Weakly Perturbed (Nonlinear) Oscillators

(I) Harmonic oscillator with periodic driving force

$$\ddot{x} + \omega_0^2 x = f(t) \quad (= \cos \omega t)$$

(1) Homogeneous version:

$$\ddot{x} + \omega_0^2 x = 0$$

General solution

$$x(t) = A \cos \omega_0 t + B \sin \omega_0 t$$

(A, B determined by init. cond. $x(0), \dot{x}(0)$)

System: $\begin{cases} \dot{x} = y \\ \dot{y} = -\omega_0^2 x \end{cases}$ $\quad \begin{pmatrix} x \\ y \end{pmatrix}' = \underbrace{\begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$

Fundamental Matrix

$$[\Phi(t)] = \begin{bmatrix} \cos\omega_0 t & \frac{1}{\omega_0}\sin\omega_0 t \\ \underbrace{-\omega_0 \sin\omega_0 t}_{\text{Solution}_1} & \underbrace{\cos\omega_0 t}_{\text{Solution}_2} \end{bmatrix}$$

$$\left([\Phi(0)] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \right)$$

② $\ddot{x} + \omega_0^2 x = \cos\omega t$ ← $T = \frac{2\pi}{\omega}$ per

General solution:

homog. soln $A\cos\omega_0 t + B\sin\omega_0 t$

$$x(t) = x_h(t) + x_p(t)$$

particular soln

If $\omega \neq \omega_0$, then $x_p(t) = \frac{1}{\omega_0^2 - \omega^2}\cos\omega t$

$$x(t) = \cancel{A}\cos\omega_0 t + \cancel{B}\sin\omega_0 t + \frac{1}{\omega_0^2 - \omega^2}\cos\omega t$$

(Choose init. cond. s.t. $A = B = 0$)

Hence there is a unique $T = \frac{2\pi}{\omega}$ per soln:

$$\boxed{x(t) = \left(\frac{1}{\omega_0^2 - \omega^2}\right) \cos \omega t}$$

If $\underline{\omega = \omega_0}$ (Resonance)

then $x_p(t) = t(C \cos \omega_0 t + D \sin \omega_0 t)$

$$C \overset{?}{=} \qquad D \overset{?}{=}$$

$$2(-\cancel{C}\omega_0 \sin \omega_0 t + \cancel{D}\omega_0 \cos \omega_0 t)$$

$$+ t(-\cancel{C}\omega_0^2 \cos \omega_0 t - \cancel{D}\omega_0^2 \sin \omega_0 t)$$

$$+ \omega_0^2 (t\cancel{C} \cos \omega_0 t + \cancel{tD} \sin \omega_0 t) = \cos \omega_0 t$$

$C = 0, \quad D = \frac{1}{2\omega_0}$

$$x(t) = A \cos \omega_0 t + B \sin \omega_0 t + \underline{\frac{1}{2\omega_0} t \cos \omega_0 t}$$

No periodic Solution

③ From perturbation and Monodromy matrix perspective. [H, Ch. XII, Thm 2.3]

Suppose $\frac{dX}{dt} = F(X)$ has a T-per. solution

T-per.

Then $\frac{dX}{dt} = F(X) + \mu g(t, X)$

has a T-per. solution for $|\mu| \ll 1$

if $(\Phi(T) - I)$ invertible,

i.e. $(\Phi(T) - I)^{-1}$ exists

i.e. $\lambda = 1$ is not an eigenvalue of $\Phi(T)$

Now

$$\Phi(t) = \begin{bmatrix} \cos\omega_0 t & \frac{1}{\omega_0}\sin\omega_0 t \\ -\omega_0 \sin\omega_0 t & \cos\omega_0 t \end{bmatrix}$$

$$\Phi(T) = \begin{bmatrix} \cos\omega_0 T & \frac{1}{\omega_0}\sin\omega_0 T \\ -\omega_0 \sin\omega_0 T & \cos\omega_0 T \end{bmatrix}$$

$$\lambda_1, \lambda_2 = \cos\omega_0 T \pm i \sin \omega_0 T$$

We need $\lambda_1, \lambda_2 \neq 1$ i.e. $\omega_0 T \neq 2\pi$

or $\underline{\omega \neq \omega_0}$

④ $$\ddot{x} + \omega_0^2 x = \cos \omega t$$

$(\omega \neq \omega_0)$ Periodic solution $x(t) = \dfrac{1}{\omega_0^2 - \omega^2} \cos\omega t$

