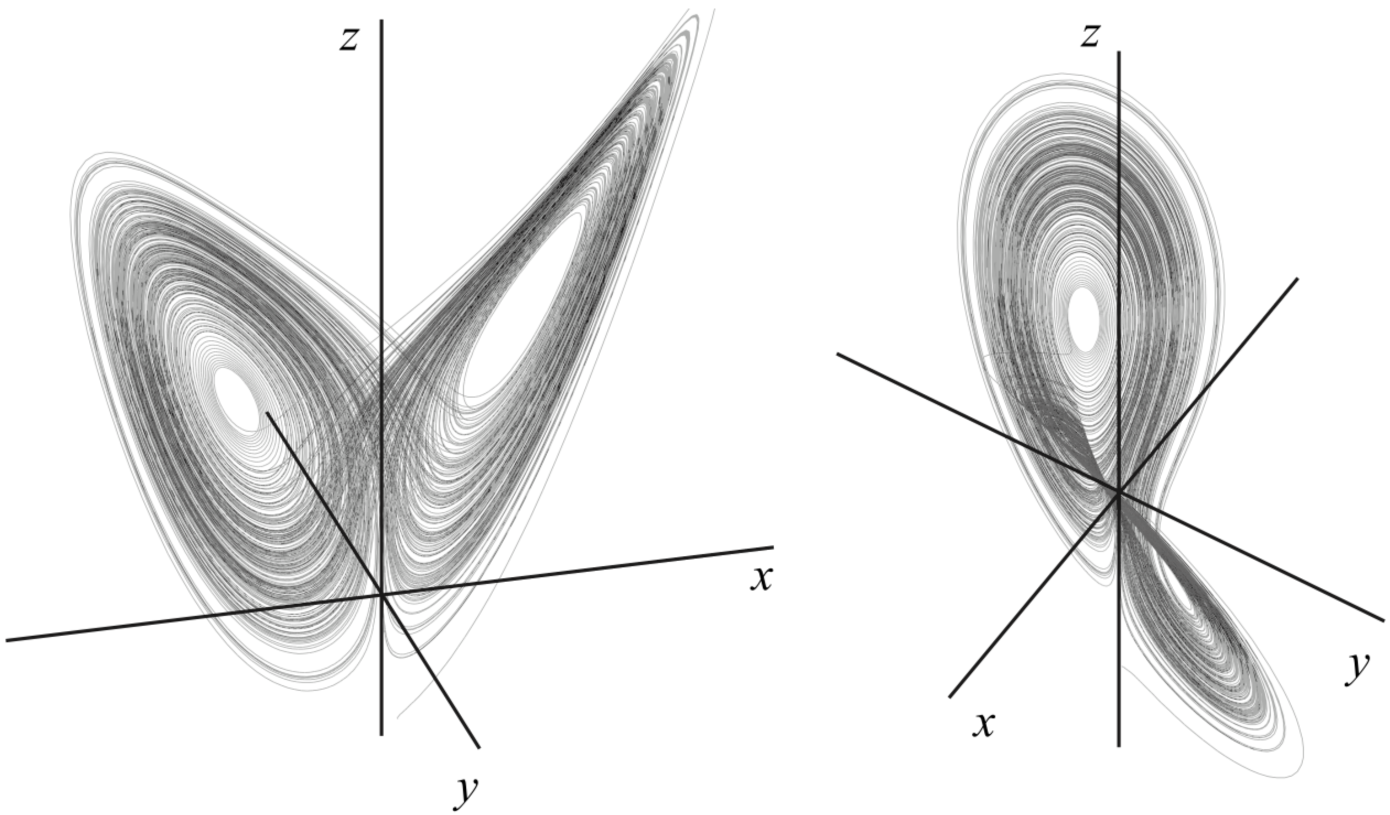


Figure 4.20. *Two views of a numerical approximation of the Lorenz Attractor for* $(\sigma, b, r) = (10, 8/3, 28)$*. The axes shown are centered at* $(0,0,20)$ *and are of length* 50.