Multiple Time Scales Expansion

$$\ddot{x} + x = \varepsilon f(t, x, \dot{x}) \qquad 0 < \varepsilon \ll 1$$

$\varepsilon = 0$

$\ddot{x} + x = 0$

(1) periodic solution can be easily destroyed

(2) how to approximate $x(t)$ for longer time intervals?

Let $x(t) = x(\tau, T)$

$$= x_0(\tau, T) + \varepsilon x_1(\tau, T) + \varepsilon^2 x_2(\tau, T) + \cdots$$

$\tau = t, \quad T = \varepsilon t, \qquad \partial_t = \partial_\tau + \varepsilon \partial_T$

$$\dot{x}(t) = x_\tau + \varepsilon x_T$$

$$= (x_0)_\tau + \varepsilon (x_1)_\tau + \varepsilon^2 (x_2)_\tau + \cdots$$

$$+ \varepsilon (x_0)_T + \varepsilon^2 (x_1)_T + \varepsilon^3 (x_2)_T + \cdots$$

$$
\begin{aligned}
\ddot{x}(t) &= x_{\tau\tau} + \varepsilon x_{\tau T} + \varepsilon x_{\tau T} + \varepsilon^2 x_{TT} \\
&= x_{\tau\tau} + 2\varepsilon x_{\tau T} + \varepsilon^2 x_{TT} \\
&= (x_0)_{\tau\tau} + \varepsilon (x_1)_{\tau\tau} + \varepsilon^2 (x_2)_{\tau\tau} + \cdots \\
&\quad + 2\varepsilon (x_0)_{\tau T} + 2\varepsilon^2 (x_1)_{\tau T} + 2\varepsilon^3 (x_2)_{\tau T} + \cdots \\
&\quad + \varepsilon^2 (x_0)_{TT} + \varepsilon^3 (x_1)_{TT} + \varepsilon^4 (x_2)_{TT} + \cdots
\end{aligned}
$$

Ex 1 $\quad \ddot{x} + (1+\varepsilon)x = 0 \qquad x(0) = 1, \quad \dot{x}(0) = 0$

(Exact Solution $\quad x(t) = \cos\left(\sqrt{1+\varepsilon}\, t\right)$)

$$x(t) = x_0(\tau, T) + \varepsilon x_1(\tau, T) + \varepsilon^2 x_2(\tau, T) + \cdots$$

$$
\begin{aligned}
&(x_0)_{\tau\tau} + 2\varepsilon (x_0)_{\tau T} + \varepsilon^2 (x_0)_{TT} + \cdots \\
&+ \varepsilon (x_1)_{\tau\tau} + 2\varepsilon^2 (x_1)_{\tau T} + \varepsilon^3 (x_1)_{TT} + \cdots \\
&+ \varepsilon^2 (x_2)_{\tau\tau} + 2\varepsilon^3 (x_2)_{\tau T} + \varepsilon^4 (x_2)_{TT} + \cdots \\
&+ (1+\varepsilon)(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) = 0
\end{aligned}
$$

$O(1)$: $\quad (x_0)_{\tau\tau} + x_0 = 0 \qquad x_0 = x_0(\tau, T)$

$$\underline{x_0(\tau, T) = A(T)\cos\tau + B(T)\sin\tau}$$

$$x(t) = x_0 + \varepsilon x_1 + \cdots \Big|_{t=0} = 1$$

$$\underbrace{x_0(0,0)}_{\substack{\shortparallel \\ 1}} + \varepsilon \underbrace{x_1(0,0)}_{\substack{\shortparallel \\ 0}} + \cdots = 1$$

$$\underline{A(0) = 1}$$

$$\dot{x}(t) = (x_0)_\tau + \varepsilon (x_0)_T + (x_1)_\tau + \varepsilon (x_1)_T + \cdots \Big|_{t=0} = 0$$

$$A(T)(-\sin\tau) + B(T)\cos\tau + \varepsilon(\cdots)\cdots \Big|_{\tau=T=0} = 0$$