Stability? $|\lambda_1| = |\lambda_2| = 1$

"Non-hyperbolic"

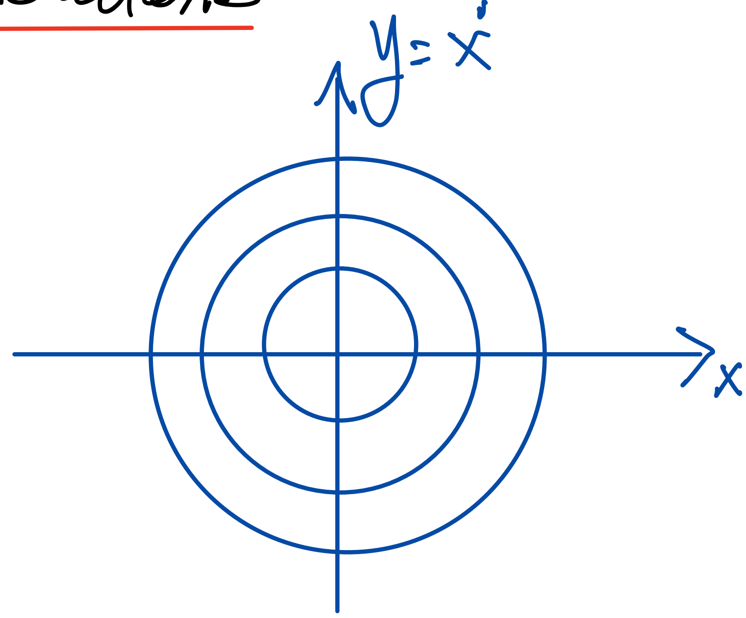
cannot be easily determined by λ_1, λ_2 only

(Use exact solution formula:

$$x(t) = A\cos\omega_0 t + B \sin \omega_0 t + \frac{1}{\omega_0^2 - \omega^2}\cos\omega t)$$

(II) Small perturbations

$$\ddot{x} + x = 0$$

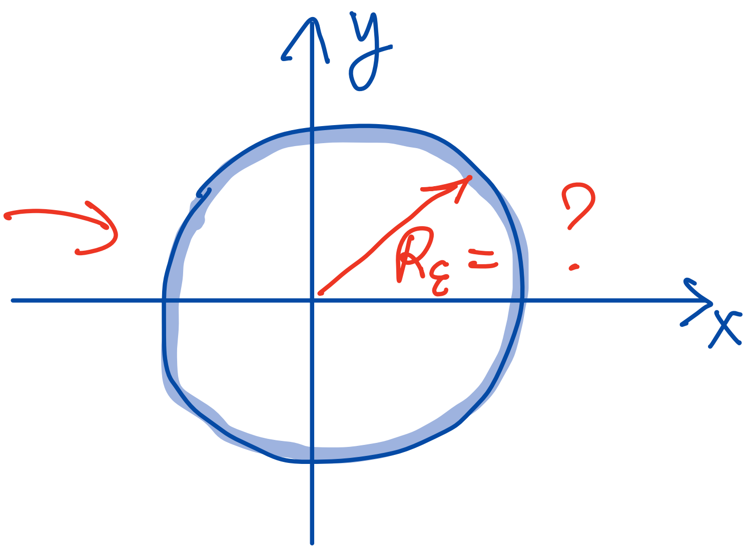


van der Pol $\qquad \ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0$

$\varepsilon \ll 1$

Unique per. soln

(Lienard Thm)



Duffing oscillator (Simple case)

