Chapter 10

Nonlinear Systems

10.1. Introduction: Darwin's finches

In 1835, Charles Darwin, as part of the second voyage of the *Beagle*, visited the Galapagos Islands where he collected 13 species of finch-like birds. These bird species were all quite similar in size and color, except for showing a remarkable variation in beak size. Different sized beaks translated into birds which had adapted to using different food sources; one species fed on seeds, another on insects, another on grubs, and so on. Darwin's finches, as they came to be known, provide one of the clearest examples of the *Principle of Competitive Exclusion*, which says that when two highly similar species living in the same environment compete for the same resources (that is, the species occupy the same ecological niche), the competition between the species for these resources will be so fierce that either

(1) one species will be driven to extinction or

(2) an evolutionary adaption will occur to move the species into different niches.

Here we will propose a mathematical model that predicts this principle and use this to introduce nonlinear systems and some of the tools we will apply to analyze them.

Consider two species, with populations $x(t)$ and $y(t)$ at time t, that separately (in isolation) would grow according to logistic models

$$\frac{dx}{dt} = ax - bx^2, \qquad \frac{dy}{dt} = cy - dy^2 \tag{10.1}$$

for positive constants a, b, c, and d. Recall from Section 2.5 that the terms $-bx^2$ and $-dy^2$ in these equations take into account the competition for resources *within* the x and y populations. We want to modify the equations to also reflect competition *between* the two species. To see how we might do this under the assumption that the species occupy the same niche, we rewrite the equations in (10.1) as

$$\frac{dx}{dt} = ax\left(\frac{K_1 - x}{K_1}\right), \qquad \frac{dy}{dt} = cy\left(\frac{K_2 - y}{K_2}\right), \tag{10.2}$$

where $K_1 = a/b$ and $K_2 = c/d$ are the carrying capacities for the logistically growing x and y populations, respectively (recall from Section 2.5 that the carrying capacity is the maximum sustainable population). Think of the factors $(K_1 - x)$ and $(K_2 - y)$ in the equations in (10.2) as the "unused capacity" in the two individual populations. Since we're assuming the species occupy the same niche, meaning that they use available resources in the same way, once we put the species in direct competition with each other, the available capacity for the x-population becomes $K_1 - x - y$ and for the y-population is $K_2 - y - x$. This says each member of the y-population takes up the space of one member of the x-population and vice versa. So to model our two competing species we propose the system

$$\frac{dx}{dt} = ax\left(\frac{K_1 - x - y}{K_1}\right), \qquad \frac{dy}{dt} = cy\left(\frac{K_2 - y - x}{K_2}\right). \tag{10.3}$$

The interaction between the two species is detrimental to both; if you multiply out the right-hand sides of the equations in (10.3), each will contain a term "xy" with a *negative* constant in front of it. The presence of the xy term, as well as the x^2 and y^2 terms, makes (10.3) a nonlinear system.

Because the independent variable t does not appear on the right-hand sides of the equations in (10.3), the system is **autonomous**. We'll deal nearly exclusively with *planar (two-dimensional) autonomous systems* in this chapter, and generally speaking our goal will be not to solve the system analytically, but rather to give a qualitative solution. This will typically mean that we want to sketch a phase portrait for the system, showing the trajectories of various solutions and making qualitative predictions about the behavior of the solutions from this picture.

The tools we will use to sketch our phase portrait fall into two broad categories:

(1) a global analysis using a nullcline-and-arrow diagram and

(2) a local analysis, focusing on the phase portrait near each equilibrium point. Recall that an equilibrium solution is a constant solution $(x(t), y(t)) = (c_1, c_2)$ to the differential equation system, and the constant value of the equilibrium solution—in this case (c_1, c_2)—is called an equilibrium point.

In this section, we focus on the global analysis, using the system in equation (10.3) to illustrate the procedure. Because of the biological meaning attached to this system, our goal is to sketch a phase portrait in the first quadrant of the xy-plane. To get started, consider the carrying capacities K_1 and K_2. We will assume that $K_1 > K_2$; that is, in isolation the maximum sustainable population for the x species is greater than that of the y species. The opposite assumption, $K_2 > K_1$, leads to a completely parallel mathematical analysis (but a different biological outcome, as we will see). The case of equality, $K_1 = K_2$, is less important from a biological perspective; it is discussed in Exercise 8. We'll also assume that the coefficients a and c, while possibly similar, are not exactly equal.

Nullclines. Recall that a **nullcline** for an autonomous system

$$\frac{dx}{dt} = f(x, y), \qquad \frac{dy}{dt} = g(x, y), \tag{10.4}$$

is a curve in the xy-plane along which either $\frac{dx}{dt} = 0$ or $\frac{dy}{dt} = 0$; these are the curves with equations $f(x, y) = 0$ and $g(x, y) = 0$, respectively. For emphasis, we will sometimes call these the x-nullclines and y-nullclines, respectively.

What are the x-nullclines for the system (10.3)? By the first equation in (10.3) we have

$$\frac{dx}{dt} = 0 \Longleftrightarrow x = 0 \quad \text{or} \quad K_1 - x - y = 0,$$

and so the derivative $\frac{dx}{dt}$ is 0 along the vertical line $x = 0$ and along the line $x + y = K_1$ as shown in Fig. 10.1.

The x-nullcline $x + y = K_1$ divides the first quadrant into two regions, as shown in Fig. 10.1. At any point in the small triangular region, $\frac{dx}{dt} = x(K_1 - x - y)$ is positive, and we indicate this by putting a right-pointing arrow in that region of the first quadrant. If we move out of this region, staying in the first quadrant but crossing over the line $x + y = K_1$, the derivative $\frac{dx}{dt}$ becomes negative. To indicate this, we mark the other region in the first quadrant with a left-pointing arrow. These arrows tell us about the right/left direction of any trajectory in the first quadrant.

Turning to the y-nullclines, we use the second equation in (10.3) to see that

$$\frac{dy}{dt} = 0 \iff y = 0 \quad \text{or} \quad K_2 - x - y = 0,$$

and the y-nullclines are the horizontal line $y = 0$ and the line $x + y = K_2$. This pair of lines is shown in Fig. 10.2. The nullcline $x + y = K_2$ divides the first quadrant into two regions. In the small triangular piece, $\frac{dy}{dt}$ is positive, and that region gets marked with an upward-pointing arrow. From here, as we cross over the line $x + y = K_2$ but stay in the first quadrant, $\frac{dy}{dt}$ becomes negative. This is indicated with the downward-pointing arrow in Fig. 10.2. The up/down arrows tell us where in the first quadrant trajectories are moving up or moving down.

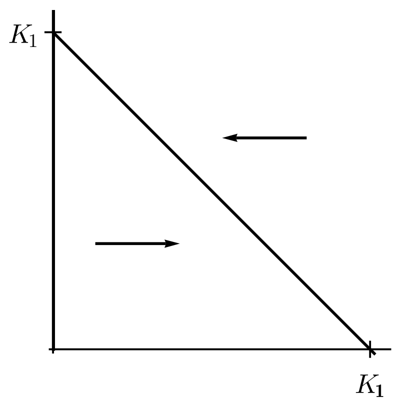


Figure 10.1. The x-nullclines.

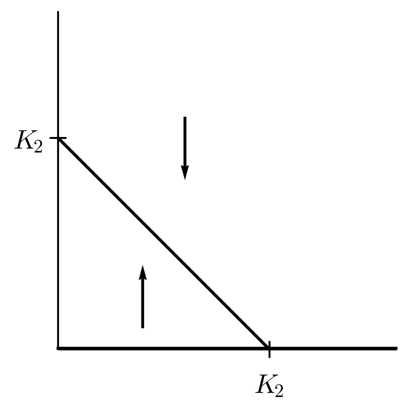


Figure 10.2. The y-nullclines.

To put the information in Figs. 10.1 and 10.2 together into a single picture, recall our assumption that $K_1 > K_2$. This tells us how the parallel lines $x + y = K_1$ and $x + y = K_2$ are placed in relation to each other; see Fig. 10.3.

The combined collection of nullclines divides the first quadrant into three basic regions, labeled I, II, and III in Fig. 10.3. The right/left and up/down direction-indicator arrows from the Figs. 10.1 and 10.2 are transferred to our combined sketch. They indicate that in region I, both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are positive, while both are negative in region III. Region II is marked with right and down arrows; there $\frac{dx}{dt}$ is positive and $\frac{dy}{dt}$ is negative. We have also added scaled field vectors at points along the nullclines $x + y = K_1$ and $x + y = K_2$. These nullcline arrows are horizontal along the y-nullcline $x + y = K_2$ and vertical along the x-nullcline $x + y = K_1$.

The direction-indicator arrows in Fig. 10.3 tell us qualitatively how to move (with increasing time) through each region along a trajectory in a phase portrait. If $\frac{dx}{dt} > 0$, then x increases with t, and similarly if $\frac{dy}{dt} > 0$, then y increases with t. Thus any portion of a trajectory that is in region I must be moving *to the right* and *up* in that region. Because region I is bounded above and to the right by the y-nullcline $y = K_2 - x$, a trajectory in region *I* must cross this nullcline (horizontally) passing into region II[1] and then begin moving down as it continues to move right. A trajectory in region III moves down and to the left. It may pass into region II, crossing the x-nullcline vertically (as indicated by the field vectors along $x + y = K_1$), and then move down and to the right. Using

[1]The trajectory cannot "stall" before or at the nullcline because the portion of the nullcline the orbit can reach by moving up and to the right, as well as the portion of region I it can reach, is free of equilibrium points; see Theorem 8.6.3.

this information, we obtain a reasonable sketch of a phase portrait in the first quadrant, as shown in Fig. 10.4. Moreover, the x-axis (a y-nullcline) and the y-axis (an x-nullcline) consist entirely of equilibrium points and trajectories that traverse some portion of the axis, as shown in Fig. 10.3. Explicit formulas for the solutions whose trajectories lie on the y-axis can be found by substituting $x = 0$ in the equation for $\frac{dy}{dt}$ and solving the resulting logistic equation; see Exercise 9. Similarly, the solutions whose trajectories lie on the x-axis are found by substituting $y = 0$ into the equation for $\frac{dx}{dt}$.

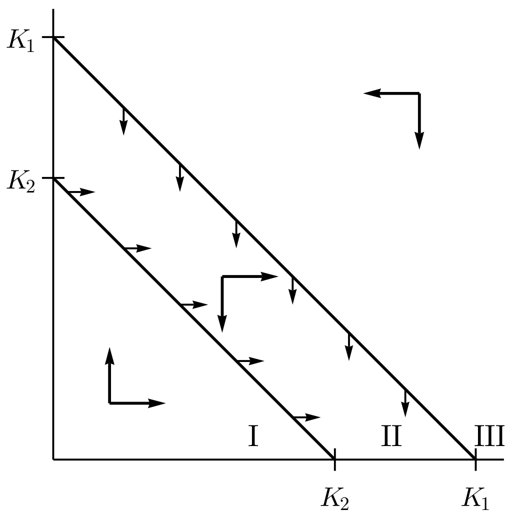


Figure 10.3. Nullcline-and-arrow diagram.

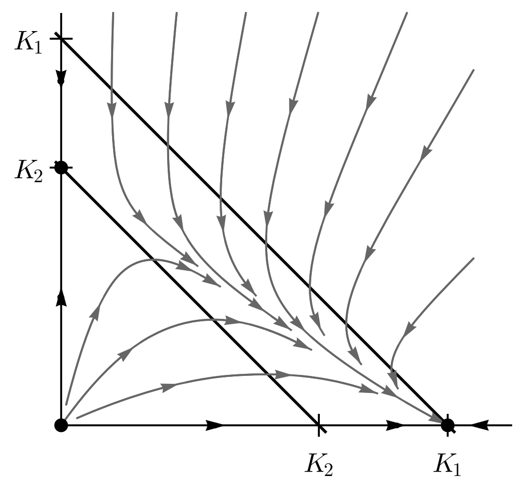


Figure 10.4. Some trajectories in the first quadrant.

Is what we see in our sketch consistent with Darwin's principle of competitive exclusion? Our information suggests that no matter where a trajectory starts in the first quadrant (in other words no matter what the initial populations of the two species), as time goes on, the x-population approaches K_1 and the y-population tends to 0. Recall we assumed at the beginning that $K_1 > K_2$. So it appears that the the species with the larger carrying capacity thrives, while the other species is driven to extinction. Exercise 10 discusses how to *verify* this prediction.

The system in equation (10.3) is an example of a **quadratic competing species model**. In general these models take the form

$$\frac{dx}{dt} = a_1x - b_1x^2 - c_1xy, \qquad \frac{dy}{dt} = a_2y - b_2y^2 - c_2xy. \tag{10.5}$$

You can easily verify that the two species occupy the "same niche" (meaning that the system can be rewritten in the form (10.3)) exactly when $c_1 = b_1$ and $c_2 = b_2$. Depending on the values of the parameters in these equations, peaceful coexistence of the two species may or may not be possible. Exercises 1–4 and 17–18 illustrate some possibilities for the behavior of such systems.

Example 10.1.1. Nullclines need not be lines. For example, consider the system

$$\begin{aligned} \frac{dx}{dt} &= 1 - x^2 - y^2, \\ \frac{dy}{dt} &= x - y. \end{aligned} \tag{10.6}$$

The x-nullcline has equation $x^2 + y^2 = 1$, the equation of a circle of radius one centered at $(0, 0)$. The y-nullcline is the line $y = x$. In Fig. 10.5 we show the x-nullcline and the corresponding right/left arrows, while Fig. 10.6 shows the y-nullcline and corresponding up/down arrows.

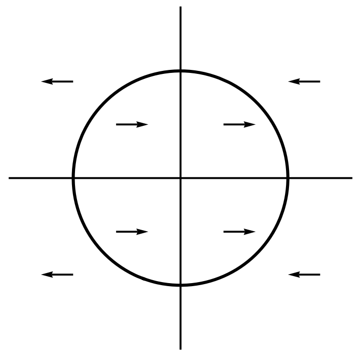


Figure 10.5. The x-nullcline.

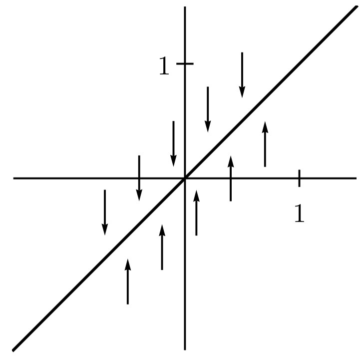


Figure 10.6. The y-nullcline.

In Fig. 10.7 we combine Figs. 10.5 and (10.6), specializing to the first quadrant, and show the direction-indicator arrows in the four basic regions together with some scaled field vectors along the nullclines. Using Fig. 10.7, we then show a few trajectories in Fig. 10.8.

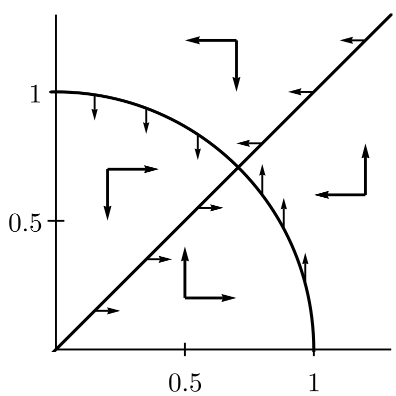


Figure 10.7. Nullcline-and-arrow diagram in first quadrant.

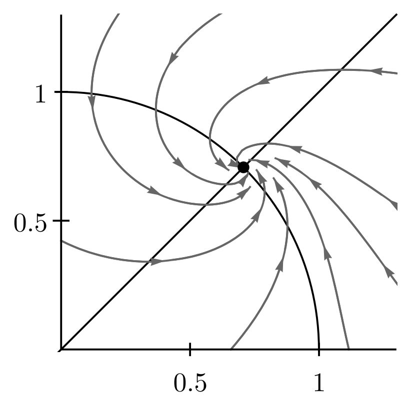


Figure 10.8. Some trajectories in first quadrant.

Vector fields and an existence and uniqueness theorem. A nullcline-and-arrow diagram is a quick substitute for a more detailed **vector field** sketch (see Section 8.2.1). Compare the vector field sketch for equation (10.6) shown in Fig. 10.9 to Fig. 10.7.

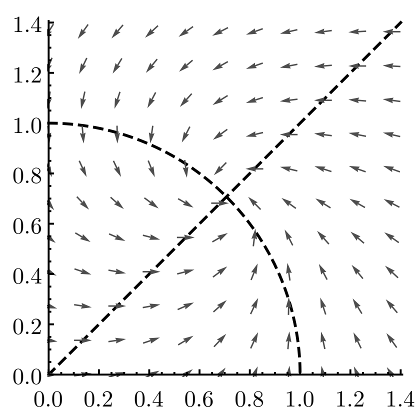


Figure 10.9. A (scaled) vector field for equation (10.6), with the nullclines dashed.

Keep in mind that because the system of Example 10.1.1 is autonomous, the vector field plot in Fig. 10.9 remains stationary over time—the arrows do not "wiggle" as time varies.

Here is one interpretation for a vector field $\mathbf{F}(x, y) = (f(x, y), g(x, y))$ (corresponding to our general autonomous system (10.4)): Think of the surface of a stream seen from above. Each point $\mathbf{X} = (x, y)$ represents an infinitesimal patch of water centered at $\mathbf{X}$ and $\mathbf{F}(\mathbf{X})$ is its *velocity vector*. The direction of the vector is the direction that patch at $\mathbf{X}$ is moving, and the length of the velocity vector, $\|\mathbf{F}(\mathbf{X})\|$, is the speed of the water in that patch. The system of differential equations (10.4) can be abbreviated in vector form as the single vector equation

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}(t)), \quad \text{or} \quad \frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}), \quad \text{or just} \quad \mathbf{X}' = \mathbf{F}(\mathbf{X}).$$

The solution $\mathbf{X}(t)$ is the position at time t of a leaf floating on the water surface. At every instant t, its velocity vector $\frac{d\mathbf{X}}{dt}$ matches the velocity vector $\mathbf{F}(\mathbf{X}(t))$ of the patch of water on which it's floating at time t. Put another way, the differential equation is specified by the velocity field $\mathbf{F}$ of the water surface. When you find a solution of the differential equation, you're finding a parametric curve followed by a floating object on that surface. As long as the vector field is sufficiently nice—for example, $\mathbf{F}$ is continuously differentiable, meaning that its coordinate functions f and g have continuous partial derivatives—then given an initial time t_0 and initial position $\mathbf{X}_0$, there is one and only one solution $\mathbf{X}(t)$ on some open interval I containing t_0, of the initial value problem

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}), \qquad \mathbf{X}(t_0) = \mathbf{X}_0.$$

This is the **Existence and Uniqueness Theorem** for autonomous systems. The notion of continuously differentiable vector fields $\mathbf{F}(x, y) = (f(x, y), g(x, y))$ is discussed further in the next section. For now we note that whenever the coordinate functions f and g are polynomials[2], then $\mathbf{F}$ will be continuously differentiable. In particular, the Existence and Uniqueness Theorem for autonomous systems applies to the system of Example 10.1.1 as well as to the competing species model given in equation (10.3).

In the following plot, we have added to the vector field sketch of Fig. 10.9 a trajectory, over $0 \leq t < \infty$, of the unique solution to the initial value problem consisting of the system of Example 10.1.1 subject to the initial condition $x(0) = 0.1, y(0) = 0.2$.

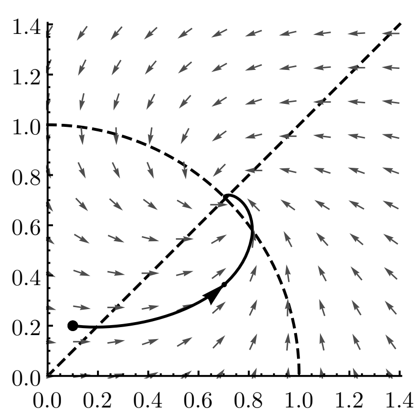


Figure 10.10. Vector field with solution curve to an initial value problem.

[2]A polynomial in the two variables x and y is a finite sum of terms of the form $cx^n y^m$ where c is a constant and n and m are nonnegative integers; for example $f(x, y) = 2x^2 + 5xy + 3y^2$ or $g(x, y) = 1 + x^3 y$.

Even though two different solutions of an autonomous system cannot take the same value at a given time t_0, can the trajectories followed by two different solutions of

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X})$$

touch (by passing through the same point at different times)? Yes, but only if one solution is merely a time translate of the other. This means that both solutions travel the same path and pass each point on that path at the same speed; the only difference is that one solution follows this path ahead of the other in time. The next theorem makes this precise.

Theorem 10.1.2. *Let* $\mathbf{F} = (f(x, y), g(x, y))$ *be a continuously differentiable vector field and let* $\mathbf{X}(t)$ *and* $\mathbf{Y}(t)$ *be solutions of the autonomous system*

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}) \tag{10.7}$$

satisfying $\mathbf{X}(t_0) = \mathbf{Y}(t_1)$. *Let* $T = t_0 - t_1$. *Then* $\mathbf{Y}(t) = \mathbf{X}(t+T)$ *for all* t *for which both sides are defined.*

Proof. Let $\mathbf{Z}(t) = \mathbf{X}(t + T)$. Notice that $\mathbf{Z}'(t) = \mathbf{X}'(t + T) = \mathbf{F}(\mathbf{X}(t + T)) = \mathbf{F}(\mathbf{Z}(t))$, so $\mathbf{Z}(t)$ solves the system (10.7). Also, $\mathbf{Z}(t_1) = \mathbf{X}(t_1 + T) = \mathbf{X}(t_0) = \mathbf{Y}(t_1)$. By the uniqueness part of the Existence and Uniqueness Theorem for autonomous systems just discussed, $\mathbf{Z}(t) = \mathbf{Y}(t)$ for all t for which both sides are defined. This is the desired conclusion. $\square$

A consequence of the preceding theorem is that if the trajectories of two solutions $\mathbf{X} = \mathbf{X}(t)$ and $\mathbf{Y} = \mathbf{Y}(t)$ of the system (10.7) (with $\mathbf{F}$ continuously differentiable) have a point in common, then the trajectories must be identical (assuming the domains of the solutions are not restricted). In terms of phase portraits, the preceding theorem tells us that given any point $\mathbf{X}_0 = (x_0, y_0)$ in the phase plane of (10.7), there is one and only one phase portrait curve passing through $\mathbf{X}_0$; every trajectory passing through $\mathbf{X}_0$ follows this curve.