$$\underline{B(0) = 0}$$

$O(\varepsilon)$: $\quad 2(x_0)_{\tau T} + (x_1)_{\tau\tau} + x_0 + x_1 = 0$

$$(x_1)_{\tau\tau} + x_1 = -2(x_0)_{\tau T} - x_0$$

$$(x_1)_{\tau\tau} + x_1$$

$$= -2\left(-A'(T)\sin\tau + B'(T)\cos\tau\right)$$

$$- A(T)\cos\tau - B(T)\sin\tau$$

$$= \left(2A'(T) - B(T)\right)\sin\tau$$

$$-\left(2B'(T) + A(T)\right)\cos\tau$$

$$2A'(T) - B(T) = 0$$

and $2B'(T) + A(T) = 0$

$$2A''(T) - B'(T) = 0$$

$$\Longrightarrow A''(T) + \frac{1}{4}A(T) = 0$$

$$A(T) = a\cos\frac{T}{2} + b\sin\frac{T}{2}$$

$$\left(A(0) = 1, \quad A'(0) = \frac{1}{2}B(0) = 0 \right)$$

$$A(T) = \cos\frac{T}{2}, \quad B(T) = 2A'(T) = -\sin\frac{T}{2}$$

$$x(t) = x_0 + \varepsilon x_1 + \cdots$$

$$= \left(\cos\frac{T}{2}\right)\cos\tau - \left(\sin\frac{T}{2}\right)\sin\tau + O(\varepsilon)$$

$$= \cos\left(\tau + \frac{T}{2}\right)$$

$$= \cos\left(t + \frac{\varepsilon}{2}t\right)$$

$$= \underline{\cos\left(\left(1+\frac{\varepsilon}{2}\right)t\right) + O(\varepsilon)}$$

compare with exact solution:

$$x(t) = \cos\left(\sqrt{1+\varepsilon}\, t\right)$$

$$= \cos\left(\left(1 + \frac{\varepsilon}{2} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\varepsilon^2 + \cdots\right)t\right)$$

$$= \cos\left(\left(1+\frac{\varepsilon}{2}\right)t - \frac{\varepsilon^2 t}{8} + \cdots\right)$$

$$= \cos\left(\left(1+\frac{\varepsilon}{2}\right)t\right)\cos\frac{\varepsilon^2 t}{2} \leftarrow 1 + \varepsilon^2 t$$

$$+ \sin\left(\left(1+\frac{\varepsilon}{2}\right)t\right)\sin\frac{\varepsilon^2 t}{2} \leftarrow \varepsilon^2 t$$

$$\approx \cos\left(\left(1+\frac{\varepsilon}{2}\right)t\right) + O(\varepsilon^2 t) \qquad \underline{t \ll \frac{1}{\varepsilon^2}}$$

Ex 2 $\quad \ddot{x} + 2\varepsilon\dot{x} + x = 0 \qquad x(0)=0,\ \dot{x}(0)=1$

~~Exact~~ solution:

$$r^2 + 2\varepsilon r + 1 = 0$$

$$r = \frac{-2\varepsilon \pm \sqrt{4-4\varepsilon^2}\, i}{2}$$

$$= -\varepsilon \pm \sqrt{1-\varepsilon^2}\, t$$

$$x(t) = \underset{0}{\cancel{A}}\, e^{-\varepsilon t}\cos\left(\sqrt{1-\varepsilon^2}\, t\right) + \underset{(1-\varepsilon^2)^{-1/2}}{\cancel{B}}\, e^{-\varepsilon t}\sin\left(\sqrt{1-\varepsilon^2}\, t\right)$$

$$\boxed{x(t) = (1-\varepsilon^2)^{-1/2} e^{-\varepsilon t}\sin\left((1-\varepsilon^2)^{1/2} t\right)}$$

Multiple Time Scale Expansion:

$$x(t) = x_0(\tau, T) + \varepsilon x_1(\tau, T) + \varepsilon^2 x_2(\tau, T) + \cdots$$

$$\underline{(x_0)_{\tau\tau}} + \underline{2\varepsilon (x_0)_{\tau T}} + \varepsilon^2 (x_0)_{TT} + \cdots$$

$$+ \varepsilon\, \underline{(x_1)_{\tau\tau}} + 2\varepsilon^2 (x_1)_{\tau T} + \varepsilon^3 (x_1)_{TT} + \cdots$$

$$+ 2\varepsilon\left(\underline{(x_0)_\tau} + \varepsilon (x_0)_T + \varepsilon (x_1)_\tau + \varepsilon^2 (x_1)_T + \cdots \right)$$

$$+ \underline{x_0} + \underline{\varepsilon x_1} + \varepsilon^2 x_2 + \cdots \qquad = 0$$