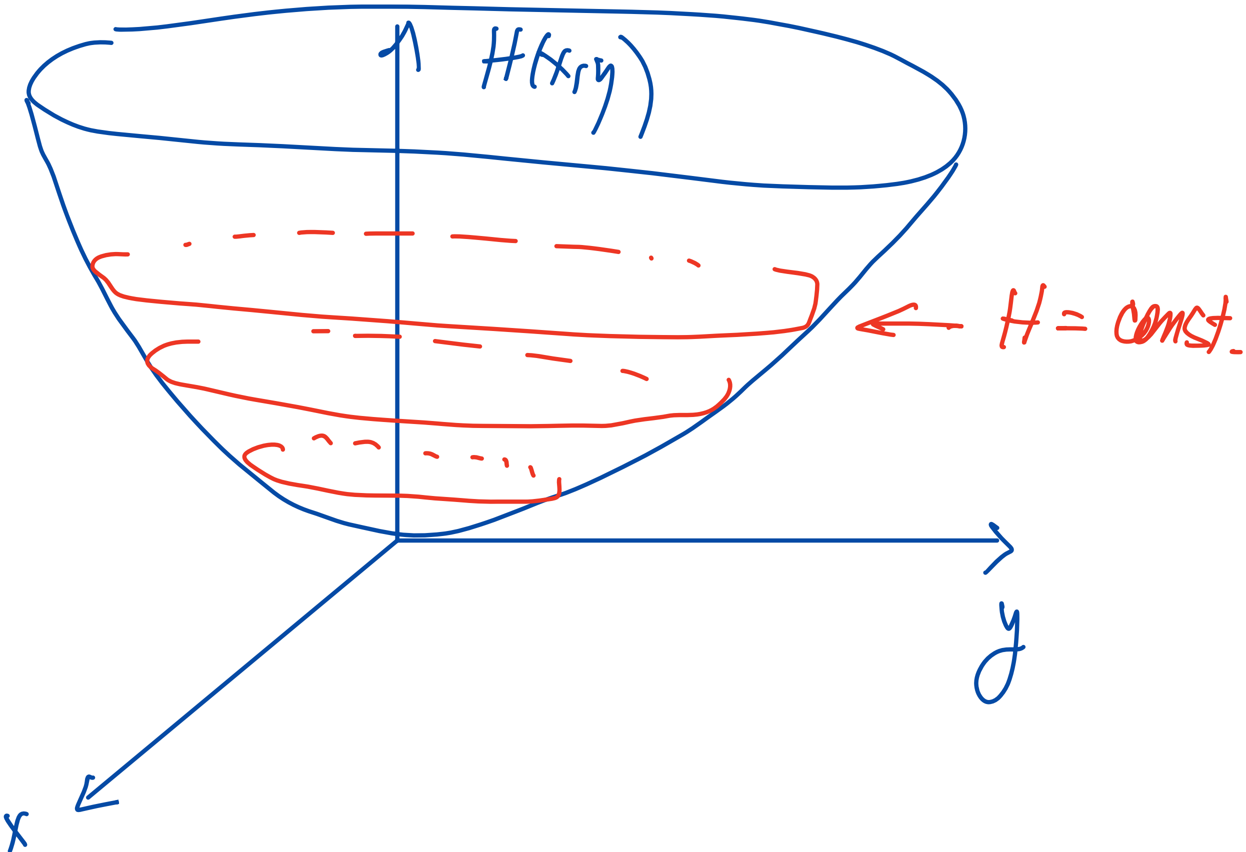
$$\ddot{x} + x + \varepsilon x^3 = 0$$

still a Hamiltonian System

$$H(x,y) = \frac{1}{2}y^2 + \left(\frac{1}{2}x^2 + \frac{1}{4}\varepsilon x^4\right)$$

$$\dot{x} = H_y \quad (= y)$$

$$\dot{y} = -H_x \quad (= x + \varepsilon x^3)$$



Can we compare the periodic orbits of

$$\ddot{x} + x = 0 \quad \text{and} \quad \ddot{x} + x + \varepsilon x^3 = 0$$

Regular Perturbation (and its failure)

1. Consider $\ddot{x} + x = 0$ (1)

(Solution is denoted by $x_0(t)$)

vs $\ddot{x} + (1+\varepsilon)x = 0$ (2)

We "hope" to write/expand solution (2) as a Taylor series in ε

$$(*) \quad x(t) = x_0(t) + \varepsilon \underbrace{x_1(t)} + \varepsilon^2 \underbrace{x_2(t)} + \cdots$$

solution of (1)

uniformly small for all t

plug it in:

$$\ddot{x} + (1+\varepsilon)x = 0$$

$$(\ddot{x}_0 + \varepsilon \ddot{x}_1 + \varepsilon^2 \ddot{x}_2 + \cdots)$$

$$+ (1+\varepsilon)(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) = 0$$

$O(1)$ $\qquad \ddot{x}_0 + x_0 = 0 \qquad x_0(t) \overset{\text{eg.}}{=} A\sin t$