Equilibrium points. An equilibrium point of the system $\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X})$ is a point $\mathbf{X}_e = (x_e, y_e)$, where $\mathbf{F}(\mathbf{X}_e) = 0$; that is, $f(x_e, y_e) = 0$ and $g(x_e, y_e) = 0$. Then the constant curve $\mathbf{X}(t) = \mathbf{X}_e$ solves the system because

$$\frac{d\mathbf{X}}{dt} = \frac{d\mathbf{X}_e}{dt} = \frac{d(\mathbf{constant})}{dt} = \mathbf{0},$$

while $\mathbf{F}(\mathbf{X}_e) = \mathbf{0}$, too. In the analogy of the stream, a leaf at $\mathbf{X}_e$ stays at $\mathbf{X}_e$ forever; it is becalmed at a still point in the flow. Since the equilibrium solutions occur at points (x_e, y_e) where both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are simultaneously 0 they occur at any point where an x-nullcline intersects a y-nullcline. In the next section, we will focus on tools producing more accurate sketches of phase portraits near equilibrium points. This will help us refine our phase portrait sketches with local information near these key points.

Example 10.1.3. Consider the system

$$\frac{dx}{dt} = x\left(5 - \frac{1}{20}x - \frac{1}{20}y\right), \qquad \frac{dy}{dt} = y\left(4 - \frac{1}{50}y - \frac{1}{50}x\right), \tag{10.8}$$

which is obtained from equation (10.3) by setting $a = 5, K_1 = 100, c = 4$, and $K_2 = 200$. The equilibrium solutions of (10.8) are $(0, 0), (100, 0)$, and $(0, 200)$, found by solving the pair of equations

$$0 = x\left(5 - \frac{1}{20}x - \frac{1}{20}y\right) \quad \text{and} \quad 0 = y\left(4 - \frac{1}{50}y - \frac{1}{50}x\right).$$

The solutions occur when one of the factors in *each* product is equal to 0:

$$x = 0,\ y = 0 \text{ gives the equilibrium point } (0, 0),$$

$$x = 0,\ 4 - \frac{1}{50}y - \frac{1}{50}x = 0 \text{ gives the equilibrium point } (0, 200),$$

$$5 - \frac{1}{20}x - \frac{1}{20}y = 0,\ y = 0 \text{ gives the equilibrium point } (100, 0).$$

Since the fourth possible pair of equations,

$$5 - \frac{1}{20}x - \frac{1}{20}y = 0, \quad 4 - \frac{1}{50}y - \frac{1}{50}x = 0,$$

has no solution (the two lines are parallel), these are all of the equilibrium points.

Fig. 10.11 shows the solution of (10.8) satisfying the initial condition $x(0) = 8, y(0) = 15$, plotted over a (scaled) vector-field background. Observe that the solution appears to approach the equilibrium point (0, 200) as $t \to \infty$. This is consistent with Theorem 8.6.3, which says that if a solution tends to a finite limit, it must be an equilibrium point.

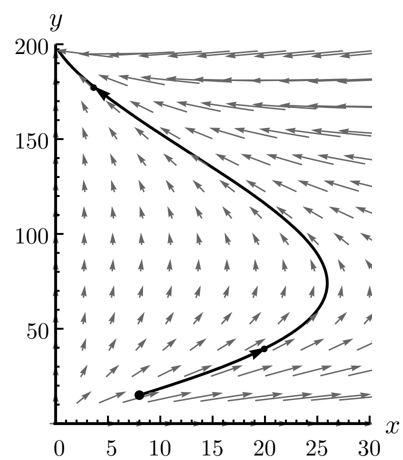


Figure 10.11. Solution to (10.8) with initial condition $x(0) = 8$, $y(0) = 15$.

10.1.1. Exercises.

In Exercises 1–4 two competing (dissimilar) species whose populations (in millions of individuals) are modeled by the given equations. For each, find the equilibrium points for the system, and do a nullcline-and-arrow diagram for a phase portrait in the first quadrant. Based on your sketch, can you make a prediction about the long-term fate of the two populations?

1. $\begin{aligned} x'(t) &= x(5 - x - 4y), \\ y'(t) &= y(2 - y - x). \end{aligned}$

2. $\begin{aligned} x'(t) &= x(40 - x - y), \\ y'(t) &= y(90 - x - 2y). \end{aligned}$

3. $\begin{aligned} x'(t) &= x(5 - 2x - 4y), \\ y'(t) &= y(7 - 3y - 4x). \end{aligned}$

4. $\begin{aligned} x'(t) &= x(6 - x - 2y), \\ y'(t) &= y(3 - x - y). \end{aligned}$

5. The following differential equation system is proposed for two competing species whose populations would grow exponentially (rather than logistically) in isolation:

$$\frac{dx}{dt} = x - xy, \qquad \frac{dy}{dt} = 2y - 2xy.$$

   (a) What are the equilibrium points for this system?

(b) Do a nullcline-and-arrow diagram for the phase portrait in the first quadrant, showing a few trajectories.

6. The following model[3] has been proposed for competition between Neanderthal man (with population $N(t)$ at time t in years) and early modern man (with population $E(t)$ at time t):

$$\frac{dN}{dt} = N[a - b - d(N + E)], \quad \frac{dE}{dt} = E[a - sb - d(N + E)]$$

where a, b, and d are positive constants with $a > b$ and s is a positive constant with $0 < s < 1$. The constant s is sometimes called the "parameter of similarity" and it reflects the different average life spans of Neanderthal man and early man. It is estimated to be about 0.995.

(a) Show that these equations have the same form as other quadratic competing species models; that is, they reflect logistic growth for each species in the absence of the other and a (mutually detrimental) interaction term that is proportional to the product of the two species.

(b) Show that

$$\frac{d}{dt}\left(\frac{N}{E}\right) = \frac{Nb(s-1)}{E}.$$

(c) Set $R = N/E$, so that the differential equation in (b) is

$$\frac{dR}{dt} = Rb(s-1).$$

Solve this equation for R to show that

$$\frac{N}{E} = \frac{N(0)}{E(0)}e^{-b(1-s)t}.$$

(d) Explain how your work in (c) and the assumption that $b > 0$ and $0 < s < 1$ show that Neanderthal man is driven to extinction by early modern man.

(e) Use $s = 0.995$ and $b = 0.033 = 1/30$ (this reflects an average life span of 30 years for Neanderthal man) to determine how long it takes until the ratio $N(t)/E(t)$ is 10 percent of its initial ratio $N(0)/E(0)$. Compare this to the estimated extinction time of Neanderthal man of 10,000 years.

7. The time from infection with HIV to the clinical diagnosis of AIDS is only partially known. To study this, patients for whom an estimate of date of infection with HIV can be given are important.[4] Suppose a group of K hemophiliacs are known to have been given an HIV-infected blood product at time $t = 0$. The number $x(t)$ of people who are HIV-positive but do not yet have AIDS and the number $y(t)$ who have converted from being HIV-positive to having AIDS are modeled by the differential equations

$$\frac{dx}{dt} = -atx, \quad \frac{dy}{dt} = atx$$

(a is a positive constant, and t is in years), with initial condition $x(0) = K, y(0) = 0$. Notice that this is **not** an autonomous system. However, it can be solved explicitly because the first equation, which doesn't involve y, is a first-order separable equation.

(a) Solve the equation

$$\frac{dx}{dt} = -atx$$

explicitly for x as a function of t, using the initial condition $x(0) = K$.

[3] This exercise is based on the article *A mathematical model for Neanderthal extinction* by J. C. Flores, J. Theor. Biol. 191 (1998), 295–298. See also [**31**, p. 115].

[4] This exercise is a simplification of a problem that appears in [**31**, p. 394].

(b) Substitute your answer for (a) into the second equation

$$\frac{dy}{dt} = atx$$

and then solve the resulting separable equation explicitly for y in terms of t, using the initial condition $y(0) = 0$.

(c) If 1/3 of the initial population of K has converted from HIV-positive status to AIDS after 6 years, how long does it take for 90% of the initial population to convert?

8. What does the nullcline-and-arrow diagram for the system in equation (10.3) look like if $K_1 = K_2$? Show that there are infinitely many equilibrium points in this case.

9. (a) Show that in the competing species system (10.3), the x-axis and the y-axis consist entirely of equilibrium points and trajectories that traverse some portion of the axis (as shown in Fig. 10.4). Explicit formulas for the solutions whose trajectories lie on the y-axis can be found by substituting $x = 0$ in the equation for $\frac{dy}{dt}$ and solving the resulting logistic equation. A similar comment applies to find explicit formulas for the solutions whose trajectories lie on the x-axis.

(b) Give a careful argument to show that the trajectory of any solution of the competing species system (10.3) satisfying $x(0) > 0$, $y(0) > 0$ must lie entirely in the first quadrant.

10. Some of the observations "suggested" by a nullcline-and-arrow diagram can be verified using the following properties, which apply to an autonomous system

$$\frac{dx}{dt} = f(x, y), \qquad \frac{dy}{dt} = g(x, y)$$

when f and g are continuous.

Fact (i). If a solution $x(t)$ is bounded and monotone increasing or monotone decreasing for all $t \geq t_0$, then $\lim_{t\to\infty} x(t)$ exists as a finite number. The same applies to $y(t)$. To say that an increasing function $x(t)$ is bounded means $x(t) \leq c$ for some constant c. A decreasing function is bounded if it is greater than or equal to some constant.

Fact (ii). According to Theorem 8.6.3, if a trajectory $(x(t), y(t))$ has a limit (x_*, y_*) as $t \to \infty$, then (x_*, y_*) must be an equilibrium point.

Assuming $K_1 > K_2$, explain how Fact (i) and Fact (ii) help justify the assertion that in the system in (10.3), one species is driven to extinction while the other tends to the carrying capacity for that population. You may also use the fact that any trajectory that starts in region II of Fig. 10.3 must stay in that region for all later times.

11. In this problem we consider the competing species system (10.8) of Example 10.1.3. Recall $(0, 0)$ is an equilibrium solution and that on the positive half of the x-axis the system has equilibrium point $x = 100$, $y = 0$, while on the positive half of the y-axis it has equilibrium point $x = 0$, $y = 200$.

(a) Give an exact solution to the system with initial condition $x(0) = 2,\ y(0) = 0$ using the solution in equation (2.87) of the logistic equation (2.80) in Section 2.5. Describe the behavior of your solution as $t \to \infty$ and as $t \to -\infty$, and verify that the trajectory of this solution is the interval $0 < x < 100$ on the x-axis, traversed from left to right as t increases from $-\infty$ to ∞.

(b) Again using equation (2.87), give an exact solution of the system (10.8) with initial condition $x(0) = 200,\ y(0) = 0$. Show that this solution tends to 100 as $t \to \infty$ and traces out the segment $100 < x < \infty,\ y = 0$ on the x-axis as t ranges from $-\frac{\ln 2}{5}$ to ∞.

(c) By parts (a) and (b) the portion of the positive half of the x-axis to the left of the equilibrium point $(100, 0)$ and the portion to the right of this point are each traversed by trajectories of solutions. A similar result holds on the positive half of the y-axis: The portion of the positive half of the y-axis below the equilibrium point $(0, 200)$ and the portion above this point are traversed by solutions of the system satisfying the initial condition $x(0) = 0,\ y(0) = y_0$ for $0 < y_0 < 200$ and $200 < y_0$, respectively (you are not required to verify this). Explain why a trajectory for the system (10.8) whose initial point (x_0, y_0) lies in the open first quadrant (so that $x_0 > 0$ and $y_0 > 0$) can never leave the open first quadrant. Hint: Can trajectories cross?

12. Sam and Sally have each obtained a numerical solution of the initial value problem
$$\frac{dx}{dt} = (x + y^2)/10, \quad \frac{dy}{dt} = xy - x, \qquad x(0) = 1, \quad y(0) = 2.$$
According to Sam, $x(1) \approx 2.11$ and $y(1) \approx 5.05$, while Sally says $x(1) \approx 2.11$ and $y(1) \approx 1.22$. Which of the two solutions **must** be incorrect? Explain.

13. Suppose we know that $\mathbf{X}(t) = (2e^{-3t}, -e^{-3t})$ is a solution of some autonomous system $\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X})$ where $\mathbf{F}$ is a continuously differentiable vector field.
    (a) Sketch the trajectory of this solution for $-\infty < t < \infty$, and show that $\mathbf{Y}_1(t) = (6e^{-3t}, -3e^{-3t})$, $\mathbf{Y}_2(t) = (2e^{-5t}, -e^{-5t})$, and $\mathbf{Y}_3(t) = (2e^{-3(t-1)}, -e^{-3(t-1)})$ all have the same trajectory as $\mathbf{X}(t)$.
    (b) Which of the functions $\mathbf{Y}_1(t)$, $\mathbf{Y}_2(t)$, or $\mathbf{Y}_3(t)$ are merely "time translates" of $\mathbf{X}(t)$; that is, which can be written as $\mathbf{X}(t + T)$ for some constant T (and all t)?
    (c) Which of $\mathbf{Y}_1(t)$, $\mathbf{Y}_2(t)$, or $\mathbf{Y}_3(t)$ *cannot* be a solution of the same system $\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X})$? Justify your answer.
    (d) *Could* $(0, 0)$ be an equilibrium point of the system $\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X})$? *Must* $(0, 0)$ be an equilibrium point of the system $\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X})$? Justify your answer.

14. Suppose we know that $\mathbf{X}(t) = (2\cos t, 3\sin t)$ is a solution of some autonomous system $\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X})$ where $\mathbf{F}$ is a continuously differentiable vector field.
    (a) Sketch the trajectory of this solution for $-\infty < t < \infty$, and show that $\mathbf{Y}_1(t) = (-2\sin t, 3\cos t)$, $\mathbf{Y}_2(t) = (2\cos(2t), 3\sin(2t))$, and $\mathbf{Y}_3(t) = (2\cos(t/2), 3\sin(t/2))$ all have the same trajectory as $\mathbf{X}(t)$.
    (b) Which of the functions $\mathbf{Y}_1(t)$, $\mathbf{Y}_2(t)$, or $\mathbf{Y}_3(t)$ are merely "time translates" of $\mathbf{X}(t)$; that is, which can be written as $\mathbf{X}(t + T)$ for some constant T (and all t)?
    (c) Which of $\mathbf{Y}_1(t)$, $\mathbf{Y}_2(t)$, or $\mathbf{Y}_3(t)$ *cannot* be a solution of the system $\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X})$? Justify your answer.

15. In this problem you will work with the system
$$\frac{dx}{dt} = y - \frac{x^2}{2}, \quad \frac{dy}{dt} = y\left(y - \frac{1}{2}x - 1\right).$$
    (a) Sketch the nullcline-and-arrow diagram for this system in the first quadrant, and identify any equilibrium points in your sketch. Include the y-nullcline $y = 0$ forming the lower boundary of your diagram.
    (b) Suppose that $x = x(t), y = y(t)$ is a solution of the system such that $x(0) = 2$ and $y(0) = 1$. Use your diagram of (a) and Theorem 8.6.3 to predict $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$.
    (c) (CAS) Use a computer algebra system to produce a phase portrait of the system over the rectangle $0 < x < 3, 0 < y < 3$. Include nullclines in your portrait. Check that your nullcline-and-arrow diagram of (a) and your solution of (b) are consistent with the portrait.

16. Consider the system
$$\frac{dx}{dt} = 4 - x^2 - y, \qquad \frac{dy}{dt} = x(1 - y^2).$$
    (a) Sketch the nullcline-and-arrow diagram for this system in the first quadrant and find any equilibrium points there.

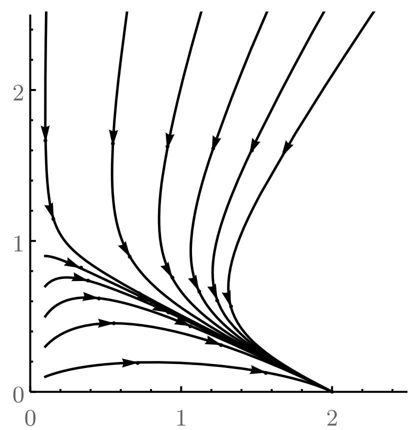
(b) Suppose that $x = x(t), y = y(t)$ is a solution of the system such that $x(0) = 1$ and $y(0) = 3$. Use your diagram of (a) and Theorem 8.6.3 to predict $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$.

(c) (CAS) Use a computer algebra system to produce a phase portrait of the system over the rectangle $0 < x < 3, 0 < y < 3$. Include nullclines in your portrait. Check that your nullcline-and-arrow diagram of (a) and your solution of (b) are consistent with the portrait.

17. (CAS) The figure below shows a phase portrait, in the first quadrant, for the system

$$\frac{dx}{dt} = x(2 - x - y), \quad \frac{dy}{dt} = y(1 - x - y). \tag{10.9}$$

Notice this is the competing species system (10.3) with $a = 2, c = 1, K_1 = 2$, and $K_2 = 1$.



From (10.9) we have

$$\frac{dy}{dx} = \frac{y(1 - x - y)}{x(2 - x - y)}. \tag{10.10}$$

After some computation and simplification we can also obtain

$$\frac{d^2y}{dx^2} = \frac{y(2 - x - 2y)}{x^2(-2 + x + y)^3}. \tag{10.11}$$

(a) Use the second derivative (10.11) to create a concavity chart for the system (10.9) in the first quadrant, and check that your chart is consistent with phase portrait given above.

(b) Verify by substitution that $y = 1 - \frac{x}{2}$ is a solution of the first-order equation (10.10). The line with equation $y = 1 - \frac{x}{2}$ is also a "zero-concavity line" that you should have obtained in (a). Draw some field vectors *for the system* at points on this line (in the first quadrant). Finally, find an explicit formula for a solution to the system (10.9), for $-\infty < t < \infty$, whose trajectory traverses the portion of this line in the first quadrant by solving the initial value problem

$$\frac{dx}{dt} = x(2 - x - y), \quad \frac{dy}{dt} = y(1 - x - y), \qquad x(0) = 1, \quad y(0) = 1/2.$$

To do this, substitute $1 - x/2$ for y in the first equation and $2 - 2y$ for x in the second equation to obtain

$$\frac{dx}{dt} = x\left(1 - \frac{x}{2}\right), \quad \frac{dy}{dt} = y\,(y - 1)\,.$$

Solve both of these first-order equations, with the respective initial conditions $x\,(0) = 1$ and $y(0) = 1/2$, by separation of variables or as Bernoulli equations or using a CAS.

(c) Suppose we have any solution to the system (10.9) satisfying $x(0) > 0$ and $y(0) > 0$. Show that the trajectory of the solution for $t \geq 0$ has at most one inflection point.

18. (CAS) Consider the system

$$\frac{dx}{dt} = x(2-x-y), \quad \frac{dy}{dt} = 4y(1-x-y). \tag{10.12}$$

(a) Using

$$\frac{dy}{dx} = \frac{4y(1-x-y)}{x(2-x-y)}$$

and

$$\frac{d^2y}{dx^2} = \frac{4y(-4+14x-12x^2+3x^3+(16-27x+9x^2)y+3(-5+3x)y^2+3y^3)}{x^2(-2+x+y)^3},$$

produce a concavity chart in the first quadrant for the system (10.12). Hint: Use a CAS to determine the curves in the first quadrant where $-4+14x-12x^2+3x^3+(16-27x+9x^2)y+3(-5+3x)y^2+3y^3=0$ and also sketch the line $y=2-x$ (where the second derivative is undefined) in this quadrant. These divide the first quadrant into regions where the concavity can be determined using test values in the formula for $\frac{d^2y}{dx^2}$.

(b) Using a CAS, add to your concavity chart from (a) a trajectory having two points of inflection (in the first quadrant).

10.2. Linear approximation: The major cases

The basic idea of this section is both simple and appealing: We can often predict what the phase portrait of a nonlinear system looks like *near an isolated equilibrium point* by relating it to an associated linear system whose phase portrait we know. Our first task is to explain how to find this associated linear system.

We begin with a brief review of linear approximation from calculus. Suppose we have a differentiable function $f(x)$ and we pick a point $(x_0, f(x_0))$ on the graph of f. To get the best straight line approximation to the graph of f near the point $(x_0, f(x_0))$, we use the tangent line passing through $(x_0, f(x_0))$. This line is the graph of the function

$$y = f(x_0) + f'(x_0)(x-x_0), \tag{10.13}$$

and (10.13) is the best linear approximation to f near x_0. We have

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + R_f(x), \tag{10.14}$$

where the "error term" $R_f(x)$ (the difference between $f(x)$ and the linear approximation (10.13)) is not just small when x is near x_0, but small in comparison to the distance $|x-x_0|$ between x and x_0. By this we mean

$$\frac{|R_f(x)|}{|x-x_0|} \to 0 \quad \text{as } x \to x_0. \tag{10.15}$$

The condition in (10.15) is just a restatement of the familiar definition of the derivative of f at x_0 from calculus.

For a differentiable function of two variables, say, $f(x,y)$, the graph of $z=f(x,y)$ is a surface in $\mathbb{R}^3$. The best linear approximation to this surface near a point $P=(x_0,y_0,f(x_0,y_0))$ is obtained by using the *tangent plane* to the surface passing through P. This tangent plane has equation

$$z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0).$$

Keep in mind that for a given point P, $\frac{\partial f}{\partial x}(x_0, y_0)$ and $\frac{\partial f}{\partial y}(x_0, y_0)$ are just numbers. Thus, the best linear approximation to f near (x_0, y_0) should be given by the function that takes (x, y) to

$$f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$

If the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous at and near (x_0, y_0), this is a good approximation in the following sense: If we move away from (x_0, y_0) to the point (x, y), then

$$f(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) + R_f(x, y), \tag{10.16}$$

where the error term $R_f(x, y)$, now a function of two variables, is not only small when (x, y) is near (x_0, y_0), but small compared to the distance

$$\|\mathbf{X} - \mathbf{X}_0\| = \|(x, y) - (x_0, y_0)\| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

between the points $\mathbf{X} = (x, y)$ and $\mathbf{X}_0 = (x_0, y_0)$. This means that

$$\frac{|R_f(\mathbf{X})|}{\|\mathbf{X} - \mathbf{X}_0\|} \to 0 \quad \text{as } \mathbf{X} \to \mathbf{X}_0, \tag{10.17}$$

which is the two-variable analogue of (10.15).

Verifying (10.17) requires more work than the one-variable case and really does use the assumption that the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are themselves continuous functions of (x, y). This assumption on a function $f(x, y)$ is important enough that we give it an official name.

Definition 10.2.1. A function $f(x, y)$ is *continuously differentiable* if the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at all points of the domain of f and are themselves continuous functions of (x, y).

