Fast/Slow Dynamics

(van der Pol Oscillator)

$$\ddot{x} + \varepsilon(x^2-1)\dot{x} + x = 0$$

$$\varepsilon \gg 1$$

$$\Big(\underbrace{\dot{x} + \varepsilon\Big(\frac{x^3}{3} - x\Big)}_{y}\Big)^{\cdot} + x = 0$$

$$F(x) = \frac{x^3}{3} - x$$

$$y = \dot{x} + \varepsilon F(x), \qquad \dot{y} + x = 0$$

$$\begin{cases} \dot{x} = y - \varepsilon F(x) \\ \dot{y} = -x \end{cases}$$

$$\varepsilon \gg 1, \qquad \varepsilon = \frac{1}{\mu} \qquad \mu \ll 1$$

$$\begin{cases} \dot{x} = y - \frac{1}{\mu} F(x) \\ \dot{y} = x \end{cases}$$

Let $z = \mu y$ or $y = \frac{z}{\mu}$

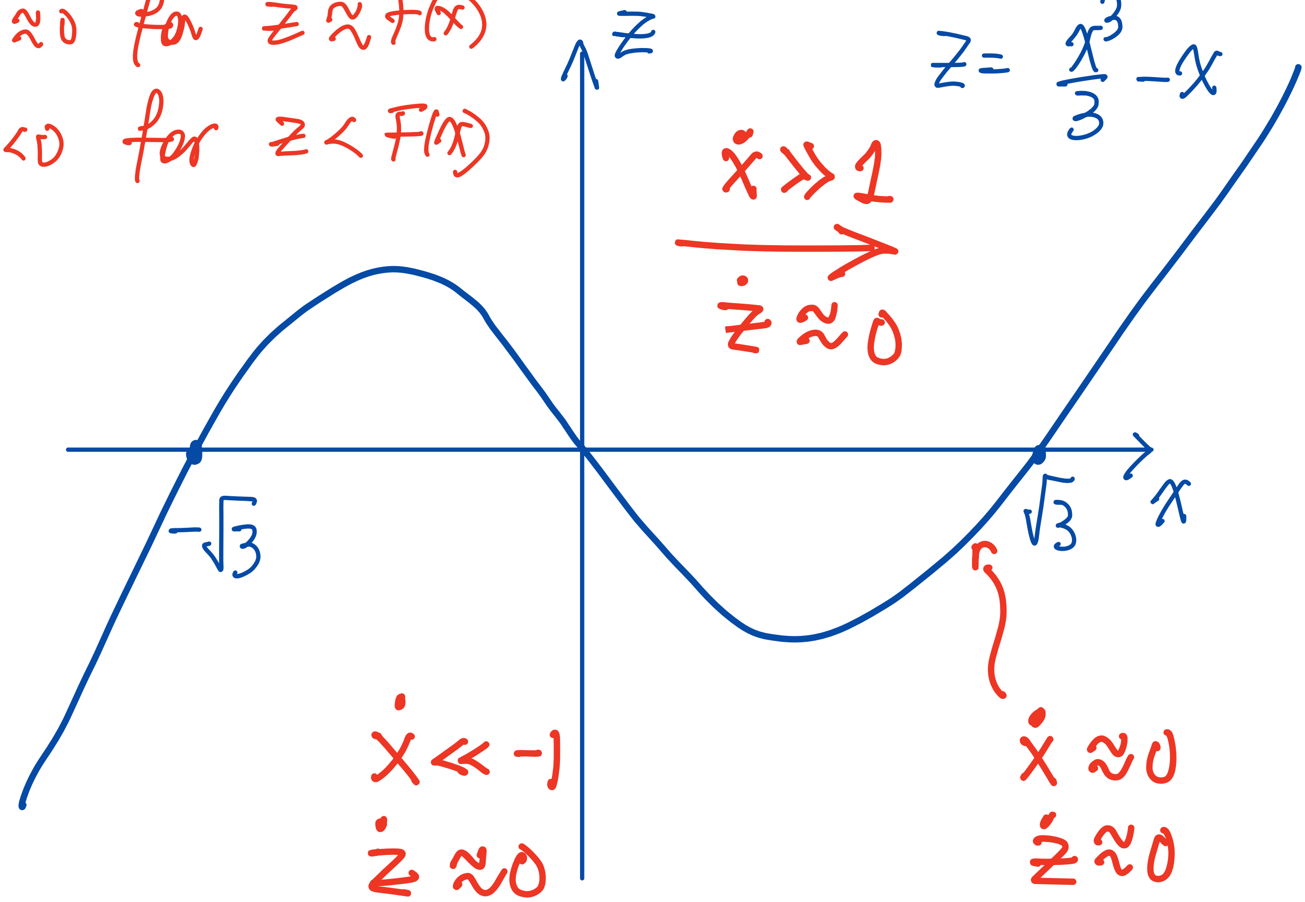
Then
$$\boxed{\begin{aligned} \dot{x} &= \frac{1}{\mu}\left(z - F(x)\right) \\ \dot{z} &= -\mu x \end{aligned}}$$