$O(\varepsilon)$ $\qquad \ddot{x}_1 + x_1 + x_0 = 0$

$$\ddot{x}_1 + x_1 = -x_0 = -A\sin t$$

resonance

$$x_1(t) = \frac{A}{2}\, t\cos t$$

Hence:

$$x(t) = x_0(t) + \varepsilon\, x_1(t) + \cdots$$

$$= A\sin t + \underbrace{\varepsilon \frac{A}{2}\, t\cos t}_{} + \cdots$$

① is small for t small, $\ll \frac{1}{\varepsilon}$

② Can be big for $t \sim \frac{1}{\varepsilon}$

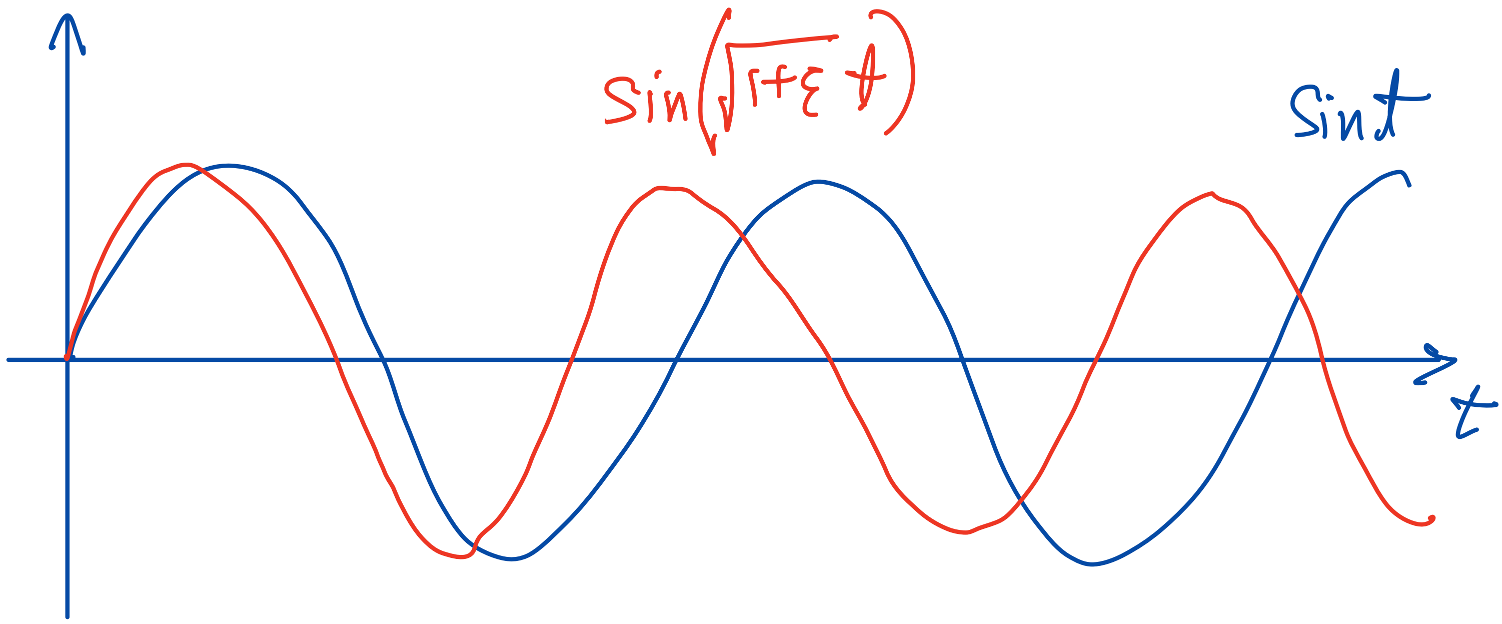
We "know" that the expansion (*) will not work.