Now, let's consider a vector field $\mathbf{F}(x, y) = (f(x, y), g(x, y))$. We say that $\mathbf{F}$ is continuously differentiable if both coordinate functions f and g are continuously differentiable. In this case the remarks above apply to the second coordinate function g, as well as to the function f. Specifically,

$$z = g(x_0, y_0) + \frac{\partial g}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial g}{\partial y}(x_0, y_0)(y - y_0)$$

is the best linear approximation to g near $\mathbf{X}_0 = (x_0, y_0)$, and its graph is the tangent plane to the graph of g at the point $(x_0, y_0, g(x_0, y_0))$. Moreover,

$$g(x, y) = g(x_0, y_0) + \frac{\partial g}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial g}{\partial y}(x_0, y_0)(y - y_0) + R_g(x, y) \tag{10.18}$$

where the error term R_g satisfies

$$\frac{|R_g(\mathbf{X})|}{\|\mathbf{X} - \mathbf{X}_0\|} \to 0 \quad \text{as } \mathbf{X} \to \mathbf{X}_0. \tag{10.19}$$

Let's put equations (10.16) and (10.18) together in vector form. Writing vectors as columns, we have

$$\mathbf{F}(\mathbf{X}) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{pmatrix} + \begin{pmatrix} \dfrac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \dfrac{\partial f}{\partial y}(x_0,y_0)(y-y_0) \\[2ex] \dfrac{\partial g}{\partial x}(x_0,y_0)(x-x_0) + \dfrac{\partial g}{\partial y}(x_0,y_0)(y-y_0) \end{pmatrix} + \begin{pmatrix} R_f(x,y) \\ R_g(x,y) \end{pmatrix}. \tag{10.20}$$

The first and third terms on the right-hand side of equation (10.20) can be written concisely in vector form as

$$\begin{pmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{pmatrix} = \mathbf{F}(x_0,y_0) = \mathbf{F}(\mathbf{X}_0)$$

(where $\mathbf{X}_0 = (x_0, y_0)$) and

$$\begin{pmatrix} R_f(x,y) \\ R_g(x,y) \end{pmatrix} = \mathbf{R}(x,y) = \mathbf{R}(\mathbf{X}),$$

where the "error" vector field $\mathbf{R}$ is defined by this last equation.

Since $\|\mathbf{R}(x,y)\| = \sqrt{R_f(x,y)^2 + R_g(x,y)^2}$, the limit requirements (10.17) and (10.19) together say that

$$\frac{\|\mathbf{R}(\mathbf{X})\|}{\|\mathbf{X}-\mathbf{X}_0\|} \to 0 \quad \text{as } \mathbf{X} \to \mathbf{X}_0. \tag{10.21}$$

On the other hand, the middle term on the right-hand side of equation (10.20) is the product of a 2×2 matrix $\mathbf{J}$ and a 2×1 vector $\mathbf{X} - \mathbf{X}_0$, where

$$\mathbf{J} = \begin{pmatrix} \dfrac{\partial f}{\partial x}(x_0,y_0) & \dfrac{\partial f}{\partial y}(x_0,y_0) \\[2ex] \dfrac{\partial g}{\partial x}(x_0,y_0) & \dfrac{\partial g}{\partial y}(x_0,y_0) \end{pmatrix} \tag{10.22}$$

and

$$\mathbf{X} - \mathbf{X}_0 = \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}.$$

The matrix $\mathbf{J}$ is called the **Jacobian matrix** of $\mathbf{F}$ at $\mathbf{X}_0 = (x_0, y_0)$. To emphasize that the partial derivatives in equation (10.22) are evaluated at $\mathbf{X}_0 = (x_0, y_0)$ we sometimes write $\mathbf{J}(\mathbf{X}_0)$ instead of just $\mathbf{J}$.

On putting this together, we can rewrite equation (10.20) succinctly as

$$\mathbf{F}(\mathbf{X}) = \mathbf{F}(\mathbf{X}_0) + \mathbf{J}(\mathbf{X}_0)(\mathbf{X}-\mathbf{X}_0) + \mathbf{R}(\mathbf{X}), \tag{10.23}$$

where the error term $\mathbf{R}(\mathbf{X})$ satisfies (10.21). This is the analogue for a continuously differentiable vector field of equation (10.14) for a one-variable differentiable function. The Jacobian matrix $\mathbf{J}(\mathbf{X}_0)$ for $\mathbf{F}$ at (x_0, y_0) plays the same role as the derivative $f'(x_0)$ in equation (10.14).

Now let's return to our nonlinear system

$$\frac{dx}{dt} = f(x,y), \qquad \frac{dy}{dt} = g(x,y),$$

where f and g are continuously differentiable. In terms of the vector field $\mathbf{F} = (f, g)$, we can write this system in vector form as

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}). \tag{10.24}$$

Suppose $\mathbf{X}_e = (x_e, y_e)$ is an equilibrium point of the system (10.24), so that $\mathbf{F}(\mathbf{X}_e) = \mathbf{0} = (0,0)$. Since $\mathbf{F}$ is continuously differentiable, we can invoke equation (10.23) with $\mathbf{X}_0$ chosen to be this equilibrium point $\mathbf{X}_e$. Since $\mathbf{F}(\mathbf{X}_e) = \mathbf{0}$, (10.23) reduces to

$$\mathbf{F}(\mathbf{X}) = \mathbf{J}(\mathbf{X}_e)(\mathbf{X} - \mathbf{X}_e) + \mathbf{R}(\mathbf{X}).$$

If the determinant of $\mathbf{J}(\mathbf{X}_e)$ is not zero, the Jacobian matrix $\mathbf{J}(\mathbf{X}_e)$ is invertible and $\|\mathbf{J}(\mathbf{X}_e)(\mathbf{X} - \mathbf{X}_e)\|$ tends to zero as $\mathbf{X} \to \mathbf{X}_e$ at the same rate as $\|\mathbf{X} - \mathbf{X}_e\|$. However, the limit condition (10.21) says that the length of $\mathbf{R}(\mathbf{X})$ tends to zero even faster than $\|\mathbf{X} - \mathbf{X}_e\|$, leaving $\mathbf{J}(\mathbf{X}_e)(\mathbf{X} - \mathbf{X}_e)$ as the dominant part of $\mathbf{F}(\mathbf{X})$ when $\mathbf{X}$ is near $\mathbf{X}_e$. Thus it is reasonable to expect the trajectories of the system (10.24) to resemble those of the system

$$\frac{d\mathbf{X}}{dt} = \mathbf{J}(\mathbf{X}_e)(\mathbf{X} - \mathbf{X}_e) \tag{10.25}$$

for $\mathbf{X}$ near $\mathbf{X}_e$.

The system (10.25) is "translated linear" with an equilibrium point at $\mathbf{X}_e$ and is easily solved: If $\mathbf{Y}$ is a solution of the truly linear system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{J}(\mathbf{X}_e)\mathbf{Y}, \tag{10.26}$$

then it's easy to check that $\mathbf{X}(t) = \mathbf{Y}(t) + \mathbf{X}_e$ satisfies (10.25). Conversely, if $\mathbf{X}$ satisfies (10.25), then $\mathbf{Y}(t) = \mathbf{X}(t) - \mathbf{X}_e$ satisfies (10.26). Note that we can quickly solve the linear system (10.26) using techniques discussed in Chapter 8 or Chapter 9.

Recall that when $\mathbf{J}(\mathbf{X}_e)$ is invertible, $\mathbf{0} = (0,0)$ is the only equilibrium point of the linear system (10.26); in addition, not only is $\mathbf{X}_e$ the only equilibrium point of the translated linear system (10.25), but also it is possible to show that $\mathbf{X}_e$ must be an *isolated* equilibrium point of the *original nonlinear system*. The equation $\mathbf{X}(t) = \mathbf{Y}(t) + \mathbf{X}_e$ relating the solutions of (10.25) and (10.26) shows the equilibrium solution $\mathbf{Y}(t) \equiv \mathbf{0}$ for (10.26) is translated to the equilibrium solutions $\mathbf{X}(t) = \mathbf{X}_e$ of (10.25). In fact, all solutions of (10.26) become solutions of (10.25) when translated (shifted) by $\mathbf{X}_e$, and all solutions of (10.25) become solutions of (10.26) when translated by $-\mathbf{X}_e$. Thus the phase portrait of (10.25) is obtained simply by translating the phase portrait of (10.26) by $\mathbf{X}_e$. For example, if $(0,0)$ is a saddle point for (10.26), $\mathbf{X}_e$ will be a saddle point for the translated linear system (10.25); see Figs. 10.12 and 10.13. Except for being translated, the phase portrait of (10.25) is identical to that of (10.26). Moreover, because the translated linear system (10.25) approximates the original nonlinear system (10.24) near $\mathbf{X}_e$, we expect the phase portrait of (10.25) to *resemble*, near $\mathbf{X}_e$, the phase portrait of the nonlinear system (10.24). We now explore to what extent this expectation is valid through examples and theorems.

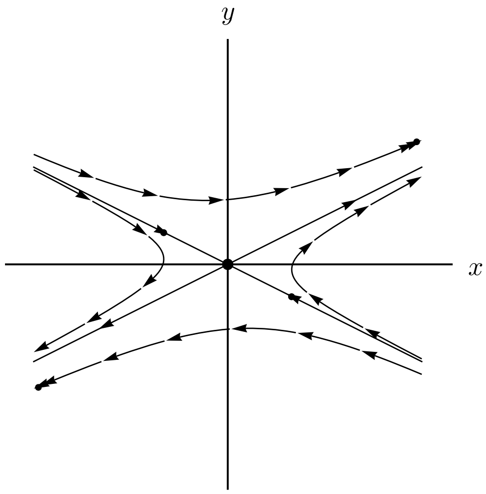


Figure 10.12. A possible phase portrait for the linear system in equation (10.26).

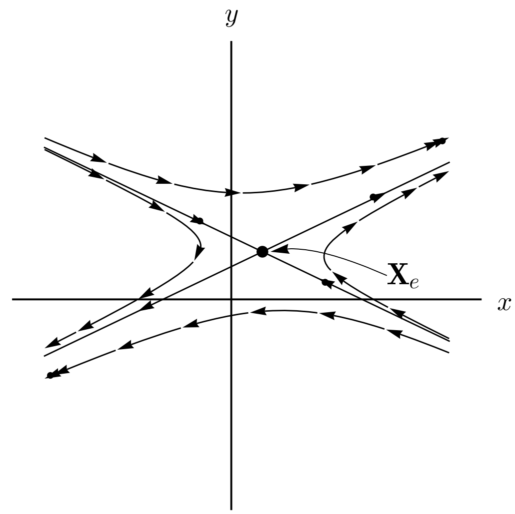


Figure 10.13. A phase portrait for the corresponding translated linear system in equation (10.25).

Example 10.2.2. Consider the nonlinear system

$$\frac{dx}{dt} = 1 - y, \qquad \frac{dy}{dt} = x^2 - y^2.$$

The equilibrium points, which occur where $1-y$ and x^2-y^2 are simultaneously zero, are $(1,1)$ and $(-1,1)$. In the notation from our discussion above we have $f(x,y) = 1-y$ and $g(x,y) = x^2 - y^2$. Computing the first-order partial derivatives of f and g gives

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = -1, \qquad \frac{\partial g}{\partial x} = 2x, \quad \frac{\partial g}{\partial y} = -2y.$$

Evaluating these derivatives at $(1,1)$ and $(-1,1)$, we see that the Jacobian matrix $\mathbf{J}_1$ at $(1,1)$ and the Jacobian matrix $\mathbf{J}_2$ at $(-1,1)$ are

$$\mathbf{J}_1 = \begin{pmatrix} 0 & -1 \\ 2 & -2 \end{pmatrix} \quad \text{and} \quad \mathbf{J}_2 = \begin{pmatrix} 0 & -1 \\ -2 & -2 \end{pmatrix}.$$

Let's focus on the equilibrium point $(1,1)$. The eigenvalues of $\mathbf{J}_1$ are the roots of the characteristic polynomial

$$\det(\mathbf{J}_1 - z\mathbf{I}) = \det\begin{pmatrix} -z & -1 \\ 2 & -2-z \end{pmatrix} = z^2 + 2z + 2,$$

which are easily calculated to be the complex conjugate pair $-1 \pm i$. The point $(0,0)$ is thus a stable spiral point of the linear system $\mathbf{X}' = \mathbf{J}_1\mathbf{X}$. We can reach the same conclusion with less computational work by applying the methods of Section 8.9. The matrix $\mathbf{J}_1$ has determinant Δ equal to 2 and trace τ equal to -2. Since $\tau^2 - 4\Delta < 0$, Table 8.2 (or Fig. 8.74) in Section 8.9 tells us that the linear system $\mathbf{X}' = \mathbf{J}_1\mathbf{X}$ has a stable spiral point at the origin. In a similar way you can either compute that $\mathbf{J}_2$ has eigenvalues $-1 \pm \sqrt{3}$ (one positive and one negative value) or simply observe that the determinant of $\mathbf{J}_2$ is negative, to conclude that the linear system $\mathbf{X}' = \mathbf{J}_2\mathbf{X}$ has a saddle point at the origin. We'll see next how this information carries over to the original nonlinear system.

Major types. Roughly speaking, the next result says that under most conditions the phase portrait for a nonlinear system $\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X})$ near one of its equilibrium points $\mathbf{X}_e$ will resemble that of the related linear system $\frac{d\mathbf{Y}}{dt} = \mathbf{J}(\mathbf{X}_e)\mathbf{Y}$ near its equilibrium point $(0,0)$, where $\mathbf{J}(\mathbf{X}_e)$ is the Jacobian matrix of $\mathbf{F}$ evaluated at $\mathbf{X}_e$. We use the familiar classification terminology from Chapter 8 of "saddle point", "node", and "spiral point", and so on to capture this idea. Recall that nondegenerate nodes correspond to two different real eigenvalues with the same sign, while a planar linear system has a degenerate node at $(0,0)$ if the associated matrix has a repeated eigenvalue. We will refer to saddle points, spiral points, and nondegenerate nodes as **major types** and get the best information in these cases:

Theorem 10.2.3. *Suppose we have an autonomous nonlinear system*

$$\frac{dx}{dt} = f(x,y), \qquad \frac{dy}{dt} = g(x,y), \tag{10.27}$$

where $f(x,y)$ and $g(x,y)$ are continuously differentiable. Suppose (x_e, y_e) is an isolated equilibrium point for (10.27)*. Consider the related linear system*

$$\mathbf{X}' = \mathbf{J}\mathbf{X} \tag{10.28}$$

where $\mathbf{J}$ is the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f}{\partial x}(x_e, y_e) & \frac{\partial f}{\partial y}(x_e, y_e) \\ \frac{\partial g}{\partial x}(x_e, y_e) & \frac{\partial g}{\partial y}(x_e, y_e) \end{pmatrix},$$

and assume that $(0,0)$ is an isolated equilibrium point for this linear system (this is the same as saying $\det \mathbf{J} \neq 0$).

If $(0,0)$ is one of the major types (saddle point, spiral point, or nondegenerate node) for the linear system (10.28)*, then (x_e, y_e) is the same type for the nonlinear system. Moreover, for these major types, the stability of the equilibrium point is the same for the nonlinear system as it is for the linear system.*

Borderline types. Note that Theorem 10.2.3 does not apply if $(0,0)$ is a center or degenerate node for the linear system $\mathbf{X}' = \mathbf{J}\mathbf{X}$. These cases are discussed in the next theorem, which will be further explored in Section 10.3.

Theorem 10.2.4. *Under the same hypotheses as Theorem* 10.2.3*, we have the following:*

If the linear system $\mathbf{X}' = \mathbf{J}\mathbf{X}$ has a degenerate node at $(0,0)$, then the nonlinear system has either a node or spiral at (x_e, y_e), and the stability is the same for both systems.

If the linear system $\mathbf{X}' = \mathbf{J}\mathbf{X}$ has a center at $(0,0)$, then the nonlinear system has either a center, a spiral, or a hybrid center/spiral at (x_e, y_e). In this case, we cannot predict the stability of (x_e, y_e) for (10.27)*.*

It is helpful to keep the picture of the trace-determinant plane from Section 8.9 in mind here. Degenerate nodes correspond there to points on the parabola $4\Delta = \tau^2$; to either side of this parabola are regions corresponding to spirals and nodes. Centers correspond to the positive half of the Δ-axis; the bordering regions on either side of this ray correspond to asymptotically stable spirals or unstable spirals.

The terminology "hybrid center/spiral" refers to a situation where in each disk centered at the equilibrium point there are closed curve orbits surrounding the equilibrium point, but between two such orbits there may be spiral trajectories. We'll see an example in Exercise 8 of Section 10.3. This situation does not occur if the component functions $f(x,y)$ and $g(x,y)$ are sufficiently nice (as will be the case for nearly every example we consider)[5].

[5]In particular, it cannot occur if f and g are polynomials in x and y.

Tables 10.1 and 10.2 summarize the conclusions of Theorem 10.2.3 and 10.2.4. In these tables, we have a nonlinear system

$$\frac{dy}{dx} = f(x, y), \qquad \frac{dy}{dt} = g(x, y),$$

where f and g are continuously differentiable, and we have an isolated equilibrium point at (x_e, y_e). We denote the Jacobian matrix at (x_e, y_e) by $\mathbf{J}$.

Table 10.1. Major Cases

Eigenvalues of $\mathbf{J}$	Classification for $\mathbf{X}' = \mathbf{JX}$	Classification for nonlinear
Real; $r_1 < 0$, $r_2 > 0$	saddle point (unstable)	saddle point (unstable)
Real; $r_1, r_2 > 0$, $r_1 \neq r_2$	unstable node	unstable node
Real; $r_1, r_2 < 0$, $r_1 \neq r_2$	asymptotically stable node	asymptotically stable node
Complex $\alpha \pm i\beta$; $\alpha > 0$, $\beta \neq 0$	unstable spiral	unstable spiral
Complex $\alpha \pm i\beta$; $\alpha < 0$, $\beta \neq 0$	asymptotically stable spiral	asymptotically stable spiral

Table 10.2. Borderline Cases

Eigenvalues of $\mathbf{J}$	Classification for $\mathbf{X}' = \mathbf{JX}$	Classification for nonlinear
Purely imaginary $\pm i\beta$	center (stable)	center, spiral or hybrid center/spiral stability not determined
Real; $r_1 = r_2 > 0$	degenerate node; unstable	node or spiral point unstable
Real; $r_1 = r_2 < 0$	degenerate node asymptotically stable	node or spiral point; asymptotically stable

Theorem 10.2.3 asserts that when the equilibrium point $(0, 0)$ of the linear system $\mathbf{X}' = \mathbf{JX}$, where $\mathbf{J}$ is the Jacobian matrix at (x_e, y_e), is one of the major types, the corresponding nonlinear equilibrium (x_e, y_e) inherits the same type. This is very useful but a bit ambiguous since we have not given a mathematical definition of a *nonlinear* node, saddle, or spiral point. An accurate (even though heuristic) way to think about this is that trajectories near a nonlinear node, saddle, or spiral point (x_e, y_e) look like slightly distorted versions of the trajectories of the corresponding linear system near its equilibrium $(0, 0)$. The resemblance is strongest near (x_e, y_e) but fades (or

even disappears) far from (x_e, y_e).[6] However, it is reasonable to seek information more concrete and mathematically detailed. Such results do exist, depending on the type of the equilibrium. For instance, here's a general definition of a **node**.

Nonlinear nodes. Let

$$\frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y) \tag{10.29}$$

be an autonomous system such that f and g are continuously differentiable. Roughly speaking, an equilibrium point (x_e, y_e) for (10.29) is a **stable node** provided that the trajectory of any solution that starts sufficiently close to (x_e, y_e) will approach (x_e, y_e), as $t \to \infty$, tangent to some line through (x_e, y_e) (so that it approaches in a specific direction). More precisely, (x_e, y_e) is a stable node if there is a $\rho > 0$ such that each solution $x = x(t)$, $y = y(t)$ of the system satisfying

$$\|(x(0), y(0)) - (x_e, y_e)\| < \rho \tag{10.30}$$

exists for all $t \geq 0$, has limit (x_e, y_e) as $t \to \infty$, and the trajectory of the solution approaches (x_e, y_e) tangent to some line through (x_e, y_e). *Observe that a stable node is necessarily asymptotically stable.*

The analogous notion of an unstable node involves approach to (x_e, y_e) as $t \to -\infty$: An equilibrium point (x_e, y_e) for the system (10.29) is an **unstable node** provided that there is some $\rho > 0$ such that each solution $x = x(t)$, $y = y(t)$ of the system satisfying inequality (10.30) exists for all $t \leq 0$, has limit (x_e, y_e) as $t \to -\infty$, and the trajectory of the solution approaches (x_e, y_e) (as $t \to -\infty$) tangent to some line through (x_e, y_e).

In keeping with the terminology we have used to describe nodes in the linear setting, we say a node (x_e, y_e) of the system (10.29) is **proper node** provided that for *every* line L through (x_e, y_e), there is a trajectory of the system that approaches (x_e, y_e) tangent to L. Nodes that are not proper are **improper**. If the Jacobian matrix $\mathbf{J}(x_e, y_e)$ of the system (10.29) has two distinct eigenvalues having the same sign, then, just as in the linear case, there will be exactly two lines through (x_e, y_e) to which trajectories will be tangent. Thus, (x_e, y_e) will be an improper node. Moreover, just as in the linear case, one of the lines will be tangent to many distinct trajectories while the other will be tangent to only two trajectories, approaching (x_e, y_e) from opposite directions. Just as in the linear case, we will call these two tangent lines the **nodal tangents** for (10.29) at (x_e, y_e). It can be shown that these nodal tangents are precisely those of the linear system $\mathbf{X}' = \mathbf{J}(x_e, y_e)\mathbf{X}$, but shifted by x_e in the x-direction and by y_e in the y-direction.

Example 10.2.5. Consider the system

$$\frac{dx}{dt} = -5xy + 4y + x, \quad \frac{dy}{dt} = -x^2 - y^5 + 2, \tag{10.31}$$

which has $(1, 1)$ as an equilibrium point. The system's Jacobian matrix evaluated at $(1, 1)$ is

$$\mathbf{J} = \begin{pmatrix} -4 & -1 \\ -2 & -5 \end{pmatrix},$$

which has eigenvalues -3 and -6, with corresponding eigenvectors

$$\mathbf{V}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ for the eigenvalue } -3 \quad \text{and} \quad \mathbf{V}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ for the eigenvalue } -6.$$

The eigenvectors yield the nodal tangents $y = -x$ and $y = 2x$ for the linear system $\mathbf{X}' = \mathbf{J}\mathbf{X}$. Thus, those for the nonlinear system (10.31) are $y = -(x - 1) + 1$ and $y = 2(x - 1) + 1$; see Figs. 10.14 and 10.15.