$$O(1) \Rightarrow \quad (x_0)_{\tau\tau} + x_0 = 0$$

$$\underline{x_0(\tau, T) = A(T)\cos\tau + B(T)\sin\tau}$$

$$x(0)=0, \ \dot{x}(0) = 1 \Longrightarrow A(0)=0, \ B(0)=1$$

$$O(\varepsilon) \quad 2(x_0)_{\tau T} + (x_1)_{\tau\tau} + 2(x_0)_\tau + x_1 = 0$$

$$(x_1)_{\tau\tau} + x_1 = -2(x_0)_{\tau T} - 2(x_0)_\tau$$

$$= -2\Big[-A'(T)\sin\tau + B'(T)\cos\tau - A(T)\sin\tau + B(T)\cos\tau\Big]$$

$$= -2\Big[-(A'(T)+A(T))\sin\tau + (B'(T)+B(T))\cos\tau\Big]$$

$$A'(T)+A(T)=0 \qquad B'(T)+B(T)=0$$

$$A(0)=0 \Longrightarrow A(T)\equiv 0 \qquad B(0)=1 \Longrightarrow B(T)=e^{-T}$$

$$x_0(\tau, T) = e^{-T}\sin\tau = e^{-\varepsilon t}\sin t$$

$$x(t) = x_0 + \varepsilon x_1 + \cdots$$

$$= \underbrace{e^{-\varepsilon t}\sin t}_{\xrightarrow{t\to+\infty} 0} + O(\varepsilon)$$

bounded

Compare with exact solution:

$$x(t) = (1-\varepsilon^2)^{-1/2} e^{-\varepsilon t} \sin\left((1-\varepsilon^2)^{1/2} t\right)$$

$$= e^{-\varepsilon t}\left(1 + \frac{1}{2}\varepsilon^2 + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!}\varepsilon^4 + \cdots\right)$$

$$\sin\left(\left(1 - \frac{\varepsilon^2}{2} + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(-\varepsilon^4) + \cdots\right)t\right)$$

$$= e^{-\varepsilon t}\left(1 + \frac{\varepsilon^2}{2}\right) \underbrace{\sin\left(t - \frac{\varepsilon^2}{2}t + \cdots\right)}_{\sin t \cos\frac{\varepsilon^2 t}{2} - \cos t \sin\frac{\varepsilon^2 t}{2}} + O(\varepsilon^2)$$

$$= e^{-\varepsilon t}\sin t + O(\varepsilon^2 t) \qquad \underline{t < \frac{1}{\varepsilon^2}}$$

Ex 3 van der Pol

$$\ddot{x} + \varepsilon(x^2-1)\dot{x} + x = 0$$

$$x(t) = X(\tau, T)$$

$$\left(X_{\tau\tau} + 2\varepsilon X_{\tau T} + \varepsilon^2 X_{TT}\right)$$
$$+ \varepsilon\left(X^2-1\right)\left(X_\tau + \varepsilon X_T\right) + X = 0$$

$$X = X_0 + \varepsilon X_1 + \cdots$$

$O(1)$ $$X_{0\tau\tau} + X_0 = 0$$

$$\Rightarrow \quad X_0(\tau, T) = A(T)\cos\tau + B(T)\sin\tau$$
$$= \underset{\text{amplitude}}{r(T)}\cos(\tau + \underset{\text{phase}}{\phi(T)})$$

slowly varying

$O(\varepsilon)$ $$(X_1)_{\tau\tau} + 2(X_0)_{\tau T}$$
$$+ \left(X_0^2-1\right) X_{0\tau} + X_1 = 0$$

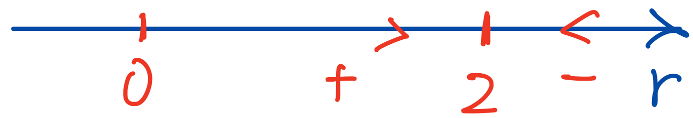
$$
\begin{aligned}
&(X_1)_{\tau\tau} + X_1 \\
&= -2(X_0)_{\tau T} - (X_0^2 - 1)X_{0\tau} \\
&= -2\left(r(T)\cos(\tau+\phi(T))\right)_{\tau T} \\
&\quad + \left(r^2(T)\cos^2(\tau+\phi(T)) - 1\right) r(T)\sin(\tau+\phi(T)) \\
&= 2\left(r(T)\sin(\tau+\phi(T))\right)_T \\
&\quad + \left(r^2(T)\cos^2(\tau+\phi(T)) - 1\right) r(T)\sin(\tau+\phi(T)) \\
&= 2r'(T)\sin(\tau+\phi(T)) + 2r(T)\phi'(T)\cos(\tau+\phi(T)) \\
&\quad + r^3(T)\cos^2\sin - r(T)\sin \\
&= (2r' - r)s + 2r\phi' c + r^3 c^2 s
\end{aligned}
$$