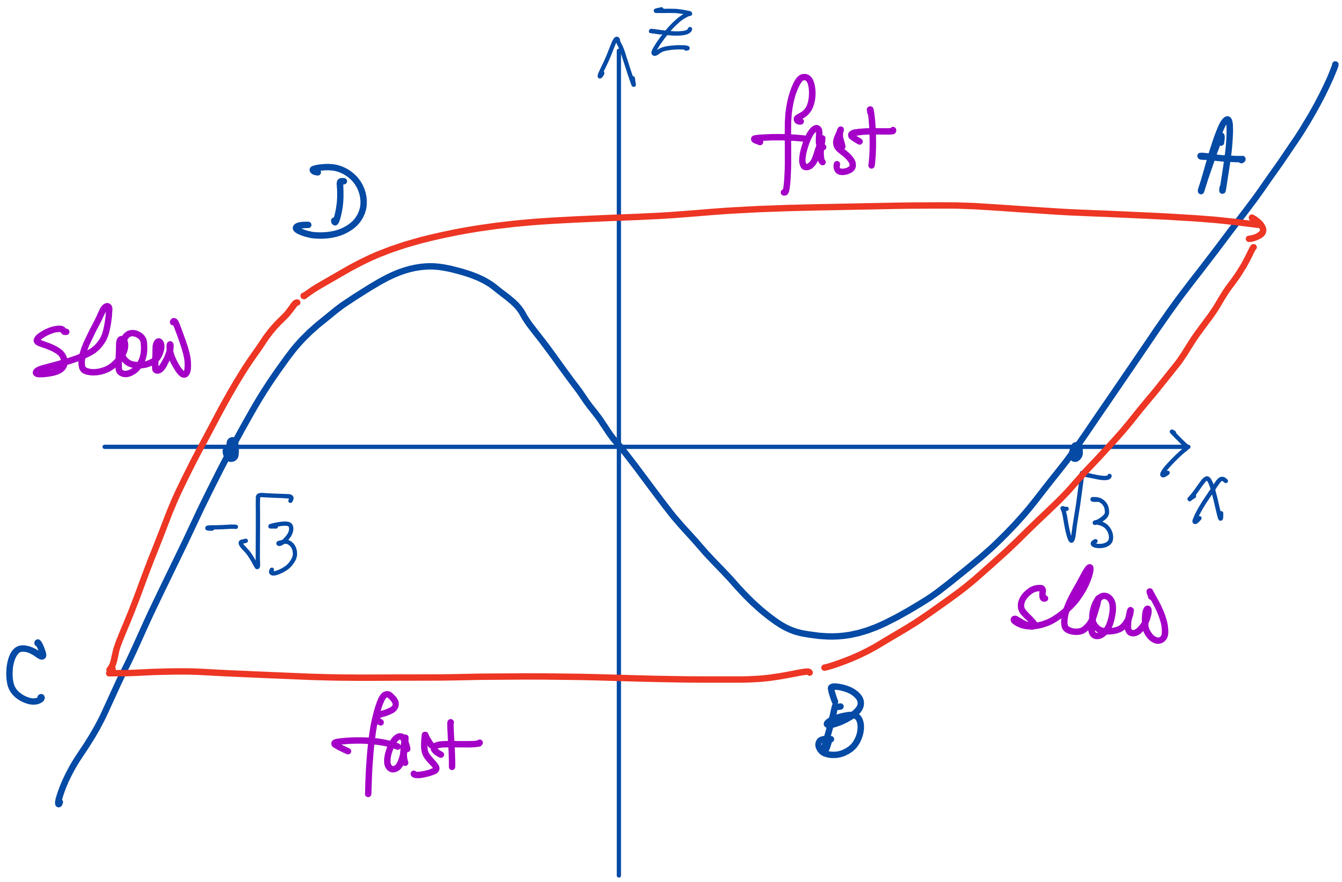
$\dot{x} > 0$ for $z > F(x)$

$\dot{x} \approx 0$ for $z \approx F(x)$

$\dot{x} < 0$ for $z < F(x)$

$\dot{z} \approx 0$



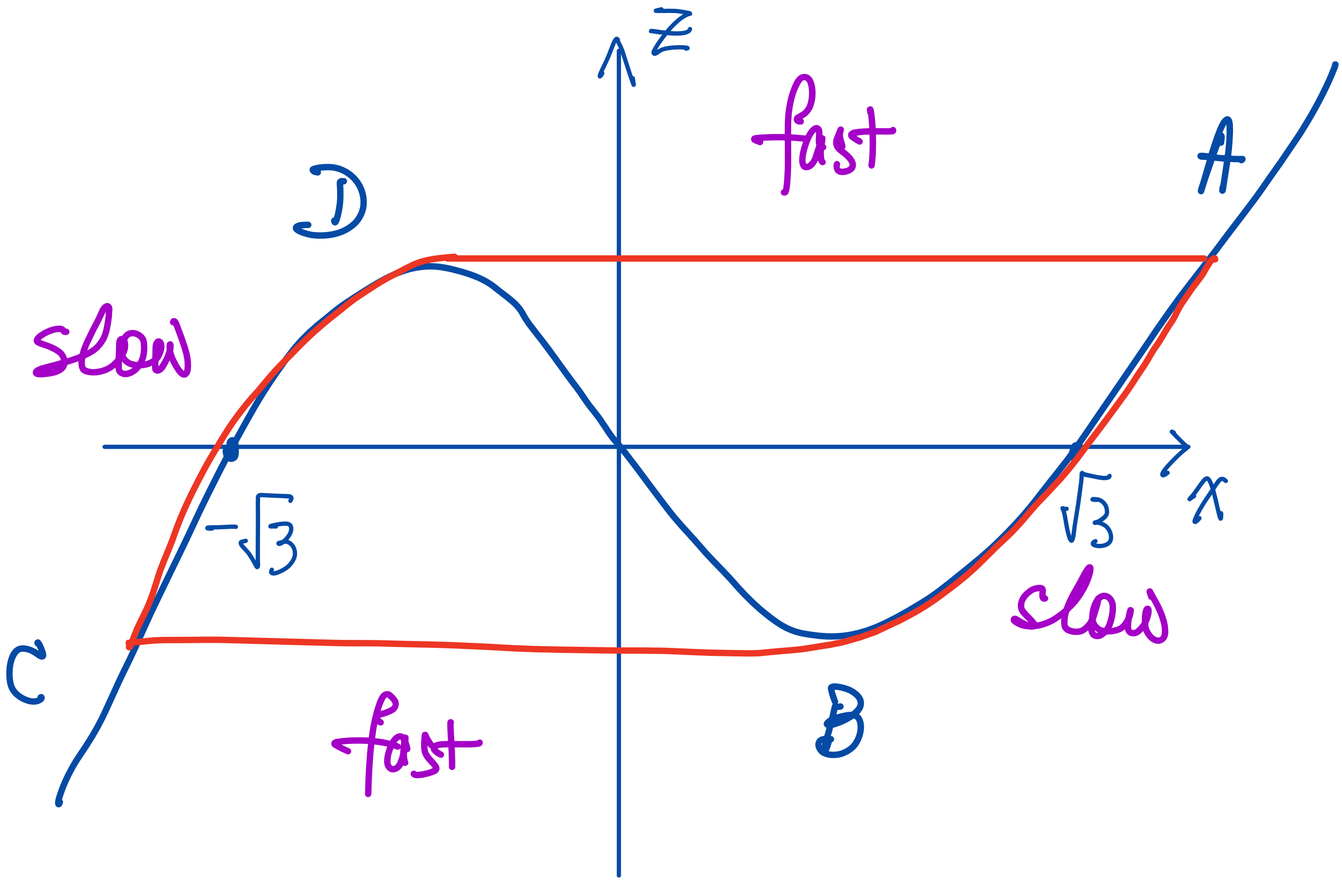


① Time to go from D to A and B to C

$$\left.\begin{aligned} \text{Speed} &\approx \frac{1}{\mu} \\ \text{Distance} &\approx O(1) \\ \text{Time} &\approx \mu \ll 1 \end{aligned}\right\} \text{ fast dynamics}$$

② Time to go from A to B and C to D

$$\left.\begin{aligned} \dot{z} &= \mu x \approx O(\mu) \\ \text{Time} &\approx \frac{1}{\mu} \end{aligned}\right\} \text{ slow dynamics} = ?$$