[6] You can see an illustration of this remark in Figs. 10.14 and 10.15 by looking at the trajectory in the lower right corner of Fig. 10.15.

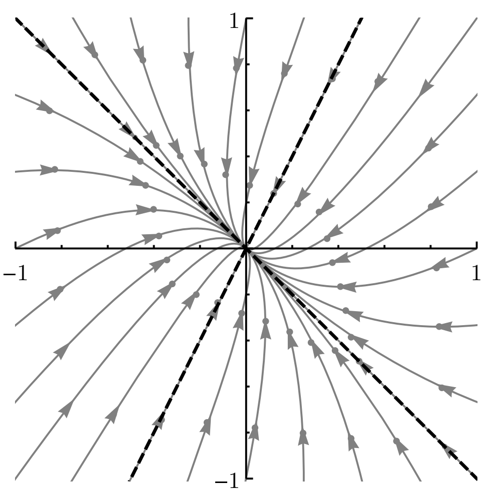


Figure 10.14. The lines $y = -x$ and $y = 2x$, dashed, are nodal tangents of $\mathbf{X}' = \mathbf{J}\mathbf{X}$, where $\mathbf{J} = \mathbf{J}(1, 1)$ for the system (10.31).

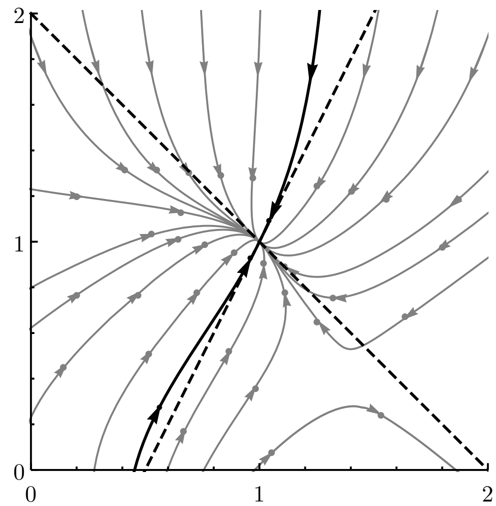


Figure 10.15. The lines $y = -(x-1)+1$ and $y = 2(x-1)+1$, dashed, are nodal tangents for the system (10.31), with the first being the preferred nodal tangent.

Note that for the linear system $\mathbf{X}' = \mathbf{J}\mathbf{X}$ of Fig. 10.14, the line $y = -x$ is the "preferred nodal tangent" in that all trajectories of the system, except those along the ray solutions $\mathbf{X} = e^{-6t}\mathbf{V}_2$ and $\mathbf{X} = -e^{-6t}\mathbf{V}_2$, are tangent to $y = -x$ at $(0, 0)$. Looking to Fig. 10.15, we see the translate $y = -(x - 1) + 1$ of $y = -x$ is the preferred nodal tangent for the nonlinear system's node at $(1, 1)$. This reflects a general result. Recall that the preferred nodal tangent for a 2×2 linear system $\mathbf{X}' = \mathbf{A}\mathbf{X}$, where $\mathbf{A}$ has distinct real eigenvalues of the same sign, is the eigenvector line for the eigenvector of $\mathbf{A}$ corresponding to the eigenvalue of smaller absolute value. Thus, we have the following general result:

Theorem 10.2.6. *Assume that the vector field $\mathbf{F} = (f, g)$ is continuously differentiable, (x_e, y_e) is an equilibrium point of the system*

$$\frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y),$$

and $\mathbf{J}$ is the Jacobian matrix of $\mathbf{F}$ at (x_e, y_e). Suppose that $\mathbf{J}$ has distinct real eigenvalues λ_1 and λ_2 of the same sign, having, respectively, corresponding eigenvectors $\mathbf{V}_1$ and $\mathbf{V}_2$. Then (x_e, y_e) is a node for this system whose nodal tangents will be the lines L_1 and L_2, passing through (x_e, y_e), and parallel to $\mathbf{V}_1$ and $\mathbf{V}_2$, respectively. Moreover, the preferred nodal tangent will be L_j if λ_j is the eigenvalue having smaller absolute value.

We can similarly give a nice description of saddle points for nonlinear systems; see below. For a mathematical description of nonlinear spiral points, see Exercise 31.

Nonlinear saddle points. Suppose that $\mathbf{F} = (f, g)$ is a continuously differentiable vector field and that (x_e, y_e) is an equilibrium point of the system

$$\frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y). \tag{10.32}$$

Suppose that the Jacobian matrix $\mathbf{J}$ at (x_e, y_e) has eigenvalues r_1 and r_2 with $r_1 < 0 < r_2$ and associated eigenvectors $\mathbf{V}_1$ for r_1 and $\mathbf{V}_2$ for r_2. The equilibrium $\mathbf{0} = (0, 0)$ for the linear system $\mathbf{X}' = \mathbf{J}\mathbf{X}$ is a saddle point. We know from Section 8.4 that the incoming ray solutions for this system all lie on the line through $\mathbf{0}$ and in the direction $\mathbf{V}_1$. Since these ray solutions tend to the equilibrium $\mathbf{0}$ as $t \to \infty$, we call this eigenvector line the **stable line**. Similarly, the outgoing ray solutions all lie on the line containing $\mathbf{0}$ and in the direction $\mathbf{V}_2$. We call this the **unstable line** since these solutions all leave the vicinity of $\mathbf{0}$ as $t \to \infty$; see Fig. 10.16.

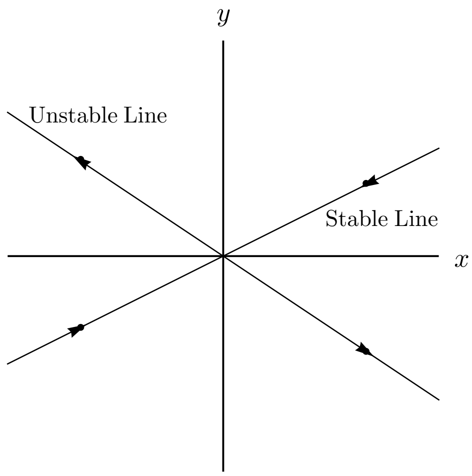


Figure 10.16. Stable and unstable lines for $\mathbf{X}' = \mathbf{J}\mathbf{X}$.

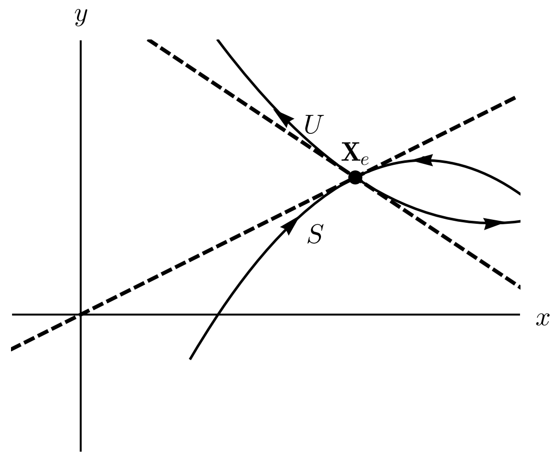


Figure 10.17. Stable and unstable curves S and U for the nonlinear system $\mathbf{X}' = \mathbf{F}(\mathbf{X})$.

The new ingredient here is that our nonlinear system possesses analogues of the stable and unstable lines, namely the **stable curve** S and **unstable curve** U associated to the equilibrium point $\mathbf{X}_e$. The stable curve S and the unstable curve U both pass through $\mathbf{X}_e$ and have continuously turning tangent lines. The stable curve S has the following properties:

(S_1) Any solution $\mathbf{X}(t)$ of the system (10.32) having initial value $\mathbf{X}(0)$ that lies on S must exist and remain on S for all times $t \geq 0$.

(S_2) $\mathbf{X}(t) \to \mathbf{X}_e$ as $t \to \infty$.

The unstable curve U has the following companion properties:

(U_1) Any solution $\mathbf{X}(t)$ of the system (10.32) with $\mathbf{X}(0)$ lying on U must exist and remain on U for all time $t \leq 0$.

(U_2) $\mathbf{X}(t) \to \mathbf{X}_e$ as $t \to -\infty$.

Note that a solution $\mathbf{X}(t)$ lying on S tends to $\mathbf{X}_e$ as time t goes forward towards ∞, while a solution $\mathbf{X}(t)$ lying on U approaches $\mathbf{X}_e$ as time t goes backwards towards $-\infty$. We have the following useful result:

Theorem 10.2.7 (Stable Curve Theorem)**.** *Assume that the vector field* $\mathbf{F} = (f, g)$ *is continuously differentiable,* (x_e, y_e) *is an equilibrium point of the system* (10.32)*, and* $\mathbf{J}$ *is the Jacobian matrix of* $\mathbf{F}$ *at* (x_e, y_e)*. Suppose that* $(0, 0)$ *is a saddle point for the linear system* $\mathbf{X}' = \mathbf{J}\mathbf{X}$*. The stable curve* S *and unstable curve* U *exist with the properties* (S_1), (S_2), (U_1), (U_2) *described above, and the tangent lines to* S *and* U *at* (x_e, y_e) *are parallel to the stable line and unstable line, respectively, of the associated linear system* $\mathbf{X}' = \mathbf{J}\mathbf{X}$*.*

Fig. 10.17 above shows possible curves S and U and their tangent lines (dashed) at $\mathbf{X}_e$ for a nonlinear system whose approximating linear system has stable and unstable lines as pictured in Fig. 10.16.

Example 10.2.8. The nonlinear system

$$\frac{dx}{dt} = -x, \qquad \frac{dy}{dt} = x^2 + y \tag{10.33}$$

has $(0,0)$ as its only equilibrium point. The Jacobian matrix there is

$$\mathbf{J} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

which has eigenvalues 1 and -1. By Theorem 10.2.3, the nonlinear system has a saddle point at $(0,0)$, and we expect its phase portrait nearby to resemble the phase portrait of the linear system $\mathbf{X}' = \mathbf{J}\mathbf{X}$. Here we can verify this directly because the nonlinear system is simple enough to permit computation of the form of all solutions. Since y does not appear in the first equation of (10.33) we can solve for x in terms of t to obtain

$$x(t) = c_1 e^{-t}.$$

Substitute this into the second equation of (10.33) to get

$$\frac{dy}{dt} = c_1^2 e^{-2t} + y,$$

a first-order linear equation which we can solve for y:

$$y(t) = -\frac{c_1^2}{3} e^{-2t} + c_2 e^t.$$

Choosing $c_1 = 0, c_2 = 1$ gives the solution $x(t) = 0, y(t) = e^t$, which describes the positive half of the y-axis, directed away from the origin with increasing time. The trajectory corresponding to the choice $c_1 = 0, c_2 = -1$ is along the negative half of the y-axis, still directed away from the origin. This trajectory lies on the unstable curve U, which consists of the y-axis (note that as t goes backward in time, that is, as $t \to -\infty$, the point $(x(t), y(t)) = (0, -e^t)$ tends to the equilibrium point $(0,0)$, in agreement with the definition of U). When $c_1 = 1$ and $c_2 = 0$ we have the solution

$$x(t) = e^{-t}, \quad y(t) = -\frac{1}{3} e^{-2t}.$$

These are parametric equations for the part of the parabola $y = -\frac{1}{3}x^2$ in the fourth quadrant (why?), and as $t \to \infty$ we approach $(0,0)$ along this parabolic curve; see Fig. 10.19. Similarly, the choice $c_1 = -1, c_2 = 0$ gives a trajectory along the other half of the same parabola, directed in to $(0,0)$. The entire parabola is the stable curve S.

A sketch of the phase portrait is shown in Fig. 10.19; notice how the overall picture near $(0,0)$ looks like a distorted version of the phase portrait in Fig. 10.18 for the approximating linear system

$$\mathbf{X}' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{X}.$$

The trajectories for the nonlinear system (Fig. 10.19) have the following properties, which are characteristic of the trajectories of a linear system with a saddle point at $\mathbf{0}$:

- All trajectories that approach the equilibrium point $\mathbf{0}$ as $t \to \infty$ lie on a single curve through $\mathbf{0}$—the stable curve $y = -x^2/3$.
- All trajectories that approach $\mathbf{0}$ as $t \to -\infty$ lie on another curve through the origin—the unstable curve $x = 0$.
- No other trajectory (aside from the equilibrium solution $\mathbf{X}(t) = \mathbf{0}$) will approach $\mathbf{0}$ as $t \to \infty$ or as $t \to -\infty$ and $\mathbf{0}$ is an unstable equilibrium point.

Finally, just as in the linear case, solutions that lie on different sides of the stable curve have very different behavior (here, tending upward without bound if above the stable curve and downward without bound if below the stable curve). For this reason, we call the stable curve a **separatrix** of the nonlinear system: It separates two different types of behavior for trajectories.

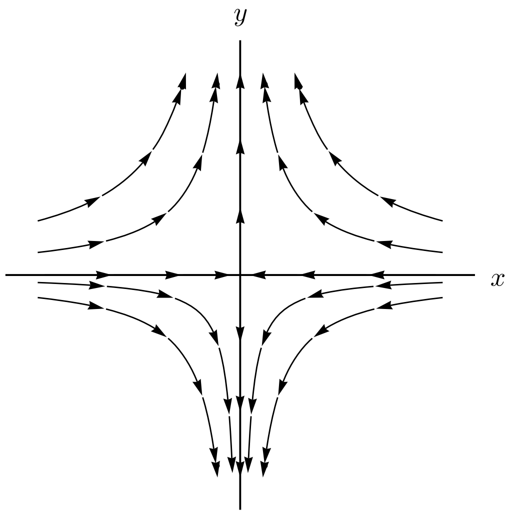


Figure 10.18. Phase portrait for approximating linear system.

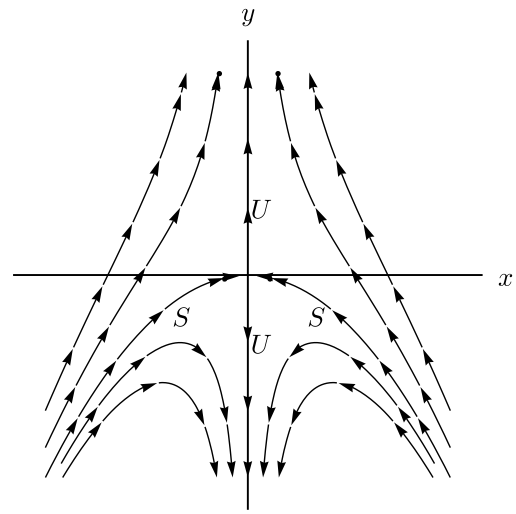


Figure 10.19. Phase portrait with stable curve S and unstable curve U.

Example 10.2.9. The nonlinear system

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -y - \sin x \tag{10.34}$$

has equilibrium points at $(\pm n\pi, 0)$ for $n = 0, 1, 2, 3 \ldots$. The Jacobian matrix is

$$\begin{pmatrix} 0 & 1 \\ -\cos x & -1 \end{pmatrix}.$$

When n is an even integer, $n = 0, \pm 2, \pm 4, \ldots$, we have $\cos(n\pi) = 1$, and when n is an odd integer, $n = \pm 1, \pm 3, \ldots$, we have $\cos(n\pi) = -1$. So when n is even, the Jacobian matrix at $(\pm n\pi, 0)$ is

$$\mathbf{J}_e = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},$$

and when n is odd, it is

$$\mathbf{J}_o = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

The matrix $\mathbf{J}_e$ has eigenvalues $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ while $\mathbf{J}_o$ has eigenvalues $-\frac{1}{2} \pm \frac{\sqrt{5}}{2}$. By Theorem 10.2.3 we expect the phase portrait of the nonlinear system to look like a saddle point at $(\pm\pi, 0)$, $(\pm 3\pi, 0)$, $(\pm 5\pi, 0), \ldots$ and like a stable spiral at $(0, 0), (\pm 2\pi, 0)$, $(\pm 4\pi, 0)$, $(\pm 6\pi, 0), \ldots$; see Fig 10.20. You should "see" the stable and unstable curves at each saddle point, as well as the different behavior of trajectories on opposite sides of the pictured stable curves (separatrices) $S_{-\pi}$ and S_{π}.

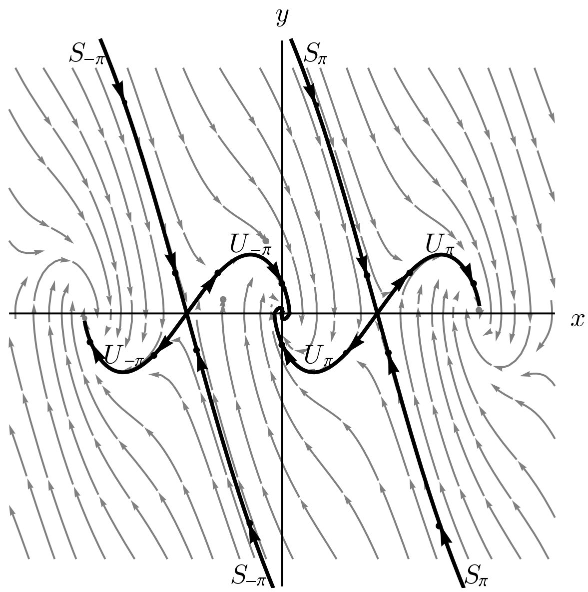


Figure 10.20. The phase portrait.

Example 10.2.10. We will revisit the competing species model introduced in the last section, now including an analysis of the equilibrium points. Our model is described by the equations

$$\frac{dx}{dt} = ax\left(\frac{K_1 - x - y}{K_1}\right), \qquad \frac{dy}{dt} = cy\left(\frac{K_2 - y - x}{K_2}\right) \tag{10.35}$$

for the populations of two similar species competing for the same resources. As in Section 10.1, we will assume $K_1 > K_2$. The equilibrium points are $(0,0)$, $(K_1,0)$, and $(0,K_2)$. The Jacobian matrix is

$$\begin{pmatrix} a - \frac{2a}{K_1}x - \frac{a}{K_1}y & -\frac{a}{K_1}x \\ -\frac{c}{K_2}y & c - \frac{2c}{K_2}y - \frac{c}{K_2}x \end{pmatrix},$$

which is obtained by computing the first-order partial derivatives of

$$f(x,y) = ax - \frac{a}{K_1}x^2 - \frac{a}{K_1}xy \quad \text{and} \quad g(x,y) = cy - \frac{c}{K_2}y^2 - \frac{c}{K_2}xy.$$

To classify the equilibrium point $(0,0)$, we evaluate the Jacobian matrix at this point, obtaining the matrix

$$\mathbf{J}_1 = \mathbf{J}(0,0) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix},$$

which has two positive eigenvalues, a and c (recall, as in Section 10.1, we assume $a \neq c$). The equilibrium point $(0,0)$ is an unstable node for our nonlinear system, and we expect the phase portrait near $(0,0)$ to look like a mildly distorted version of the phase portrait for the linear system $\mathbf{X}' = \mathbf{J}_1\mathbf{X}$.

Evaluated at the equilibrium point $(K_1,0)$, the Jacobian matrix is

$$\mathbf{J}_2 = \mathbf{J}(K_1,0) = \begin{pmatrix} -a & -a \\ 0 & c - c\frac{K_1}{K_2} \end{pmatrix}.$$

The eigenvalues of this matrix are $-a$ and $c - c\frac{K_1}{K_2}$. These are both negative since we are assuming $K_1 > K_2$ and thus $\mathbf{X}' = \mathbf{J}_2\mathbf{X}$ has stable node at $(K_1,0)$ (nondegenerate if $a \neq c(\frac{K_1}{K_2} - 1)$).

At the equilibrium point $(0, K_2)$, the Jacobian matrix is

$$\mathbf{J}_3 = \begin{pmatrix} a - a\frac{K_2}{K_1} & 0 \\ -c & -c \end{pmatrix}$$

which has eigenvalues $-c$ and $a - a\frac{K_2}{K_1}$. Using our assumption $K_1 > K_2$ we see that one eigenvalue is positive and one is negative, so that the point $(0, K_2)$ is classified as a saddle point for the original nonlinear system.

The classification of the equilibrium points helps us fill out the sketch of the phase portrait as begun in the previous section; see Fig. 10.21. With the phase portrait in hand, we predict that for any solution $(x(t), y(t))$ with $x(0) > 0$ and $y(0) > 0$ we have $x(t) \to K_1$ and $y(t) \to 0$ as $t \to \infty$. Thus our model predicts that one species is driven to extinction, while the other approaches its carrying capacity. Moreover, the surviving species is the one corresponding to the larger initial carrying capacity.

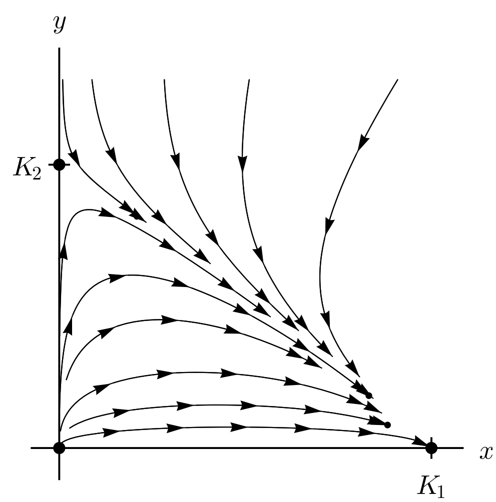


Figure 10.21. Typical phase portrait for (10.35); $K_1 > K_2$.

Example 10.2.11. The system

$$\frac{dx}{dt} = x(1 - x - 2y), \qquad \frac{dy}{dt} = y(1 - y - 3x)$$

could be given the interpretation of a competing species model—two interacting populations that in isolation would grow logistically but together compete with each other for resources. Observe that this system fits the description of a quadratic competing species model (see equation (10.5)). The units for x and y could be tens of thousands (or millions) of individuals. Note that these two populations do not occupy the same niche. Thus, although the two species compete for at least some of the same resources, they don't use them (or fight for them) in exactly the same way.

The x-nullclines are the line $x + 2y = 1$ and the vertical line $x = 0$. The y-nullclines are the line $3x + y = 1$ and the horizontal line $y = 0$. Fig. 10.22 shows the first quadrant of the phase plane, with these nullclines and the corresponding direction-indicator arrows in the four basic regions partitioned by the nullclines. The equilibrium points are $(0, 0), (1, 0), (0, 1)$, and $(1/5, 2/5)$; they occur when an x-nullcline intersects a y-nullcline.