$$
\begin{aligned}
\cos^2\alpha\sin\alpha &= \sin\alpha\left(\frac{\cos 2\alpha + 1}{2}\right) \\
&= \frac{\sin\alpha}{2} + \frac{1}{2}\sin\alpha\cos 2\alpha \\
&= \frac{\sin\alpha}{2} + \frac{1}{4}(\sin 3\alpha - \sin\alpha) \\
&= \frac{\sin 3\alpha}{4} + \frac{\sin\alpha}{4}
\end{aligned}
$$

$$= \left(2r' - r + \frac{r^3}{4}\right)\sin(-) + 2r\phi'\cos(-)$$

$$+ \frac{r^3}{4}\sin(3(\tau+\phi(\tau)))$$

$$2r' - r + \frac{r^3}{4} = 0 \Rightarrow r' = \frac{1}{2}r\left(1-\frac{r^2}{4}\right)$$



$r=2$ is a stable pt.

$$2r\phi' = 0 \Rightarrow \phi'(\tau) = 0$$

$$\phi(\tau) = \phi(0)$$

$$x(t) = X(t,\tau)$$

$$= \underbrace{r(\tau)}_{\xrightarrow{\tau\to+\infty} 2}\cos(\tau+\underbrace{\phi(\tau)}_{= \phi(0)}) + O(\varepsilon)$$

$$\approx 2\cos(t+\phi(0)) + O(\varepsilon)$$

($r=2$ is a stable limit cycle)

Ex 4 Duffing Equation

$$\ddot{x} + x + \varepsilon x^3 = 0$$

$$x(t) = X(\tau, T)$$

$$X_{\tau\tau} + 2\varepsilon X_{\tau T} + \varepsilon^2 X_{TT} + X + \varepsilon X^3 = 0$$

$O(1) \Rightarrow \quad X_{0\tau\tau} + X_0 = 0$

$$X_0(\tau, T) = r(T)\cos(\tau + \phi(T))$$

$O(\varepsilon) \Rightarrow \quad X_{1\tau\tau} + 2X_{0\tau T} + X_1 + X_0^3 = 0$

$$(X_1)_{\tau\tau} + X_1$$

$$= -2\left(r(T)\cos(\tau + \phi(T))\right)_{\tau T}$$

$$- r^3(T)\cos^3(\tau + \phi(T))$$

$$= 2\left(r'(T)\sin(\tau + \phi(T)) + r(T)\phi'(T)\cos(—)\right)$$

$$- r^3(T)\cos^3(—)$$

$$\cos^3\alpha = \cos^2\alpha\cos\alpha$$
$$= \frac{1}{2}(\cos 2\alpha + 1)\cos\alpha$$
$$= \frac{1}{2}\cos 2\alpha\cos\alpha + \frac{1}{2}\cos\alpha$$
$$= \frac{1}{4}(\cos 3\alpha + \cos\alpha) + \frac{1}{2}\cos\alpha$$
$$= \frac{\cos 3\alpha}{4} + \frac{3}{4}\cos\alpha$$

$$= 2(r'\sin(\alpha) + r\phi'\cos\alpha) - r^3\left(\frac{\cos 3\alpha}{4} + \frac{3}{4}\cos\alpha\right)$$

$$= 2r'\sin\alpha + \left(2r\phi' - \frac{3r^3}{4}\right)\cos\alpha - \frac{r^3}{4}\cos 3\alpha$$

$$r'(T) = 0 \Longrightarrow r(T) = r(0)$$

$$\phi'(T) = \frac{3}{8}r^2 = \frac{3}{8}r^2(0)$$

$$\phi(T) = \phi(0) + \frac{3}{8}r^2(0)T$$

$$\boxed{\begin{aligned} X(\tau, T) &= r(0)\cos\left(\tau + \phi(0) + \frac{3}{8} r^2(0)\, T\right) \\ &= \underbrace{r(0)}_{\text{amplitude}}\cos\Big(\underbrace{\left(1 + \frac{3}{8} r^2(0)\,\varepsilon\right)}_{\text{new frequency}} t + \underbrace{\phi(0)}_{\text{phase shift}}\Big) \end{aligned}}$$

$$\left(\begin{array}{ll} \varepsilon > 0 & \text{stiffening} \\ \varepsilon < 0 & \text{softening} \end{array}\right)$$