along A to B, $z = F(x)$

$$\dot{z} = -\mu x$$

$$F(x)^{\cdot} = -\mu x$$

$$F'(x)\,\dot{x} = -\mu x$$

$$\frac{F'(x)\,dx}{x} = -\mu\,dt$$

$$\int_{x_A}^{x_B} \frac{F'(x)}{x}\,dx = -\mu\left(T_B - T_A\right)$$

$$T_B - T_A = \frac{1}{\mu}\int_{x_B}^{x_A} \frac{F'(x)}{x}\,dx$$

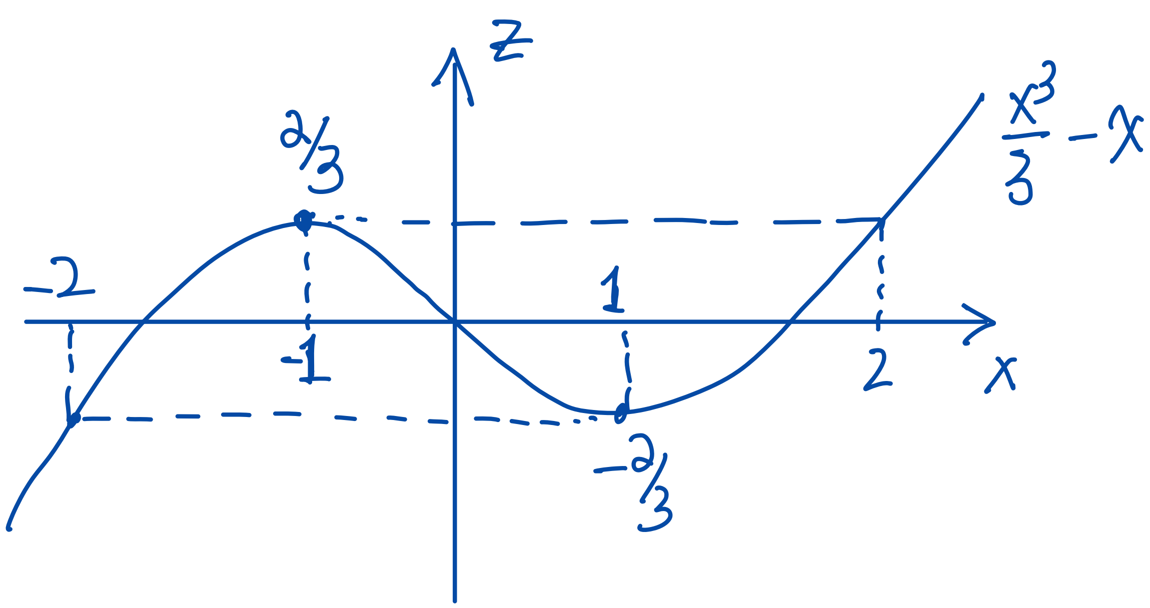
$$= \frac{1}{\mu}\int_{x_B}^{x_A} \left(\frac{x^2-1}{x}\right) dx$$

$$F(x) = \frac{x^3}{3} - x, \qquad F'(x) = x^2 - 1 = 0$$

$$x = \pm 1 \Rightarrow x_B = 1$$

$$\frac{x_A^3}{3} - x_A = \frac{(-1)^3}{3} - (-1) = -\frac{1}{3} + 1 = \frac{2}{3}$$

i.e. $x_A^3 - 3x_A = 2 \Longrightarrow x_A = 2$



$$T_B - T_A = \frac{1}{\mu}\int_1^2 \left(\frac{x^2-1}{x}\right) dx$$

$$= \frac{1}{\mu}\int_1^2 \left(x - \frac{1}{x}\right) dx$$

$$= \frac{1}{\mu}\left[\frac{3}{2} - \ln 2\right]$$

A slightly different form:

$$\varepsilon \ddot{x} + (x^2-1)\dot{x} + x = 0 \qquad \varepsilon \ll 1$$

$$\Big(\underbrace{\varepsilon \dot{x} + \frac{x^3}{3} - x^2}_{y}\Big)^{\cdot} + x = 0$$

$$y = \varepsilon \dot{x} + \frac{x^3}{3} - x^2$$

$$\begin{cases} \dot{x} = \frac{1}{\varepsilon}\left(y - F(x)\right) \\ \dot{y} = -x \end{cases}$$

or change of time scale:

$$\tau = \mu t \qquad t = \frac{\tau}{\mu}$$

$$\frac{d}{dt} = \frac{d}{d\tau}\frac{d\tau}{dt} = \frac{1}{\mu}\frac{d}{d\tau}$$

$$\varepsilon \ddot{x} + (x^2-1)\dot{x} + x = 0$$

$$\varepsilon \frac{1}{\mu^2}\frac{d^2}{d\tau^2}x + (x^2-1)\frac{1}{\mu}\frac{dx}{d\tau} + x = 0$$

Set $\mu = \sqrt{\varepsilon}$

$$\Rightarrow \quad \left(\frac{d^2}{d\tau^2}x\right) + \frac{1}{\sqrt{\varepsilon}}(x^2-1)\frac{dx}{d\tau} + x = 0$$