The Jacobian matrix is

$$\mathbf{J} = \begin{pmatrix} 1 - 2x - 2y & -2x \\ -3y & 1 - 2y - 3x \end{pmatrix}.$$

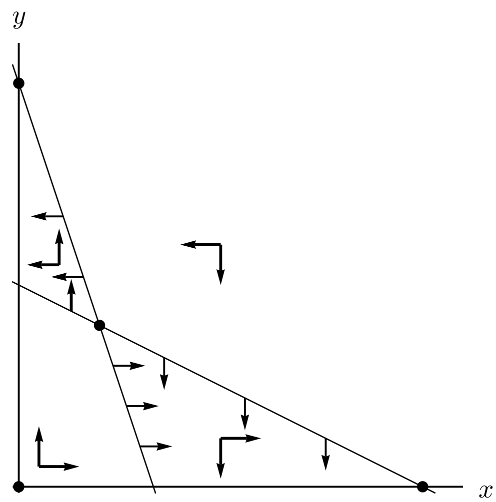


Figure 10.22. Nullcline-and-arrow diagram.

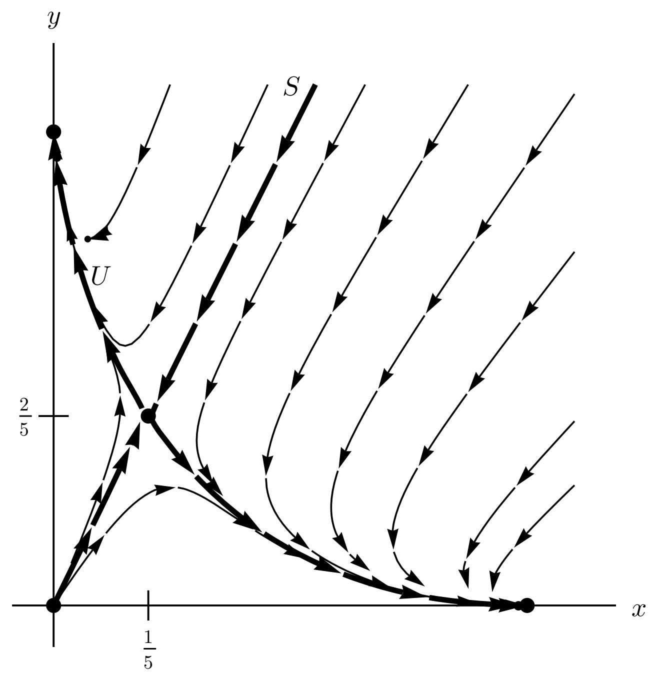


Figure 10.23. Phase portrait showing stable curve S and unstable curve U in bold.

The table below shows the eigenvalues of the Jacobian matrix $\mathbf{J}$ at each equilibrium point and the corresponding conclusion for the equilibrium point classification.

Equilibrium point classification for Example 10.2.11		
Equilibrium point	Eigenvalues of $\mathbf{J}$	Classification
$(1, 0)$	$-1, -2$	asymptotically stable node
$(0, 1)$	$-1, -1$	asymptotically stable node or spiral point
$(1/5, 2/5)$	$-1, \frac{2}{5}$	saddle
$(0, 0)$	$1, 1$	unstable node or spiral point

The Jacobian matrix at $(0, 0)$ puts us into a borderline case, and by Theorem 10.2.4 $(0, 0)$ is either an unstable spiral or an unstable node. Because the y-axis is an x-nullcline, it consists of trajectories and equilibrium points (the idea is similar to that in Exercise 9 of Section 10.1). Thus, no trajectory can cross the y-axis, preventing $(0, 0)$ from being a spiral point; hence, it is an unstable node. The other borderline case—that of the equilibrium point $(0, 1)$—is settled by the same argument: $(0, 1)$ is an asymptotically stable node (not a spiral point).

Notice that we can determine the saddle point classification at $(1/5, 2/5)$ without the computational fuss of actually determining the eigenvalues; the Jacobian matrix has determinant $-\frac{2}{5} < 0$, and by Fig. 8.74 (and Theorem 10.2.3) this tells us that we have a saddle at $(1/5, 2/5)$. There are two trajectories that approach $(1/5, 2/5)$ as $t \to \infty$ (these follow the stable curve), but unless we are so lucky to have an initial condition that puts us exactly on one of these curves, or at the equilibrium point $(1/5, 2/5)$ itself, first-quadrant trajectories of our system will be such that either $x(t) \to 1$ and $y(t) \to 0$ as $t \to \infty$ or $x(t) \to 0$ and $y(t) \to 1$ as $t \to \infty$. If these represent populations, then extinction of one species occurs. Which of these two cases occurs depends on which on which side of the separatrix the initial condition has put us. Again, we see that the separatrix separates qualitatively different behavior.

Basin of attraction. The preceding example gives a nice way to introduce the following notion: The **basin of attraction** of an asymptotically stable equilibrium point is the set of all initial points of trajectories that approach the equilibrium point as $t \to \infty$. More precisely, a point (x_0, y_0) belongs to the basin of attraction of the asymptotically stable equilibrium point (x_e, y_e) of the system $dx/dt = f(x, y), dy/dt = g(x, y)$ provided that if $x = x(t), y = y(t)$ solves the system and satisfies $x(0) = x_0$ and $y(0) = y_0$, then $\lim_{t\to\infty}(x(t), y(t)) = (x_e, y_e)$.

In Example 10.2.11, what points in the first quadrant $x > 0$, $y > 0$ lie in the basin of attraction of the equilibrium point $(0, 1)$? Fig. 10.23 provides the answer: Any point in the first quadrant that lies strictly to the left of the stable curve S is in the basin of attraction of $(0, 1)$. In Exercise 21 you are asked to describe the points in the first quadrant that lie in the basin of attraction of the equilibrium point $(1, 0)$.

Remark: When considered in the entire plane, the basin of attraction of an asymptotically stable equilibrium point must include an open disk of positive radius centered at the equilibrium point; this is a consequence of the definition of an asymptotically stable equilibrium. Notice that for a planar *linear system* with a stable node or stable spiral point at $(0, 0)$, the basin of attraction is the whole plane.

10.2.1. Exercises.

In Exercises 1–15, (i) find all equilibrium points, (ii) find the Jacobian matrix for the linear approximation at each equilibrium point, and (iii) for each equilibrium point where Theorem 10.2.3 applies, classify the point as to type (spiral, saddle, etc.) and stability. If Theorem 10.2.3 does not apply, say so, and then say what information Theorem 10.2.4 gives.

1. $\dfrac{dx}{dt} = x - y^2, \quad \dfrac{dy}{dt} = x - y.$
2. $\dfrac{dx}{dt} = x(2x+3y-7), \quad \dfrac{dy}{dt} = y(3x-4y-2).$
3. $\dfrac{dx}{dt} = x(x^2 + y^2 - 10), \quad \dfrac{dy}{dt} = y(xy - 3).$
4. $\dfrac{dx}{dt} = x - xy, \quad \dfrac{dy}{dt} = xy - y.$
5. $\dfrac{dx}{dt} = y, \quad \dfrac{dy}{dt} = -x + (1 - x^2)y.$
6. $\dfrac{dx}{dt} = y, \quad \dfrac{dy}{dt} = -x - (1 - x^2)y.$
7. $\dfrac{dx}{dt} = y, \quad \dfrac{dy}{dt} = -x + 3(1 - x^2)y.$
8. $\dfrac{dx}{dt} = x(1 - x - y), \quad \dfrac{dy}{dt} = y(2 - x - 4y).$
9. $\dfrac{dx}{dt} = \sin(x + y), \quad \dfrac{dy}{dt} = e^x - 1.$
10. $\dfrac{dx}{dt} = y + x^2 - 3, \quad \dfrac{dy}{dx} = x(1 + y).$
11. $\dfrac{dx}{dt} = y - x^3, \quad \dfrac{dy}{dt} = -x.$
12. $\dfrac{dx}{dt} = x + xy, \quad \dfrac{dy}{dt} = x - y.$
13. $\dfrac{dx}{dt} = y^3 - x^3 \quad \dfrac{dy}{dt} = 1 - y^3.$
14. $\dfrac{dx}{dt} = -3x + x^2 - xy, \quad \dfrac{dy}{dt} = -5y + xy.$
15. $\dfrac{dx}{dt} = x + y, \quad \dfrac{dy}{dt} = y + x^2y.$

16. In Exercise 1 of Section 10.1, you did a nullcline-and-arrow diagram for the system
$$\frac{dx}{dt} = x(5 - x - 4y), \qquad \frac{dy}{dt} = y(2 - y - x).$$

Show that there are four equilibrium points, and find the linear approximation at each. Use Theorem 10.2.3 to classify each equilibrium point, if possible. Put together this classification information with the nullcline-and-arrow diagram from Exercise 1 in Section 10.1 to give a sketch of a phase portrait in the first quadrant.

17. Two competing species are modeled by the system
$$\frac{dx}{dt} = x(120 - 3x - y), \qquad \frac{dy}{dt} = y(80 - x - y).$$
What is the outcome of this competition? Extinction of one species or peaceful coexistence?

18. The linear approximation for an autonomous system at an equilibrium point (x_e, y_e) has matrix $\mathbf{A}$. Under what conditions on the trace τ of $\mathbf{A}$ and the determinant Δ of $\mathbf{A}$ can you **not** apply Theorem 10.2.3 to classify (x_e, y_e)?

19. Consider the system
$$\frac{dx}{dt} = y - x, \qquad \frac{dy}{dt} = \frac{5x^2}{x^2+4} - y.$$
   (a) Find all equilibrium points in the first quadrant and the Jacobian matrix at each. Classify the equilibrium points if possible.
   (b) Do a nullcline-and-arrow diagram for the phase portrait in the first quadrant. Sketch some trajectories, keeping in mind your answers to (a).

20. Show that $(1, 1)$ is a nonisolated equilibrium point of
$$\frac{dx}{dt} = x^2 - y^2, \qquad \frac{dy}{dt} = xy - x^2,$$
and compute the determinant of the associated Jacobian matrix $\mathbf{J}(1, 1)$.

21. In Example 10.2.11, what points in the first quadrant $x > 0$, $y > 0$ are in the basin of attraction of the equilibrium point $(1, 0)$?

22. Consider the system
$$\frac{dx}{dt} = -x + y, \qquad \frac{dy}{dt} = x - xy^2. \tag{10.36}$$
   (a) Confirm that $(1, 1)$ is an asymptotically stable equilibrium point of (10.36).
   (b) Give a nullcline-and-arrow sketch in the first quadrant for the system (10.36).
   (c) Is $(3, 2)$ in the basin of attraction of $(1, 1)$? Justify your answer. You should be able to construct a convincing argument, not involving any computer-based work.

23. Consider the system
$$\frac{dx}{dt} = y - 4x, \qquad \frac{dy}{dt} = -xy - 3x. \tag{10.37}$$
   (a) Confirm that $(0, 0)$ is an asymptotically stable equilibrium point of (10.37).
   (b) Identify a point that is not in the basin of attraction of $(0, 0)$. Justify your answer.
   (c) Identify an entire region of points that are not in the basin of attraction of $(0, 0)$. Justify your answer.

24. Consider the system
$$\frac{dx}{dt} = -x + y, \qquad \frac{dy}{dt} = x - xy^2.$$
   (a) In Exercise 22 above you were asked to verify that $(1, 1)$ is an asymptotically stable equilibrium point. Confirm it is an improper node and find its nodal tangents. Which is the preferred nodal tangent?
   (b) (CAS) Use a computer algebra system to create a phase portrait for this system and check that your nodal tangent information of (a) is consistent with your portrait.

25. Consider the competing species system

$$\frac{dx}{dt} = x(1 - x - y), \quad \frac{dy}{dt} = y(2 - x - y)$$

in which x and y are measured in thousands and t in years.

(a) Show that $(0, 2)$ is an improper node for the system and find its nodal tangents. Which is the preferred nodal tangent?

(b) Suppose that $x(0) = 2$ and $y(0) = 0.2$. Suppose the population level of y has risen to 1.9 (i.e., 1,900); use your preferred nodal tangent information to estimate corresponding x-population level.

(c) (CAS) Use a computer algebra system to create a phase portrait for this system and check that your nodal tangent information of (a) as well as your answer to (b) is consistent with your portrait.

(d) (CAS) Solve numerically the initial value problem consisting of the given competing species system with $x(0) = 2$ and $y(0) = 0.2$. According to your numerical solution when does the y reach 1.9? What is x at that time?

26. Consider the system

$$\frac{dx}{dt} = 5 - x^2 - y, \quad \frac{dy}{dt} = x(1 - y^2).$$

(a) Find the only equilibrium point of the system with $x > 0$ and $y > 0$.

(b) Use Theorems 10.2.3 and 10.2.4 to say as much as you can about the equilibrium point you found in (a).

(c) Sketch a nullcline-and-arrow diagram for the system in the first quadrant and use it to argue that the equilibrium point you found in (a) cannot be a spiral point.

(d) (CAS) Use a computer algebra system to create, near the equilibrium point from (a), a phase portrait for both the original nonlinear system as well as for the approximating linear system there.

(e) Comment on the extent to which your phase portraits are consistent with the information provided in response to (b) and (c).

27. Consider the system

$$\frac{dx}{dt} = x - xy + (x - 2)^3, \quad \frac{dy}{dt} = -2y + xy + (y - 1)^3.$$

(a) Confirm that $(2, 1)$ is an equilibrium point of the system.

(b) Use Theorems 10.2.3 and 10.2.4 to say as much as you can about the equilibrium point $(2, 1)$.

(c) (CAS) Use a computer algebra system to create, near $(2, 1)$, a phase portrait for both the original nonlinear system as well as for the approximating linear system there.

(d) Comment on the extent to which your phase portraits are consistent with the information provided in response to (b).

28. Consider the system

$$\frac{dx}{dt} = y + x^2 - 3, \quad \frac{dy}{dt} = x(1 + y).$$

(a) Verify that $(2, -1)$ is an unstable improper node of this system and find its nodal tangents. Which is the preferred nodal tangent?

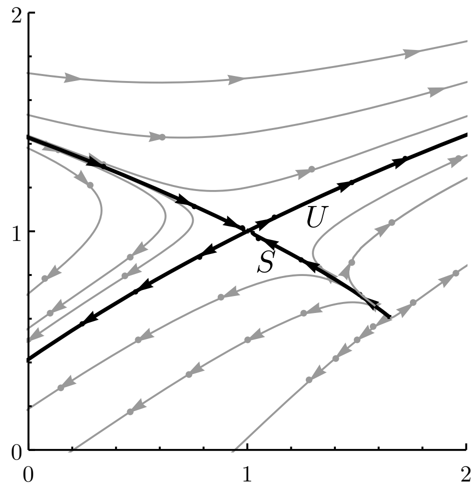
(b) (CAS) Use a computer algebra system to create a phase portrait for this system and confirm that your nodal tangent information from (a) is consistent with the portrait.

29. The population levels (in thousands) of two competing species on a small island are modeled by the system

(10.38) $$\frac{dx}{dt} = x(1 - 2x - 2y), \quad \frac{dy}{dt} = y(1 - x - y),$$

where time t is measured in years.
(a) What is the outcome of this competition? Extinction of one species or peaceful coexistence?
(b) (CAS) Solve numerically the initial value problem consisting of (10.38) with $x(0) = 0.1, y(0) = 0.02$, so that the initial x-population is 100 and the initial y-population is 20. After approximately how many years will x reach its maximum value? What is the approximate maximum value?

30. Pictured below is a phase portrait for the system $\frac{dx}{dt} = \frac{1}{2}x^2 + y^4 - \frac{3}{2}$, $\frac{dy}{dt} = xy - 1$ near its equilibrium point $(1, 1)$.



(a) Verify that $(1, 1)$ is a saddle point of the system using Theorem 10.2.3.
(b) The stable curve S and unstable curve U for $(1, 1)$ are shown in the figure above. Find equations of the lines through $(1, 1)$ that are tangent at $(1, 1)$ to S and to U, respectively.

31. **Nonlinear spiral points.** Consider a nonlinear system

(10.39) $$\frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y)$$

with continuously differentiable vector field $\mathbf{F} = (f, g)$, for which the origin $(0, 0)$ is an isolated equilibrium point. Suppose this equilibrium point is also an asymptotically stable spiral point for the corresponding linear system

$$\frac{d\mathbf{X}}{dt} = \mathbf{J}\mathbf{X},$$

where

$$\mathbf{J} = \mathbf{J}(0, 0) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is the Jacobian matrix of $\mathbf{F}$ at $(0, 0)$. Thus $\mathbf{J}$ is assumed to have complex eigenvalues $\alpha \pm i\beta$ where $\alpha < 0$ and $\beta \neq 0$. By Theorem 10.2.3, $(0, 0)$ is also an asymptotically stable spiral point for the nonlinear system (10.39). A reasonable mathematical description of what this means is that the following two conditions should hold:
(I) $(0, 0)$ is asymptotically stable for the system (10.39). In particular, if $(x(t), y(t))$ is a solution with $(x(0), y(0))$ sufficiently close to $(0, 0)$, then $(x(t), y(t)) \to (0, 0)$ as $t \to \infty$.

(II) Any solution $(x(t), y(t))$ with $(x(0), y(0))$ sufficiently close to $(0,0)$ will "spiral around" $(0,0)$ an infinite number of times as $t \to \infty$. More specifically, if $\theta(t)$ is a continuous choice of polar angle for $(x(t), y(t))$, then $\theta(t)$ tends either to $+\infty$ or $-\infty$ as $t \to \infty$.

In this exercise you will prove (II). In doing so, you may assume that (I) is true.

(a) Show that b and c have opposite signs.

(b) For definiteness, assume $c > 0$ so that $-b > 0$ by (a). Also assume throughout that $(x(t), y(t))$ is a solution to (10.39) with $(x(0), y(0))$ sufficiently close to $(0,0)$ so that $(x(t), y(t)) \to (0,0)$ as $t \to \infty$. A continuous choice of polar angle $\theta(t)$ for the point $(x(t), y(t))$ satisfies

$$\tan(\theta(t)) = \frac{y(t)}{x(t)}.$$

Writing x, y, and θ for $x(t)$, $y(t)$, and $\theta(t)$, show that

$$\theta' = \frac{xy' - yx'}{r^2},$$

where primes denote derivatives with respect to t and $r^2 = x^2 + y^2$.

(c) We have

$$x' = f(x, y) = ax + by + R_f(x, y)$$

and

$$y' = g(x, y) = cx + dy + R_g(x, y)$$

where R_f and R_g are the respective error terms. Show that

$$\theta' = \frac{cx^2 + (d-a)xy - by^2}{r^2} + \frac{xR_g(x,y) - yR_f(x,y)}{r^2}.$$

(d) Show that

$$\frac{|xR_g(x,y) - yR_f(x,y)|}{r^2} \to 0$$

as $(x, y) \to (0,0)$, that is, as $r \to 0$. Hint: $xR_g(x,y) - yR_f(x,y) = (x, -y) \cdot (R_g(x,y), R_f(x,y))$, a dot product in $\mathbb{R}^2$. Recall that for any two vectors $\mathbf{A}$ and $\mathbf{B}$ in $\mathbb{R}^2$, $|\mathbf{A} \cdot \mathbf{B}| \le \|\mathbf{A}\| \|\mathbf{B}\|$. Don't forget the crucial property (10.21) for the error term.

(e) Consider two regions in $\mathbb{R}^2$, the first defined by $|y| \le |x|$ and the second by $|y| \ge |x|$. (You may find it helpful to draw a picture.) Show that in the first region

$$\frac{cx^2 + (d-a)xy - by^2}{r^2} \ge \frac{1}{2} p\left(\frac{y}{x}\right),$$

where $p(u) = -bu^2 + (d-a)u + c$, while in the second region

$$\frac{cx^2 + (d-a)xy - by^2}{r^2} \ge \frac{1}{2} q\left(\frac{x}{y}\right),$$

where $q(u) = cu^2 + (d-a)u - b$.

(f) Since $-b$ and c are positive, the graphs of the two quadratic polynomials $p(u)$ and $q(u)$ are concave up parabolas. Show that the minimum values of $p(u)$ and $q(u)$ are both positive. Hint: Can $p(u)$ or $q(u)$ have a real zero?

(g) Combine the above results to show that there exist positive numbers B and t_0 such that $\theta'(t) \ge B$ for all $t > t_0$.

(h) Deduce from (g) that there is a real number C such that $\theta(t) \ge Bt + C$ for all $t > t_0$. Thus $\theta(t) \to \infty$ as $t \to \infty$ and $(x(t), y(t))$ spirals counterclockwise. Remark: If at the start you had instead assumed $b > 0$ and $c < 0$, then a similar argument would lead to $\theta(t) \to -\infty$ as $t \to \infty$, giving a clockwise spiral.

10.3. Linear approximation: The borderline cases

We'll look at some examples to illustrate what can happen when the classification of the equilibrium point for the approximating linear system falls into one of the borderline cases. Of these various borderline cases, centers are the most important, because they appear in key applications, and so we concentrate on these.

Example 10.3.1. Consider the nonlinear system

$$\frac{dx}{dt} = -y - 2xy, \qquad \frac{dy}{dt} = x + 2x^2, \tag{10.40}$$

which has an equilibrium point at $(0,0)$. The Jacobian matrix for the linear approximation near $(0,0)$ is

$$\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since $\mathbf{J}$ has eigenvalues $\pm i$, $(0,0)$ is a center for the approximating linear system. According to Theorem 10.2.4 the nonlinear system could have either a center or a spiral at $(0,0)$, and if a spiral, it could be either asymptotically stable or unstable.[7] Can we determine which? Our original equations are simple enough that we can write a separable first-order equation for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{x+2x^2}{-y-2xy} = -\frac{x}{y}$$

so that

$$\int y\,dy = \int -x\,dx,$$

or $x^2 + y^2 = C$. The orbits of (10.40) are therefore circles, and this nonlinear system has a center at $(0,0)$. Centers are stable, but not asymptotically stable.

It's the rare equation that is simple enough to be able to analyze as we did in the last example. A tool that is useful for some other borderline cases is a **change of variables to polar coordinates** (r,θ), where we assume $r \geq 0$. The next example illustrates this.

Example 10.3.2. We start with the nonlinear system

$$\frac{dx}{dt} = -y - x(x^2+y^2), \qquad \frac{dy}{dt} = x - y(x^2+y^2), \tag{10.41}$$

which has the same linear approximation at $(0,0)$ as in the preceding example. So again the question is whether this nonlinear system has a center or spiral at $(0,0)$. We will make a change of variables to polar coordinates to decide which.