Exact Solutions:

$x_0(t) = A\sin t$, $\quad x(t) = A\sin\left(\sqrt{1+\varepsilon}\,t\right)$

It cannot be true that

$$\left|\sin\left(\sqrt{1+\varepsilon}\,t\right) - \sin t\right| \lesssim \varepsilon \text{ for all } t$$



2. $\ddot{x} + x = -\varepsilon \dot{x} \qquad x(0) = 1, \quad \dot{x}(0) = -\frac{\varepsilon}{2}$

Exact Solution:

$$\ddot{x} + \varepsilon \dot{x} + x = 0$$

$$r^2 + \varepsilon r + 1 = 0$$

$$r = \frac{-\varepsilon \pm \sqrt{4 - \varepsilon^2}\, i}{2}$$

$$x(t) = e^{-\frac{\varepsilon}{2} t} \cos\left(\sqrt{1 - \frac{\varepsilon^2}{4}}\, t\right)$$

Regular series:

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots$$

$$\ddot{x}_0 + \varepsilon \ddot{x}_1 + \varepsilon^2 \ddot{x}_2 + \cdots$$

$$+ x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$$

$$= -\varepsilon \left(\dot{x}_0 + \varepsilon \dot{x}_1 + \varepsilon^2 \dot{x}_2 + \cdots \right)$$

$O(1) \Rightarrow \quad \ddot{x}_0 + x_0 = 0$

$$x_0(t) = \cos t$$

$O(\varepsilon) \Rightarrow \quad \ddot{x}_1 + x_1 = + \sin t$

$$x_1(t) = A\cos t + B\sin t + Ct\cos t + Dt\sin t$$

$\Rightarrow$

$$C\left(2(1)(-\sin t) - \cancel{t\cos t}\right) + \cancel{Ct\cos t}$$

$$+\cancel{D}\left(2(1)\cos t - \cancel{t\sin t}\right) + \cancel{Dt\sin t}$$

$D = 0 \quad C = -\frac{1}{2} \qquad = \sin t$

$$x_1(t) = A\cos t + B\sin t - \frac{1}{2}t\cos t$$

$$x(t) = \cos t + \varepsilon\left(\cancel{A\cos t} + B\sin t - \frac{t}{2}\cos t\right)$$

$A = 0 \ (x(0) = 1)$

$$\dot{x}(t) = -\sin t + \varepsilon\left(\cancel{B}\cos t - \frac{\cos t}{2} + \frac{t}{2}\sin t\right)$$

$B = 0 \ \left(\dot{x}(0) = -\frac{\varepsilon}{2}\right)$

$$x(t) = \cos t - \frac{\varepsilon}{2} t \cos t + O(\varepsilon^2)$$

comp

$$x(t) = e^{-\frac{\varepsilon t}{2}} \cos\left(\sqrt{1-\frac{\varepsilon^2}{4}}\, t\right)$$

Exact solution goes to zero as $t \to +\infty$

while

approximate solution is unbounded as $t \to +\infty$

③ Consider van der Pol oscillator:

$$\ddot{x} + \varepsilon(x^2-1)\dot{x} + x = 0$$

Suppose $\nearrow$ plug in

$$x(t) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots\cdots$$

$$\left(\ddot{x}_0 + \varepsilon \ddot{x}_1 + \varepsilon^2 \ddot{x}_2 + \cdots\cdots\right)$$

$$+ \varepsilon\left(\left(x_0 + \varepsilon x_1 + \varepsilon x_2 + \cdots\right)^2 - 1\right)\left(\dot{x}_0 + \varepsilon \dot{x}_1 + \cdots\right)$$

$$+ x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots = 0$$

$$O(1) \Rightarrow \quad \ddot{x}_0 + x_0 = 0 \qquad x_0(0) = A, \quad \dot{x}_0(0) = 0$$

$$x_0(t) = A\cos t$$

$$O(\varepsilon) \Rightarrow \quad \ddot{x}_1 + (x_0^2 - 1)\dot{x}_0 + x_1 = 0$$

$$
\begin{aligned}
\ddot{x}_1 + x_1 &= (1 - x_0^2)\,\dot{x}_0 \\
&= (1 - A^2\cos^2 t)(-A\sin t) \\
&= (1 - A^2 + A^2\sin^2 t)(-A\sin t) \\
&= A(A^2-1)\sin t - A^3\,\underline{\sin^3 t} \\
&= A(A^2-1)\sin t - A^3\,\underline{\left(\frac{3\sin t - \sin 3t}{4}\right)} \\
&= \left(\frac{A^3}{4} - A\right)\sin t + \frac{A^3}{4}\sin 3t \\
&= \underbrace{\frac{1}{4}A(A^2-4)\sin t} + \frac{A^3}{4}\sin 3t
\end{aligned}
$$

resonance with LHS

unless $A(A^2-4) = 0$

or $\underline{A=2}$