Differentiation with respect to t on both sides of the polar coordinate change of variable equation $r^2 = x^2+y^2$ gives $2rr' = 2xx'+2yy'$. Thus, denoting the derivative with respect to t with primes,

$$\boxed{r' = \frac{xx'+yy'}{r}. \tag{10.42}}$$

[7] According to Theorem 10.2.4, there is a third possibility: a hybrid center/spiral. An example of a hybrid center/spiral (which we have not formally defined) is found in Exercise 8. This hybrid case occurs only rarely and never when, as in Examples 10.3.1 and 10.3.2, the component functions f and g are polynomials (in x and y).

Making the substitutions $x' = -y - x(x^2+y^2)$ and $y' = x - y(x^2+y^2)$ from (10.41) into equation (10.42) we obtain after a short calculation

$$r' = \frac{-(x^2+y^2)^2}{r} = -r^3.$$

This is a separable first-order equation for r with solution $r(t) = (2t+c_1)^{-1/2}$ for arbitrary constant c_1. Similarly, differentiating the relation $\tan\theta = y/x$ with respect to t gives

$$(\sec^2\theta)\,\theta' = \frac{xy' - yx'}{x^2}, \quad \text{or equivalently,} \quad \theta' = \frac{xy' - yx'}{x^2}\cos^2\theta.$$

Using the relation $x = r\cos\theta$ from the polar coordinate change of variables, we obtain

$$\boxed{\theta' = \frac{xy' - yx'}{r^2}.}$$

Again making the substitutions for x' and y' from the original equations in (10.41), we have

$$\theta'(t) = \frac{x^2+y^2}{r^2} = 1,$$

so that $\theta(t) = t + c_2$, for arbitrary constant c_2.

What does a solution to (10.41) that starts out near the origin look like? Let's interpret the statement "starts near the origin" to mean $r(0) < 1$. This says $c_1 > 1$, and as t increases from 0, $r(t) = (2t+c_1)^{-1/2}$ decreases, while $\theta(t) = t + c_2$ increases. Thus a trajectory with $r(0) < 1$ spirals into the origin in a counterclockwise fashion. Now observe that in fact all orbits have this behavior, including those for which $r(0) \geq 1$. The equilibrium point $(0,0)$ is a stable spiral point for this nonlinear system. A sketch of some orbits is shown in Fig. 10.24.

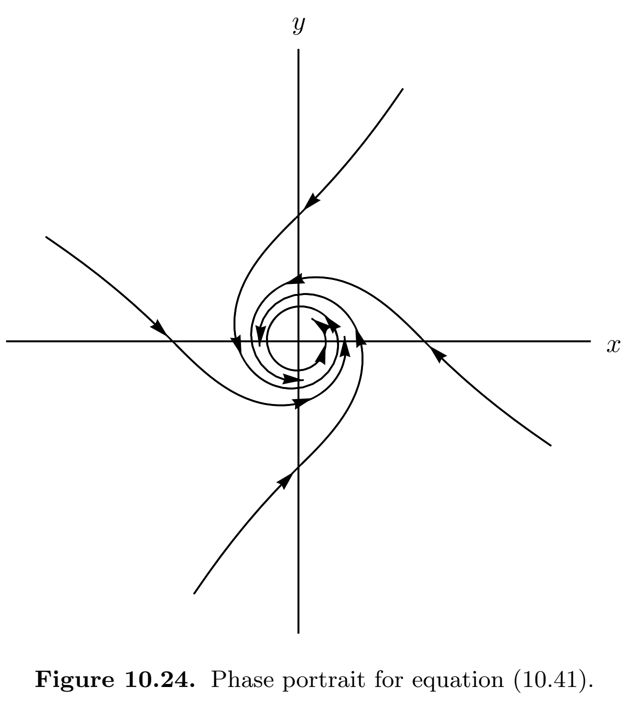


Figure 10.24. Phase portrait for equation (10.41).

Our final example, while not a borderline case, continues the theme of polar coordinate change of variable and also serves to emphasize the fact that the linear approximation gives *local information* only.

Example 10.3.3. Consider the nonlinear system

$$\frac{dx}{dt} = -y + x(1 - x^2 - y^2), \qquad \frac{dy}{dt} = x + y(1 - x^2 - y^2), \tag{10.43}$$

which has an equilibrium point at $(0, 0)$. The Jacobian matrix for the linear approximation there is

$$\mathbf{J} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Since the eigenvalues of $\mathbf{J}$ are $1 \pm i$, $(0, 0)$ is an unstable spiral for the approximating linear system. By Theorem 10.2.3, the same is true for the original nonlinear system. We will verify this independently by a polar coordinate change of variable. By equations (10.42) and (10.43) we have

$$r' = \frac{xx' + yy'}{r} = \frac{x(-y + x(1 - x^2 - y^2)) + y(x + y(1 - x^2 - y^2))}{r} = r - r^3.$$

Also

$$\theta'(t) = \frac{xy' - yx'}{r^2} = 1$$

so $\theta(t) = t + c$, for c an arbitrary constant. The first-order differential equation $\frac{dr}{dt} = r - r^3$ can be solved as a Bernoulli equation (or by separation of variables) to obtain

$$r(t) = \frac{1}{\sqrt{1 + ke^{-2t}}}$$

for some constant k. The constant k is determined by the value of r at $t = 0$:

$$\frac{1}{1 + k} = r(0)^2, \quad \text{or} \quad k = \frac{1}{r(0)^2} - 1.$$

If we start close to the origin (say, $r(0) < 1$), then k is positive and $r(t) \to 0$ as $t \to -\infty$. Since $\theta(t) = t + c$, we see that the trajectories that start near the origin spiral away from the origin in the counterclockwise direction with increasing t. This is what we expect from Theorem 10.2.3. However, as $t \to \infty$, $r(t) \to 1$. Notice that the nonlinear system has a periodic solution $x(t) = \cos t, y(t) = \sin t$, whose orbit is the circle $r(t) = 1$, or $x^2 + y^2 = 1$. As $t \to \infty$, our spirals approach this circle but cannot cross it; see Fig. 10.25. This is a phenomena we never see in a linear system. This emphasizes the fact that the method of linear approximation gives only *local* information *near* an equilibrium point.

In Example 10.3.3 we call the unit circle a **limit cycle** for the system (10.43). It is a closed-curve orbit that is approached by other orbits as $t \to \infty$. We will return to the notion of limit cycles in Section 10.8.

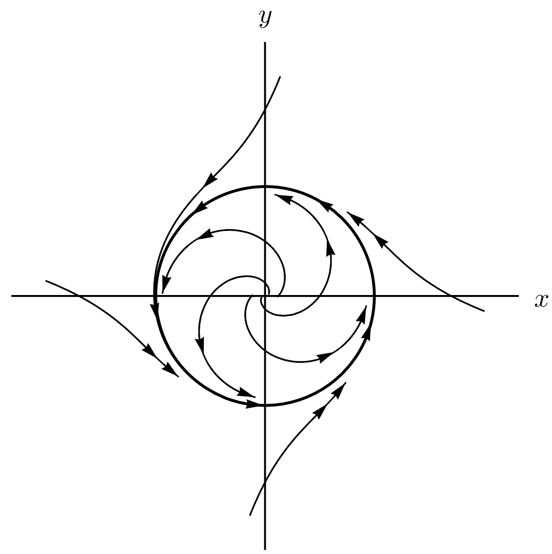


Figure 10.25. Phase portrait for the system (10.43).

10.3.1. Exercises.

1. Does the nonlinear system
$$\frac{dx}{dt} = y - xy^2, \qquad \frac{dy}{dt} = -x + x^2y$$
have a center or spiral at $(0,0)$?

2. This problem generalizes Example 10.3.2. Consider the system
$$\frac{dx}{dt} = -y + kx(x^2+y^2), \qquad \frac{dy}{dt} = x + ky(x^2+y^2),$$
where k is a constant. Show that $(0,0)$ is a stable spiral if $k < 0$, an unstable spiral if $k > 0$, and a center if $k = 0$.

3. In Example 10.3.3, suppose that $r(0) > 1$, so that
$$r(t) = \frac{1}{\sqrt{1+ke^{-2t}}},$$
where k is now negative. What happens to the corresponding orbit as $t \to \infty$?

4. Consider the solution $x(t) = (1 - \frac{3}{4}e^{-2t})^{-1/2}\cos t$, $y(t) = (1 - \frac{3}{4}e^{-2t})^{-1/2}\sin t$ of the system of Example 10.3.3. Note the solution satisfies the initial conditions $x(0) = 2, y(0) = 0$.
   (a) Show that $(-\frac{1}{2}\ln(4/3), \infty)$ is the interval of validity of this solution.
   (b) Let $T = -\frac{1}{2}\ln(4/3)$. Write the solution $x(t) = (1-\frac{3}{4}e^{-2t})^{-1/2}\cos t$, $y(t) = (1-\frac{3}{4}e^{-2t})^{-1/2}\sin t$ in polar coordinates $(r(t), \theta(t))$, and show that $r(t) \to \infty$ and $\theta(t) \to T$ as t approaches T from the right.
   (c) Part (b) suggests that the line $y = (\tan T)x$ is an asymptote for the given solution in the sense that the distance from the point
$$\left(\left(1 - \frac{3}{4}e^{-2t}\right)^{-1/2}\cos t, \left(1 - \frac{3}{4}e^{-2t}\right)^{-1/2}\sin t\right) \text{ on the trajectory} \tag{10.44}$$
to the point
$$\left(\left(1 - \frac{3}{4}e^{-2t}\right)^{-1/2}\cos(T), \left(1 - \frac{3}{4}e^{-2t}\right)^{-1/2}\sin(T)\right) \text{ on the line } y = \tan(T)x \tag{10.45}$$

approaches 0 as $t \to T^+$. Verify this. Note that each of the two points displayed above has the same distance to the origin, namely, $(1 - \frac{3}{4}e^{-2t})^{-1/2}$.

(d) (CAS) Illustrate the result in (c) by creating a parametric plot, over the interval $T + 0.01 \le t \le 5$, of the point (10.44) on the trajectory and the point (10.45) on the line.

5. Consider the nonlinear system

$$\frac{dx}{dt} = x - x^2, \qquad \frac{dy}{dt} = y. \tag{10.46}$$

(a) Find the equilibrium points.

(b) Find the Jacobian matrix $\mathbf{J}$ at $(0,0)$ and classify the equilibrium point for the linear system $\mathbf{X}' = \mathbf{J}\mathbf{X}$.

(c) Solve the separable equation

$$\frac{dy}{dx} = \frac{y}{x - x^2}$$

to find equations for the trajectories of (10.46). Sketch these trajectories near $(0,0)$.

(d) True or false: For every ray through the origin, there is a trajectory tangent to that ray.

(e) How would you classify the equilibrium point $(0,0)$ for (10.46)?

6. Consider the nonlinear system

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = x - x^3.$$

(a) Show that $(0,0), (1,0)$, and $(-1,0)$ are the equilibrium points for this system, and compute the Jacobian matrix at each of these points.

(b) Show that $(0,0)$ is a saddle point for the approximating linear system and the other equilibrium points are centers for the approximating linear systems there. On the basis of this alone, what (if anything) can you conclude about the classification of the three equilibrium points for the nonlinear system?

(c) Show that the trajectories for the nonlinear system have equations $y^2 - x^2 + \frac{x^4}{2} = C$ for some constant C. (Hint: Find a first-order separable differential equation for $\frac{dy}{dx}$ and solve it.)

(d) The graph of the surface $z = y^2 - x^2 + \frac{x^4}{2}$ is shown in Fig. 10.26. How does this help you determine whether the equilibrium points $(1,0)$ and $(-1,0)$ are centers or spiral points?

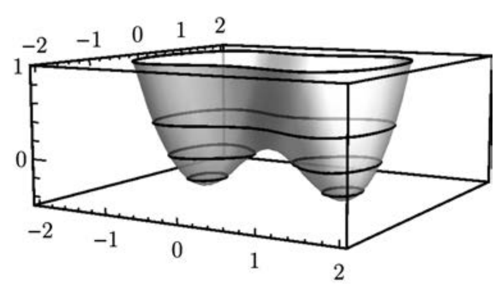


Figure 10.26. The surface $z = y^2 - x^2 + x^4/2$.

(e) Using your work in (a)–(d), sketch the phase portrait for the nonlinear system. Your picture should include the trajectories that correspond to each of the cross sections shown in Fig. 10.26.

7. Consider the nonlinear system

$$\frac{dx}{dt} = cx + y - x(x^2 + y^2), \qquad \frac{dy}{dt} = -x + cy - y(x^2 + y^2),$$

where c is a nonzero constant.

(a) Show that $(0,0)$ is the only equilibrium point of this system.
(b) Show that if $c > 0$, then $x = \sqrt{c}\sin t, y = \sqrt{c}\cos t$ is a solution.
(c) Show that if we change to polar coordinates, we have

$$\frac{dr}{dt} = r(c - r^2).$$

This is a Bernoulli equation, but instead of solving it explicitly, give a qualitative solution of this first-order equation by sketching some solutions in the tr-plane for $r > 0$. Begin your sketch by showing any equilibrium solutions $r =$ constant. Your sketch will depend on whether c is positive or negative, so you'll need two separate graphs.
(d) Show that $\theta' = -1$, so $\theta(t) = -t + k$.
(e) Make a prediction about what the phase portrait of the nonlinear system looks like if $c > 0$ and if $c < 0$.
(f) What is the stability of $(0,0)$ if $c > 0$? If $c < 0$?

8. Consider the system

$$\frac{dx}{dt} = -y + xr^3 \sin\left(\frac{\pi}{r}\right), \qquad \frac{dy}{dt} = x + yr^3 \sin\left(\frac{\pi}{r}\right),$$

where $r = \sqrt{x^2 + y^2}$.
(a) Show that $(0,0)$ is an equilibrium point by showing that $-y + xr^3\sin(\frac{\pi}{r})$ and $x + yr^3\sin(\frac{\pi}{3})$ both tend to 0 as $(x,y) \to (0,0)$.
(b) Show that in polar coordinates the system is

$$\frac{dr}{dt} = r^4 \sin\left(\frac{\pi}{r}\right), \qquad \frac{d\theta}{dt} = 1.$$

(c) Using the result of (b), show that each circle $r = \frac{1}{n}$, n a nonzero integer, is an orbit in the phase portrait for this system.
(d) A trajectory which starts between two of the circular orbits $r = \frac{1}{n}$ and $r = \frac{1}{n+1}$ must remain between these circles for all later times. What can it look like? Hint: Notice $\theta(t) = t + c$ and $\frac{dr}{dt}$ is either always positive or always negative in the region between the circles $r = \frac{1}{n}$ and $r = \frac{1}{n+1}$. The origin is called a hybrid center/spiral for the nonlinear system.

9. This problem illustrates the possibly "exotic" nature of the phase portrait when $(0,0)$ is not an isolated equilibrium point of the associated linear system.
(a) Show that the nonlinear system

$$\frac{dx}{dt} = x(x^3 - 2y^3), \qquad \frac{dy}{dt} = y(2x^3 - y^3)$$

has an isolated equilibrium point at $(0,0)$.
(b) Find the Jacobian matrix at $(0,0)$ for the system in (a), and show that the linear approximation system has a nonisolated equilibrium point at $(0,0)$.
(c) Show that the equations for the orbits of the system in (a) are given by $x^3 + y^3 = cxy$ for arbitrary constant c. To do this, solve the first-order equation

$$\frac{dy}{dx} = \frac{y(2x^3 - y^3)}{x(x^3 - 2y^3)}$$

for the orbits (substitute $v = y/x$ as in Section 2.7). Some of the orbits are shown in Fig. 10.27.

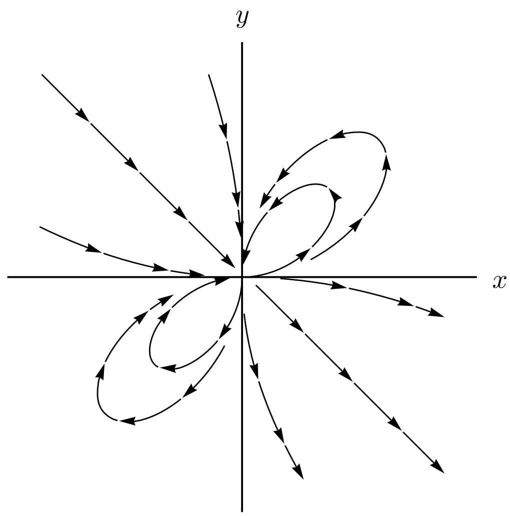


Figure 10.27. Some orbits for Exercise 9.

10.4. More on interacting populations

In Section 10.1 we proposed a model to analyze Darwin's principle of competitive exclusion. In this section we'll expand our study of interacting species, still concentrating on two-species systems.

In the 1920s the mathematician Vito Volterra was asked by his son-in-law, the Italian biologist Umberto D'Ancona, if a mathematical explanation could be found for some puzzling data he had from fish markets in three Italian cities along the Adriatic. D'Ancona had noticed that during World War I, when commercial fishing had greatly decreased in the Adriatic, there was a dramatic increase in the percentage of sharks and other predator fish among the relatively few catches that were being made. He asked Volterra if he could explain mathematically why a decrease in the overall level of fishing would benefit predator fish to a greater extent than it benefited their prey. Volterra relished the opportunity to apply differential equations to this interesting biological problem and initiated work which now appears in essentially every text on theoretical ecology. The differential equation model Volterra proposed, which goes by the name of the Lotka-Volterra equations (due to related work of a physical chemist/statistician named Alfred Lotka), investigates two interacting populations that are in a predator-prey relationship. It gives an explanation of D'Ancona's biological phenomenon, which otherwise seemed inexplicable. This model contains a warning to all gardeners contemplating control of a garden pest by a pesticide which also attacks beneficial insects; see Exercise 3.

Volterra proposed the following system of differential equations for the population $x(t)$ (the prey fish) and $y(t)$ (sharks and other predator fish) in the Adriatic Sea:

$$\frac{dx}{dt} = ax - bxy, \qquad \frac{dy}{dt} = -dy + cxy, \tag{10.47}$$

where a, b, c, and d are positive constants. In formulating these equations, Volterra made the following assumptions:

(i) Plankton, the food source for the prey fish, is abundant in the Adriatic, so there is negligible competition within the prey population for food. In the absence of predators, the prey population would thus grow, at least in the short term, according to a Malthusian model.

(ii) The predator fish are dependent upon the prey fish for their food supply; in the absence of prey, predator population would decline at an exponential rate.

These two assumptions explain the "ax" and "$-dy$" terms in our equations. The interaction terms "$-bxy$" and "cxy" occur with a positive sign for the predator (which is benefited by the interaction) and a negative sign for the prey. Think of the number of "encounters" between predator and prey to be proportional to the product of the two populations; this explains the form of the interaction terms. Another perspective on how these equations arise is to start with the basic exponential growth and decay equations

$$\frac{dx}{dt} = ax, \qquad \frac{dy}{dt} = -dy$$

and replace the constants a and d by linear functions which measure "how good life is". For the predator, a good life means abundant prey, so $-d$ is replaced by $-d + cx$, a linear function which increases as the prey population x increases. For the prey, a good life means a scarcity of predators, so a is replaced by $a - by$, a linear function which decreases as the predator population increases.

We carry out a nullcline-and-arrow analysis of the system (10.47) (in the first quadrant only because of the physical meaning of x and y as populations) and a local analysis near the equilibrium points. The x-nullclines consist of the y-axis ($x = 0$) and the horizontal line $y = a/b$. The y-nullclines are the x-axis ($y = 0$) and the vertical line $x = d/c$. Fig. 10.28 shows these nullclines and the corresponding pairs of direction-indicator arrows in the four basic regions in the first quadrant. You should verify that the various right/left and up/down configurations follow from (10.47).

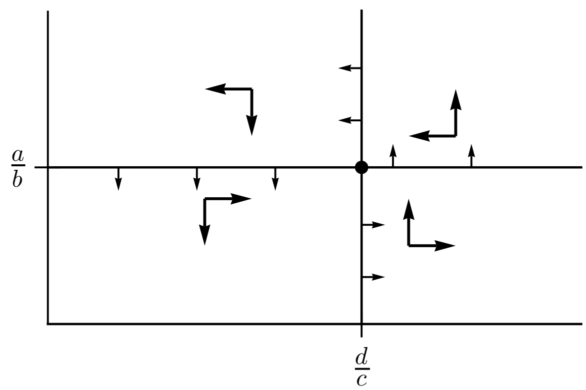


Figure 10.28. Nullcline-and-arrow diagram for (10.47).

We also see that the equilibrium points for the Lokta-Volterra system are $(0, 0)$ and $(d/c, a/b)$. The Jacobian matrix for (10.47) is

$$\begin{pmatrix} a - by & -bx \\ cy & -d + cx \end{pmatrix}.$$

Evaluated at the origin, this gives

$$\begin{pmatrix} a & 0 \\ 0 & -d \end{pmatrix},$$

a matrix with eigenvalues $a > 0$ and $-d < 0$. This puts us in a major case and we may apply Theorem 10.2.3 to see that the nonlinear Lotka-Volterra system has a saddle point at the origin. Things don't go as smoothly at the other equilibrium point, $(d/c, a/b)$. Now the Jacobian matrix is

$$\begin{pmatrix} 0 & -bd/c \\ ac/b & 0 \end{pmatrix},$$

which has purely imaginary eigenvalues $\pm i\sqrt{ad}$, and we are in a borderline case. The equilibrium point $(d/c, a/b)$ for (10.47) could be a center or a spiral, both of which are consistent with the nullcline-and-arrow diagram in Fig. 10.28. Can we decide which is correct?

We can find equations for the trajectories in the phase portrait by a familiar technique: From (10.47) we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y(-d+cx)}{x(a-by)}$$

and we separate variables obtaining

$$\int \frac{a-by}{y}\,dy = \int \frac{-d+cx}{x}\,dx$$

or $a\ln|y| - by = -d\ln|x| + cx + k$. These are equations for the trajectories we want to sketch, but we cannot solve explicitly for y in terms of x and hope to identify the orbits that way. However, we will see below that they are the equations of closed curves encircling the equilibrium point, so that $(d/c, a/b)$ is a center for the system (10.47). A sketch of the phase portrait is shown in Fig. 10.29.

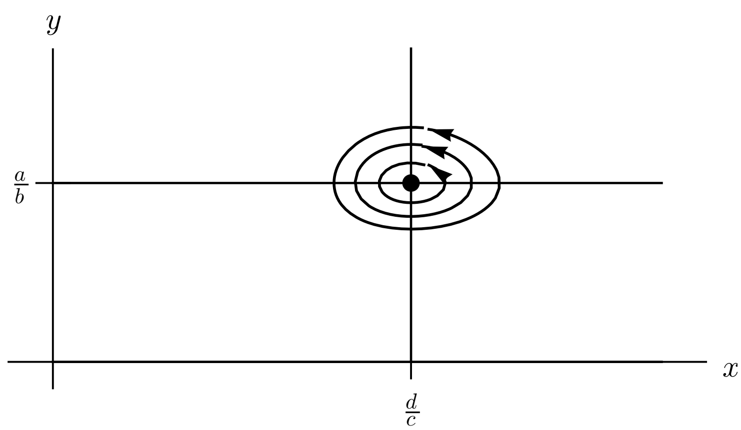


Figure 10.29. Phase portrait for equation (10.47).

Assuming this for the moment, let's return to Volterra's original puzzle: Why does a decrease in fishing benefit the predator population more than the prey population? We need to incorporate fishing into our predator/prey model, and we'll do this in the simplest natural way. Let's assume that when fishing occurs, members of both the predator and prey populations are harvested at a rate proportional to the population sizes. This leads to a modification of equation (10.47):

$$\frac{dx}{dt} = ax - bxy - hx, \qquad \frac{dy}{dt} = -dy + cxy - hy.$$

Here h is a positive constant representing fishing effort—how many are fishing, how much time they spend fishing, equipment used, and so on.[8] The larger h is, the larger the harvests, per unit of time, of prey (hx) and predator (hy). We assume that $h < a$, since otherwise the prey population will simply decrease to 0 because $\frac{dx}{dt}$ is negative. As long as $h < a$ this new system of equations has exactly the same form as the original Lotka-Volterra equations, and the only change has been in the value of the constants:

$$\frac{dx}{dt} = (a-h)x - bxy, \qquad \frac{dy}{dt} = -(d+h)y + cxy.$$

[8]Here we are using the same coefficient h in both equations, so that we are harvesting the same proportion of each population. A more general model would allow for different "harvesting coefficients" h_1 and h_2.

All of our previous analysis still applies, and a new phase portrait is sketched in Fig. 10.30. Notice that the equilibrium points are $(0,0)$ and $(\frac{d+h}{c}, \frac{a-h}{b})$.

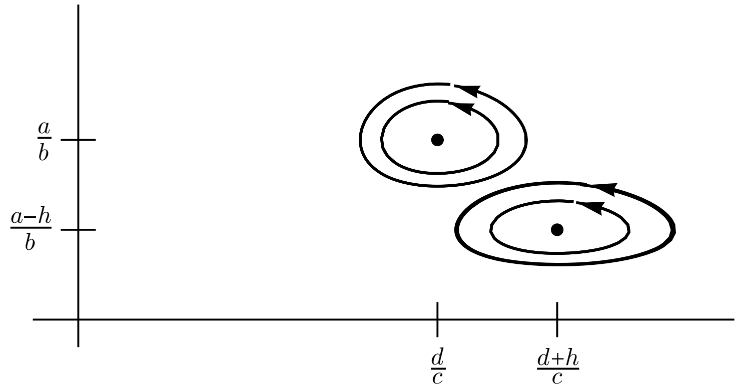


Figure 10.30. The effect of fishing intensity.

The effect of including fishing in the model has been to move the equilibrium point from $(d/c, a/b)$ to $((d+h)/c, (a-h)/b)$. This means that as h decreases, the equilibrium point for the fishing-included model moves leftward and upward. Thus when fishing intensity was lowered during World War I, the orbits circled around a point with a smaller x-coordinate and a larger y-coordinate. It seems plausible to expect that the data on numbers of predator and prey fish caught, which reflects the total populations, is related to these equilibrium values. This is indeed the case. The closed loop orbits in the phase portrait come from *periodic* functions $x(t)$ and $y(t)$. The average value of $x(t)$ over one period is exactly the x-coordinate of the equilibrium point, and the average of $y(t)$ over one period is the y-coordinate of the equilibrium point; see Exercise 2. Thus Volterra's model predicts that when fishing intensity is decreased we will see an increase in the percentage of predator fish, and a decrease in the percentage of prey fish, among the total catch.

We finish our analysis of the Lotka-Volterra equations by discussing why the orbits are closed curves.

Why the orbits in the Lotka-Volterra system must be closed curves. Here we will give an ad hoc argument that will show the curves with equation $a \ln y - by + d \ln x - cx = k$ are either closed curves or a single point for the relevant values of the constant k. (In Section 10.6 we will develop some machinery to analyze this system in a different way.) We begin with some elementary analysis of the two functions

$$f(x) = d \ln x - cx \quad \text{and} \quad g(y) = a \ln y - by.$$

A calculus argument shows that on their domains $x > 0$ and $y > 0$, the graphs of f and g are as depicted below, with f strictly increasing from $-\infty$ to its maximum at $x = \frac{d}{c}$ and then strictly decreasing to $-\infty$ as $x \to \infty$. We'll denote the maximum value of f by M_1. Similarly, g strictly increases from $-\infty$ to its maximum at $y = \frac{a}{b}$ and then strictly decreases to $-\infty$ as $y \to \infty$. Denote the maximum value of g by M_2. Whether the graphs of f and g look like those in Figs. 10.31–10.32 or Figs. 10.33–10.34 depend on the relative sizes of a and b and of c and d. Our analysis will be the same in either case.

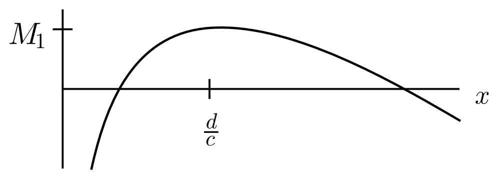


Figure 10.31. $f(x) = d \ln x - cx$.

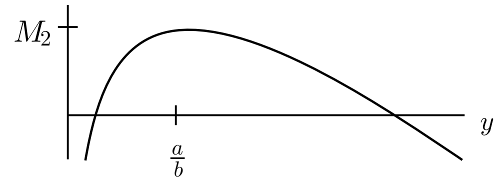


Figure 10.32. $g(y) = a \ln y - by$.

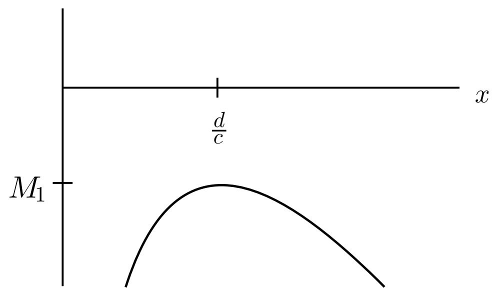


Figure 10.33. $f(x) = d \ln x - cx$.

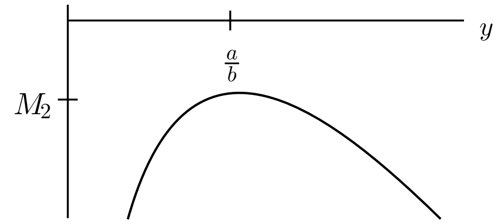


Figure 10.34. $g(y) = a \ln y - by$.

Consider an orbit with equation $f(x) + g(y) = k$. If $k > M_1 + M_2$, then there are no values of x and y for which $f(x)+g(y) = k$. The only solution to $f(x)+g(y) = M_1+M_2$ is $x = \frac{d}{c}, y = \frac{a}{b}$, so the orbit given by $f(x)+g(y) = M_1+M_2$ is just the equilibrium point $(\frac{d}{c}, \frac{a}{b})$. Thus we are mainly interested in the solutions to

$$f(x) + g(y) = k \quad \text{where } k < M_1 + M_2.$$

Note that a value of k less than $M_1 + M_2$ can be described as $M_1 + M_2 - \alpha$ for $\alpha > 0$.

Pick a positive constant α and notice that the equation $f(x) = M_1 - \alpha$ has exactly two solutions. Call the smaller solution x_m and the larger solution x_M; we have $x_m < \frac{d}{c} < x_M$. We make three observations (see Fig. 10.35):

(i) If $y = \frac{a}{b}$, so that $g(y) = M_2$, then the equation $f(x) + g(y) = M_1 + M_2 - \alpha$ has exactly two solutions for x, namely x_m and x_M (with, as already noted, $x_m < \frac{d}{c} < x_M$). This says that the orbit $f(x)+g(y) = M_1+M_2-\alpha$ intersects the horizontal line $y = \frac{a}{b}$ in exactly two points, $(x_m, \frac{a}{b})$ and $(x_M, \frac{a}{b})$.

(ii) If $x^* < x_m$ or $x^* > x_M$, then $f(x^*) < M_1 - \alpha$ and the equation $f(x^*)+g(y) = M_1+M_2-\alpha$ has no solution for y, since $g(y)$ is never greater than M_2. This says that the orbit $f(x) + g(y) = M_1 + M_2 - \alpha$ lies within the vertical strip $x_m \le x \le x_M$.

(iii) If $x_m < x^* < x_M$, then $f(x^*) > M_1 - \alpha$, and the equation $f(x^*) + g(y) = M_1 + M_2 - \alpha$ has exactly two solutions for y, one less than $\frac{a}{b}$ and one greater than $\frac{a}{b}$. This says that the vertical line $x = x^*$ intersects the orbit $f(x) + g(y) = M_1 + M_2 - \alpha$ in exactly two points.

Thus in the first quadrant, the orbits of the Lotka-Volterra system (10.47) are either closed curves encircling the equilibrium point $(d/c, a/b)$ or the equilibrium solution $x(t) = d/c, y(t) = a/b$ itself. We conclude that $(d/c, a/b)$ is a center for this nonlinear system.

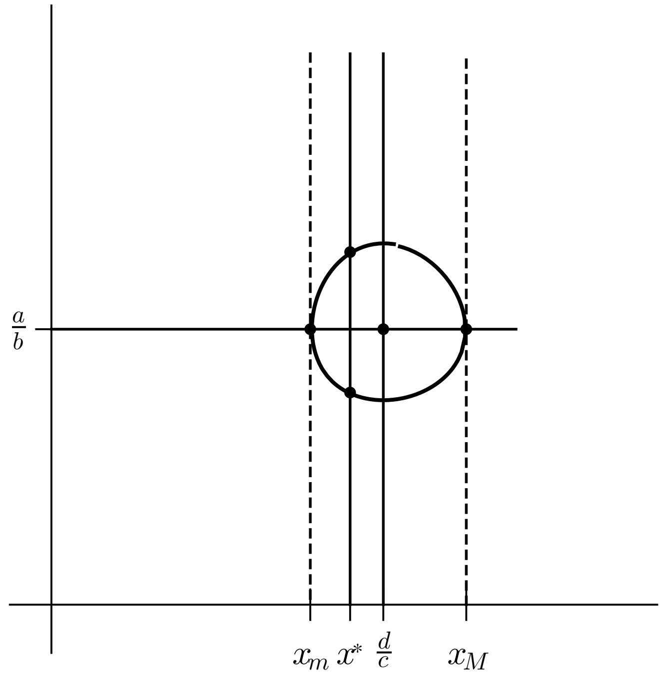


Figure 10.35. Determining an orbit for the Lotka-Volterra system.

10.4.1. Competition for space and habitat destruction. Interacting-species models are sometimes refined by specifying the nature of the competition between the species. We'll finish this section by analyzing a model for species that compete for *space*. This model has particular interest when studying the effects of habitat destruction.

Example 10.4.1. Corals in an ocean reef, barnacles growing in a rocky tide pool, plants in a tropical rain forest, different grass species in a prairie field—these are all examples of organisms that ecologists might view as "competing for space". In this example we model two species in such spatial competition. Our basic assumption is that the first species outcompetes the second, in the sense that it can colonize any space occupied by the second species just as easily as it can colonize empty space. We write $p_1(t)$ for the *proportion* of total available space occupied at time t by the first species, and $p_2(t)$ for the proportion occupied by the second species, so that $0 \leq p_1(t)$, $0 \leq p_2(t)$, and $p_1(t) + p_2(t) \leq 1$ for all times t.

The differential equation for p_1 does not depend at all on p_2, since as far as the first species is concerned, space occupied by the second species is as good as empty space. The first species colonizes at a rate jointly proportional to p_1 and the fraction of space not currently occupied by the first species, namely $1 - p_1$. There is also ordinary mortality, or local extinction, of sites occupied by the first species, which occurs at a rate proportional to p_1. The resulting equation for $\frac{dp_1}{dt}$ is

$$\frac{dp_1}{dt} = a_1 p_1(1 - p_1) - m p_1. \tag{10.48}$$

The constant a_1 is called the **colonization-rate constant** for the first species, and m is called the **mortality rate**.

For the second species, the only space available for colonization is the currently empty space, so the portion of space available for its colonization is $1 - p_1 - p_2$. It colonizes at a rate jointly proportional to p_2 and the proportion of available space, $1 - p_1 - p_2$. Some of the space currently occupied by the second species is lost due to colonization by the first species, this occurs at a rate

$a_1p_1p_2$. Putting this all together we have

$$\frac{dp_2}{dt} = a_2p_2(1 - p_1 - p_2) - a_1p_1p_2 - mp_2, \tag{10.49}$$

where the last term represents the ordinary mortality of the second species. Notice we use the same mortality rate m as in the equation for $\frac{dp_1}{dt}$; a more general model would allow for two different mortality constants.

The p_1-nullclines are the pair of lines

$$p_1 = 0 \quad \text{and} \quad p_1 = 1 - \frac{m}{a_1}.$$

We will make the assumption that $m < a_1$, so that the p_1-nullclines are situated as shown in Fig. 10.36. The left and right arrows indicate the regions in the first quadrant in which $\frac{dp_1}{dt}$ is positive or negative.

The p_2-nullclines consist of the line $p_2 = 0$ and the line

$$p_2 = 1 - \frac{m}{a_2} - \left(1 + \frac{a_1}{a_2}\right) p_1.$$

We assume that $m < a_2$. The p_2-nullclines are shown in Fig. 10.37, along with the up and down arrows, indicating the regions in the first quadrant in which $\frac{dp_2}{dt}$ is positive or negative.

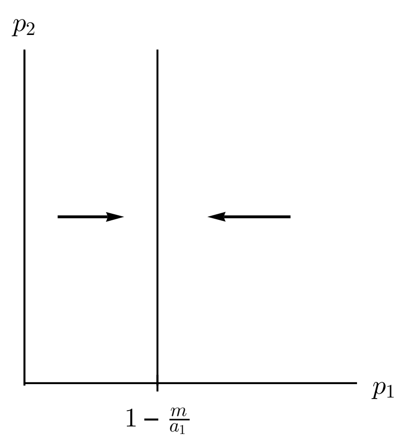


Figure 10.36. The p_1-nullcline; $m < a_1$.

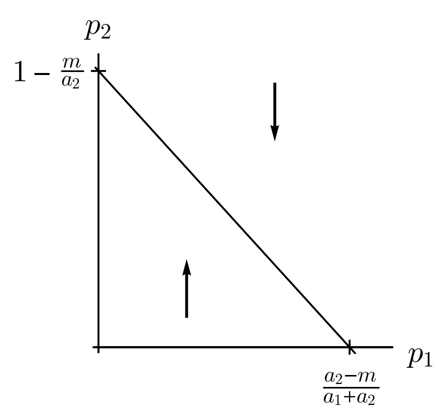


Figure 10.37. The p_2-nullcline; $m < a_2$.

How Fig. 10.36 and Fig. 10.37 fit together depends on the size of the parameters a_1, a_2, and m. The two possibilities are shown in Fig. 10.38 and Fig. 10.39. We can distinguish them by thinking about equilibrium points.

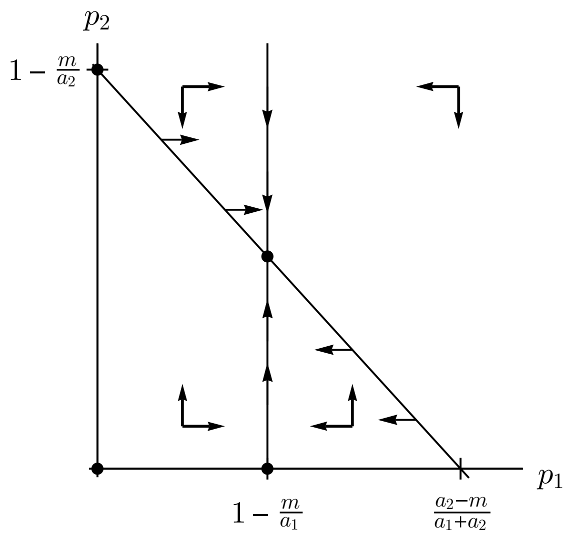


Figure 10.38. Case 1.

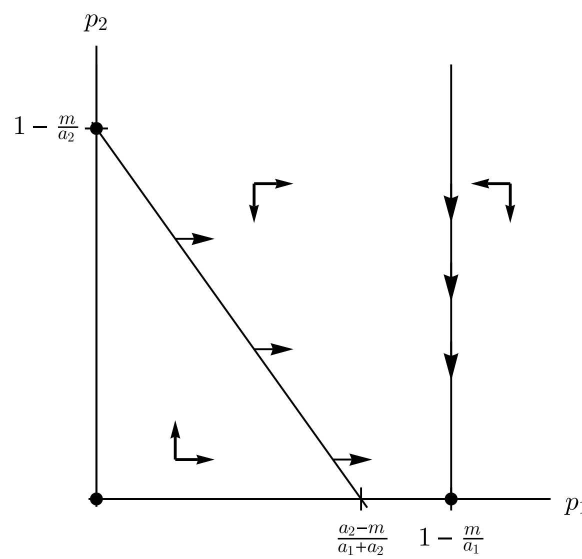


Figure 10.39. Case 2.

One equilibrium point is clearly $(0,0)$. There are equilibrium points at

$$\left(0, 1-\frac{m}{a_2}\right) \quad \text{and} \quad \left(1-\frac{m}{a_1}, 0\right),$$

and these are meaningful if $m < a_2$ and $m < a_1$. Under what conditions do we have an equilibrium point with positive values for p_1 and p_2? Some algebra shows that the lines

$$p_1 = 1 - \frac{m}{a_1} \quad \text{and} \quad p_2 = 1 - \left(1+\frac{a_1}{a_2}\right)p_1 - \frac{m}{a_2}$$

intersect in the point

$$P = \left(\frac{a_1-m}{a_1}, \frac{ma_2-a_1^2}{a_1a_2}\right). \tag{10.50}$$

This will be a biologically meaningful equilibrium point if

$$a_1 > m \quad \text{and} \quad ma_2 > a_1^2. \tag{10.51}$$

For these two conditions to hold, we must have $a_2 > a_1$ (see Exercise 15), so that the weaker competitor (species 2) must be able to more effectively colonize empty space. This is sometimes called the "competition-colonization trade off". For the rest of this example we will assume that conditions (10.51) hold. This also means that the nullcline-and-arrow diagram is as in Fig. 10.38.

The Jacobian matrix for equations (10.48)–(10.49) is

$$\begin{pmatrix} a_1 - 2a_1p_1 - m & 0 \\ -a_2p_2 - a_1p_2 & a_2 - a_2p_1 - 2a_2p_2 - a_1p_1 - m \end{pmatrix}.$$

At the equilibrium point $(0,0)$, this is

$$\mathbf{J}_1 = \begin{pmatrix} a_1 - m & 0 \\ 0 & a_2 - m \end{pmatrix}$$

and our assumption on the parameters tells us that the eigenvalues of $\mathbf{J}_1$ are both positive and $(0,0)$ is an unstable node.

Some computation shows that the Jacobian matrix at the equilibrium point P is

$$\mathbf{J}_2 = \begin{pmatrix} m - a_1 & 0 \\ (-a_1 - a_2)(\frac{m}{a_1} - \frac{a_1}{a_2}) & \frac{a_1^2 - a_2 m}{a_1} \end{pmatrix}.$$

This lower triangular matrix has eigenvalues

$$m - a_1 \quad \text{and} \quad \frac{a_1^2 - a_2 m}{a_1}.$$

Since we are assuming the conditions in equation (10.51), both of these eigenvalues are negative and this equilibrium point is a stable node. In Exercise 15 you are asked to show that the equilibrium points $(0, 1 - \frac{m}{a_2})$ and $(1 - \frac{m}{a_1}, 0)$ are saddle points. Using all of this information and the nullcline-and-arrow analysis above, we sketch the phase portrait in Fig. 10.40.

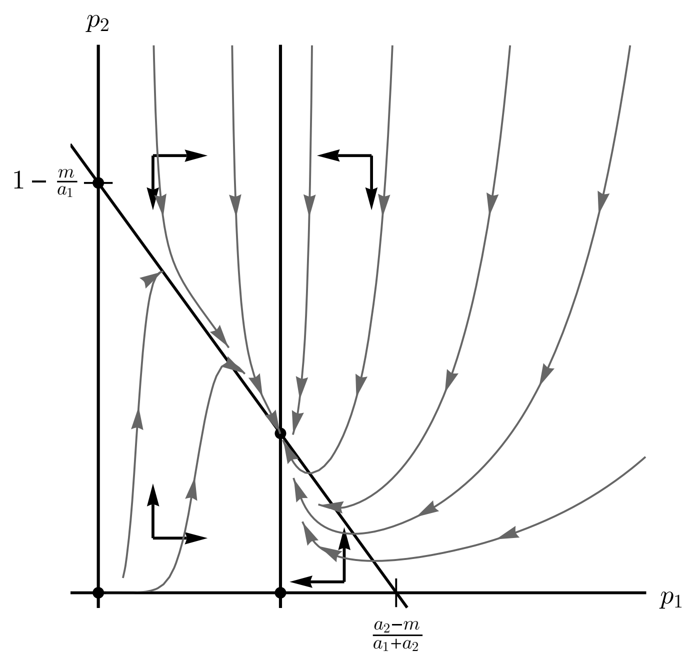


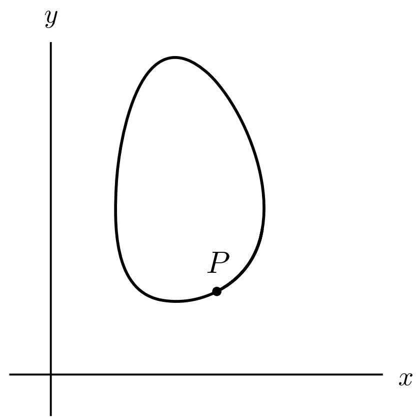
Figure 10.40. Phase portrait when (10.51) holds.

The model predicts coexistence of the two species under our assumptions on the parameters in (10.51).

Habitat destruction. The preceding example gives the simplest model for the utilization of space by species that interact in a hierarchical web (meaning, in general, that species 1 outcompetes species 2, which outcompetes species 3, etc.). There has been recent interest in incorporating a "habitat destruction" feature into such a model to see how loss of habitat affects the various species in the community. The results are unexpected and warn that a surprising "selective extinction" may occur. This is explored in Exercise 19.

10.4.2. Exercises.

1. Consider an orbit in the phase portrait for the predator-prey system in (10.47), as shown below. Suppose the point P corresponds to $t = 0$ and the orbit is traversed counterclockwise as t increases. Which of the following two graphs could show x and y as functions of t for this orbit?



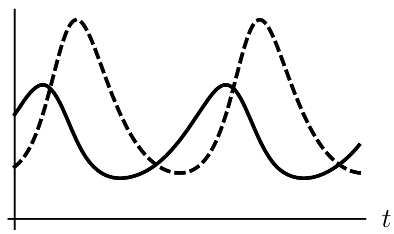


Figure 10.41. Predator population curve shown dashed.

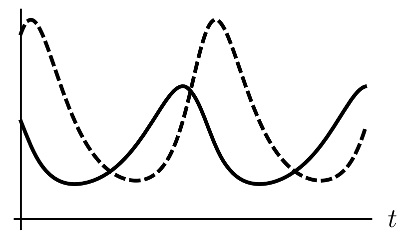


Figure 10.42. Predator population curve shown dashed.

2. Since the first-quadrant orbits in the phase portrait for the Lotka-Volterra equations (10.47) are closed loops, the solution functions $x(t)$ and $y(t)$ are periodic. The purpose of this problem is to show: *The average value of $x(t)$ over one period is d/c, the x-coordinate of the equilibrium point, and the average value of $y(t)$ is a/b, the y-coordinate of the equilibrium point.*

   Suppose T denotes the period of $y(t)$; then the average value of $y(t)$ is defined to be
$$\frac{1}{T}\int_0^T y(t)dt.$$
   Our goal is to show this is a/b.
   (a) By (10.47) we have $x'(t) = x(a - by)$, so that
$$\int_0^T \frac{x'(t)}{x(t)}dt = \int_0^T (a - by)\, dt. \tag{10.52}$$
   Using the fact that $x(0) = x(T)$, show that the *left-hand* side of equation (10.52) is 0.
   (b) From (a) you now know that
$$\int_0^T a\ dt = \int_0^T by\ dt.$$
   From this, show
$$\frac{1}{T}\int_0^T y(t)dt = \frac{a}{b}.$$
   (c) With analogous computations show
$$\frac{1}{T}\int_0^T x(t)dt = \frac{d}{c}.$$

3. Aphids are eating the tomato plants in your garden. The aphid population (prey) is kept in check by ladybugs (predator). You decide to use a pesticide to try to reduce the aphid population.

The pesticide also kills ladybugs, at the same rate that it kills the aphids. Modify the Lotka-Volterra predator-prey equations for this scenario, and then comment on the wisdom of your decision.

4. The Lotka-Volterra predator-prey equations are modified by assuming logistic growth for the prey in the absence of the predator:

$$x'(t) = x(a - bx - cy), \qquad y'(t) = y(-d + fx).$$

   (a) Which is the prey population, x or y?
   (b) Sketch the phase portrait for the system

$$x'(t) = x(1 - x - y), \qquad y'(t) = y(-1 + 2x).$$

   Identify any equilibrium points and give their stability.

5. Here's another modification of the Lotka-Volterra predator-prey equations:

$$x'(t) = ax - by\sqrt{x}, \qquad y'(t) = -cy + dy\sqrt{x}.$$

   (a) Which is the prey population, x or y?
   (b) Sketch the phase portrait for the system

$$x'(t) = x - 2y\sqrt{x}, \qquad y'(t) = -2y + y\sqrt{x}.$$

   Identify any equilibrium points in the open first quadrant $x > 0$, $y > 0$, and give their stability.

6. Two competing fish species are modeled by the equations

$$x'(t) = x(60 - 2x - y), \qquad y'(t) = y(100 - x - 4y).$$

   (a) Show there is an asymptotically stable equilibrium at $(20, 20)$.
   (b) The y-population begins to be harvested with "constant effort"—this means that fishing removes members of the y-population at a rate proportional to the y-population; the proportionality constant is called the effort coefficient. If the effort coefficient is E, what is the new system of equations for the two populations?
   (c) Find the equilibrium points for your answer to (b) if $E = 10$ and if $E = 80$.
   (d) For what values of $E < 100$ are there no equilibrium points in the open first quadrant ($x > 0$ and $y > 0$)?

7. Lionfish, a favorite for salt water aquariums due to its exotic and beautiful appearance, is also a voracious predator. Speculation is that lionfish were accidentally introduced into the Atlantic at Biscayne Bay, Florida, when a small number of fish escaped from an aquarium during Hurricane Andrew in 1992. Since then, lionfish have spread rapidly into the Caribbean and along the Eastern Seaboard, at great detriment to a variety of coral reef fishes, upon which lionfish prey. Losses of nearly 80% of native reef fishes in just 5 weeks have been documented in some locations. Lionfish have been observed eating both large quantities of smaller fish (in one case, of 20 small wrasses in 30 minutes) as well as fish up to 2/3 of their own length. They have few natural predators in the Atlantic and Caribbean.

Figure 10.43. A common variety of lionfish in the Atlantic and Caribbean. Photo by the authors.

A recent article in the Food section of the Washington Post[9] highlighted a campaign to promote lionfish as a tasty human food option, with the hope that by aggressively encouraging commercial fishing of lionfish, they might be removed from these locations in which they are not native. A follow-up article in the Washington Post indicates that this strategy has had considerable success.[10] Indeed, some Whole Foods stores in Florida and elsewhere now sell lionfish for food.

(a) Suppose that a model for lionfish population $y(t)$ in an Atlantic coral reef and their prey $x(t)$ is given by

$$x'(t) = x(r - ax - by), \quad y'(t) = y(s + cx - dy)$$

for positive constants $a, b, c, d, r,$ and s. Do a nullcline-and-arrow diagram in the first quadrant of the phase portrait for this system under the assumption that $s/d > r/b$. Show all equilibrium points.

(b) Evaluate the Jacobian matrix at each equilibrium point from (a), and classify them if possible. You may assume $r \neq s$.

(c) Using your work in (a) and (b), what do you predict for the long-term behavior of the two populations?

(d) Suppose now we selectively fish for just the lionfish, in such a way that they are harvested at a rate proportional to their population. This gives rise to a new system with equations

$$x'(t) = x(r - ax - by), \quad y'(t) = y(s + cx - dy) - fy.$$

Suppose that f is less than s but is large enough that $(s - f)/d < r/b$. What happens to the populations now? To answer this, sketch the phase portrait for the new system.

8. A predator-prey model is proposed with the equations

$$x'(t) = ax\left(1 - \frac{x}{K}\right) - bxy \text{ (prey)}, \quad y'(t) = cy\left(1 - \frac{y}{dx}\right) \text{ (predator)}.$$

These equations are based on the following principles:

The prey population grows logistically, with carrying capacity K, in the absence of predators; encounters with predators are detrimental to the prey. The predator population grows according to a modified logistic equation with *variable* carrying capacity that is proportional to the amount of available prey.

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[9]Juliet Eilperin, The Washington Post, July 7, 2010.

[10]Ramit Masti, *Can U.S. crush invasive species by eating them?*, The Washington Post, May 26, 2014.

(a) Explain how these lead to the form of the differential equations in the model. If y is the population of whales and x is the population of krill (small shrimp-like creatures), do you expect the constant d to be small or large? Hint: Think of d as a measure of how many whales can be sustained by one krill.

(b) Give a nullcline-and-arrow analysis in the first quadrant for the system

$$x'(t) = x - 2x^2 - xy, \quad y'(t) = y - 3\frac{y^2}{x},$$

which is obtained from particular values of a, b, c, d, and K.

(c) What are the equilibrium points with $x > 0$ and $y > 0$ of the system in (b)? Classify them according to type and stability.

(d) On the basis of your work in (b) and (c), sketch the phase portrait and predict the long-term fate of the two populations modeled by the equations in (b).

9. Rabbits ($x(t)$) and foxes ($y(t)$) are in a predator-prey relationship governed by the Lotka-Volterra equations

$$x'(t) = ax - bxy, \qquad y'(t) = -dy + cxy.$$

Assume that a superpredator (for example, human hunters) with a fixed population size begins to prey on both rabbits and foxes, so that the new differential equation system is

$$x'(t) = ax - bxy - h_1 x,$$

$$y'(t) = -dy + cxy - h_2 y,$$

where h_1 and h_2 are positive constants, possibly different.

(a) If $a - h_1 > 0$, show that the presence of the superpredator benefits the rabbits.

(b) What happens to the rabbit and fox populations if $a - h_1 < 0$?

10. *Reptilian dominance in Australia and New Guinea.*[11] On the continent of Australia, neighboring New Guinea, and adjacent islands there are large reptiles, such as monitor lizards and the Komodo dragon, that are predators, but few large mammalian predators. This is true not just in present times, but going back millions of years. The only carnivore mammals weighing more than 10 pounds that can be found today in Australia are the Tasmanian devil (a marsupial hyena) and the spotted-tailed quoll, which resembles a weasel. Now extinct, there was previously also a dog-like marsupial and a marsupial lion. By contrast, in the United States even today one can find dozens of kinds of mammalian carnivores and more many that existed previously but are now extinct.

What can account for this difference? The land in Australia is, by comparison with the Americas, Europe, and Asia, relatively infertile, as measured by the quality of the soil, the potential for prolonged droughts, and other climatic factors. This impedes the growth of high quality vegetation which in turn means that fewer large herbivores will be available as prey for carnivores. Carnivorous reptiles, being cold-blooded, generally require less food and energy than their warm-blooded counterparts. Thus the low productivity of the land may favor the development of meat-eating reptiles over meat-eating mammals. In this problem, we explore this theory.

Begin with the modified Lotka-Volterra model

$$x'(t) = x(r - ax - by), \qquad y'(t) = -sy + cxy$$

with positive constants a, b, c, r, and s for interacting predator (y) and prey (x) populations in Australia.

(a) Show that if $r/a < s/c$, there is no equilibrium point (x_e, y_e) with x_e and y_e both positive.

[11] The ideas in this problem are adapted from Clifford Taubes, *Modeling Differential Equations in Biology*, 2nd ed., Cambridge University Press, New York, 2008, Chapter 12.

(b) Show $(r/a, 0)$ is an equilibrium point and that under the same condition $r/a < s/c$, this equilibrium point is stable.

(c) Show by contrast that if $r/a > s/c$, then there is an equilibrium point (x_e, y_e) with both x_e and y_e positive, and moreover, this equilibrium point is stable.

(d) Assume that the constants r, a which appear in the differential equation for the prey do not change when the specific type of predator changes. If there is indeed a greater chance for stable, positive equilibrium values of both prey and predator with a reptilian predator than with a mammalian predator, would you expect the ratio s/c to be larger or smaller for a reptilian predator, as compared with a mammalian one? Do you expect that a species with lower food energy needs would have a larger or smaller value of c?

11. Sperm whales in the Southern Ocean eat squid, and the squid in turn eat krill. The following equations are proposed to model the three populations:

$$x'(t) = r_1 x(1 - b_1 x - cz), \quad y'(t) = r_2 y(1 - b_2 y/z), \quad z'(t) = r_3 z(1 - b_3 z/x - dy),$$

where all constants are positive. These equations reflect the assumption that the "carrying capacity" of the two predator populations (sperm whale and squid) is proportional to the population of their respective prey. Which variables go with which species?

12. Start with the predator-prey equations

$$x'(t) = x(-d - kx + my), \qquad y'(t) = y(a - bx - cy),$$

where the parameters a, b, c, d, k, and m are positive. Suppose a pesticide is going to be used to try to control the prey population. Assume the pesticide targets both predator and prey in an equal fashion, so that a term hx is subtracted from the first equation and a term hy from the second.

(a) Which is the predator population, x or y?

(b) Is there any condition (on the various coefficients in the differential equations) under which the prey will be driven to extinction?

(c) Show that if

$$h > \frac{ma - dc}{m + c},$$

then the predator will be driven to extinction.

13. (CAS) A small island supports a population of rabbits and foxes. The rabbit and fox populations, in thousands, are given at time t, in years, by $x(t)$ and $y(t)$, respectively. Suppose that these populations are modeled by the predator-prey system

$$x'(t) = \frac{1}{4}x - xy, \quad y'(t) = -3y + xy$$

and that $x(0) = 2.4$ and $y(0) = 0.2$.

(a) Solve this initial value problem numerically, and use your solution to find the approximate maximum and minimum values of the rabbit and fox populations. Also, approximate the period of this predator-prey system—the number of years that pass before the rabbit and fox populations first return to the levels $x = 2.4$ and $y = 0.2$.

(b) Plot your solution of (a) over one period.

(c) Find an implicit solution of the initial value problem

$$\frac{dy}{dx} = \frac{-3y + xy}{\frac{1}{4}x - xy}, \quad y(2.4) = 0.2.$$

Then plot your solution and compare it to your plot for (b).

14. (CAS)
   (a) Construct a concavity chart, over the square $-4 < x < 4, -4 < y < 4$ for the Lokta-Volterra system
$$\frac{dx}{dt} = x - xy, \quad \frac{dy}{dt} = -2y + xy.$$
   Hint: Differentiation with respect to x of $\frac{dy}{dx} = \frac{-2y+xy}{x-xy}$ gives
$$\frac{d^2y}{dx^2} = -\frac{y(6 - 4x + x^2 - 4y + 2y^2)}{x^2(-1+y)^3}.$$
   Complete the square on the x and on the y terms to analyze $\frac{d^2y}{dx^2}$.
   (b) Produce a phase portrait over the same square. Include in your portrait any curves on which $\frac{d^2y}{dx^2}$ is 0 or undefined.
15. (a) Show why the conditions in equation (10.51) of Example 10.4.1 are exactly what is needed for there to be an equilibrium point with positive coordinates for the differential equation system of Example 10.4.1, and verify that this equilibrium point lies in the biologically meaningful triangle $p_1 \geq 0$, $p_2 \geq 0$, $p_1 + p_2 \leq 1$.
   (b) Explain why $a_2 > a_1$ is a consequence of the conditions in equation (10.51).
   (c) At these equilibrium values of p_1 and p_2, is there any empty (uncolonized) space in the environment?
   (d) Still assuming the conditions of equation (10.51), compute the Jacobian matrix at the equilibrium points
$$\left(0, 1 - \frac{m}{a_2}\right) \quad \text{and} \quad \left(1 - \frac{m}{a_1}, 0\right),$$
   and show they are saddle points.
16. (CAS) Using the values $a_1 = 1$, $a_2 = 2$, and $m = 0.25$ in equations (10.48)–(10.49), verify that Fig. 10.39 applies and find and classify the equilibrium points having $p_1 \geq 0$ and $p_2 \geq 0$. Then use a computer algebra system to produce a phase portrait.
17. Suppose in the model of Example 10.4.1 we have $a_1 = 0.2$/year, $a_2 = 0.8$/year, and $m = 0.1$/year. If both species start at some positive value, what do you expect to happen to p_1 and p_2 over a long period of time? Is there any empty space?
18. (CAS) Consider the initial value problem consisting of the model of Example 10.4.1 with constants a_1, a_2, and m as in the preceding exercise and the initial condition $p_1(0) = 0.8$, $p_2(0) = 0.05$.
   (a) Using a computer algebra system, solve this initial value problem numerically and plot the solutions for $p_1(t)$ and $p_2(t)$ over the time interval $[0, 100]$ (time measured in years).
   (b) Using your numerical solution, estimate the values of p_1 and p_2 at $t = 5, 10, 30, 50$, and 100 years.
19. This problem continues the ideas of Example 10.4.1, which modeled the competition for space by species that interact in a hierarchical web (in a hierarchical web, species 1 outcompetes species 2, which outcompetes species 3, etc.). In this problem we incorporate a "habitat destruction" feature into the model. We assume that some proportion D of the sites available for colonization is randomly and permanently destroyed. Modifying equations (10.48)–(10.49) to reflect this, the new system is
$$\frac{dp_1}{dt} = a_1p_1(1 - p_1 - D) - mp_1 = p_1(a_1 - a_1p_1 - a_1D - m),$$
$$\frac{dp_2}{dt} = a_2p_2(1 - p_1 - p_2 - D) - a_1p_1p_2 - mp_2 = p_2(a_2 - a_2p_1 - a_2p_2 - a_2D - a_1p_1 - m).$$

We will see that an unanticipated consequence of habitat destruction is the different ways in which the dominant competitor (species 1) and the weaker competitor (species 2) are affected by the loss of habitat.

(a) Find the equilibrium point P_D obtained by solving the pair of equations

$$a_1 - a_1 p_1 - a_1 D - m = 0 \quad \text{and} \quad a_2 - a_2 p_1 - a_2 p_2 - a_2 D - a_1 p_1 - m = 0.$$

(When $D = 0$ your answer should agree with equation (10.50).)

(b) We know from Example 10.4.1 that when $D = 0$, under certain conditions on the parameters a_1, a_2, and m, this equilibrium point will lie in the open first quadrant $p_1 > 0$, $p_2 > 0$. Assuming these conditions hold, how large can D be so that the equilibrium point P_D from (a) has both coordinates positive? You may find it helpful to write the coordinates of P_D in the form

$$\text{first coordinate} = \frac{a_1 - m}{a_1} + \text{an expression in terms of } D$$

and

$$\text{second coordinate} = \frac{ma_2 - a_1^2}{a_1 a_2} + \text{an expression in terms of } D$$

so that the relationship between the coordinates of P_D and the equilibrium point in equation (10.50) (corresponding to $D = 0$) is clear.

(c) True or false: The dominant competitor is predicted to become extinct at a lower level of habitat destruction than the weaker competitor.

(d) Does the phrase "the enemy of my enemy is my friend" have any relevance to this model?

10.5. Modeling the spread of disease

Communicable diseases have had profound impacts on the course of history. The Antonine plague (probably smallpox or measles or both) in AD 165–180 contributed to the demise of the Roman Empire. Bubonic plague caused the death of up to one third of the population of Europe in the fourteenth century. It reappeared again in London in 1665, killing one fifth of the population there. It forced the closure of Cambridge University for a while, where coincidently Issac Newton was a student. Returning home, Newton had a spectacularly prolific period of scientific work during the university's closure. The 1918–1919 pandemic influenza affected one third of the world's population and killed more (up to 50 million) than died in World War I (16 million); healthy young adults were particularly hard hit. HIV, SARS, swine or bird flu, the possibility of weaponized anthrax, or a terrorist release of the smallpox virus are just a few of the current concerns of public health officials.

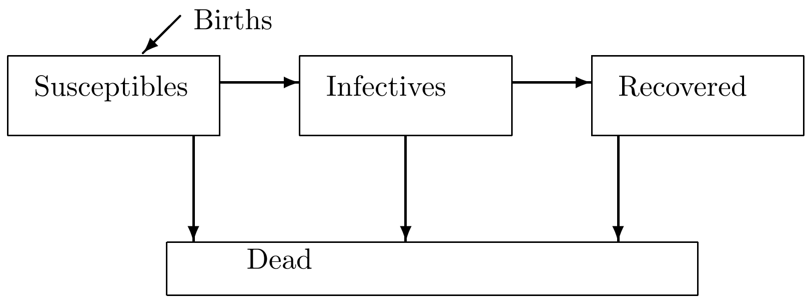
The use of mathematical models to analyze the spread of diseases goes back centuries. In 1760, the mathematician Daniel Bernoulli used a mathematical argument to show that cowpox vaccination against smallpox would significantly increase the average life span—his model showed that if deaths from smallpox could be eliminated, the average life span would increase by about 3 years (from the then-current value of 26.5 years). His work (see Exercise 4) represents the first important use of a mathematical model to address a practical vaccination proposal. In the early twentieth century, differential equation compartment models for disease propagation started to appear and win acceptance. As with other types of mathematical models, there is a trade-off between simple models which incorporate only a few broad details and more refined models whose solutions may be more difficult to obtain and to analyze. In this section we will look at several compartment models and see how even simple models give interesting predictions which can be tested against observed phenomena.

Example 10.5.1. Diseases with permanent immunity.

We'll model the spread of a disease—like measles—which confers immunity. This means, once infected and recovered, a person can never get the disease again. We will also modify this basic model, called an SIR model, to include the possibility of vaccination against the disease, and look at the idea of "herd immunity".

Setting up the SIR model. Imagine our population divided into three nonoverlapping compartments: the susceptibles, with population $S(t)$ at time t; the infectives, with population $I(t)$; and the recovered, with population $R(t)$, consisting of those people who have had the disease and recovered (and are forevermore immune from it). There is also a fourth compartment, consisting of people who have died (either from the disease or from other causes). The diagram below shows how people move from one compartment to another. Notice the model allows for births, and all newborns begin life in the susceptible compartment. We will set up our model so that total size of the population is a fixed value K. This means we assume that deaths balance out births. Here are the assumptions we make to describe the movement between the compartments:

- Infectives recover from the disease at a rate proportional to the number of infectives.
- Susceptibles become infective at a rate jointly proportional to the number of infectives and the number of susceptibles.
- The birth rate is constant and proportional to the total size K of the population.
- The total death rate is equal to the birth rate, and the death rate from each compartment is proportional to the size of the compartment, with the same proportionality constant μ for each of the three compartments, S, I, and R.



These assumptions give us the following equations:

$$\frac{dS}{dt} = \mu K - \beta SI - \mu S, \quad \frac{dI}{dt} = \beta SI - \mu I - \gamma I, \quad \frac{dR}{dt} = \gamma I - \mu R, \tag{10.53}$$

where μ, β, and γ are positive constants. Check that $\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$, reflecting the assumption that $S + I + R = K$ for all times. Since the equations for $\frac{dS}{dt}$ and $\frac{dI}{dt}$ involve only S and I, we can focus on those so as to have a planar system to analyze.

Nullcline-and-arrow diagrams. From the equation for $\frac{dI}{dt}$ in (10.53) we see that the I-nullclines are the lines with equations $I = 0$ and $S = (\gamma + \mu)/\beta$, shown in Fig. 10.44, along with the relevant up/down arrows. Using the equation for $\frac{dS}{dt}$ in (10.53) we see that the S-nullcline has equation

$$I = \frac{\mu K}{\beta}\frac{1}{S} - \frac{\mu}{\beta},$$

which is shown in Fig. 10.45 along with the relevant right/left arrows. How Figs. 10.45 and 10.44 fit together depend on whether $K \leq (\gamma + \mu)/\beta$ or $K > (\gamma + \mu)/\beta$.

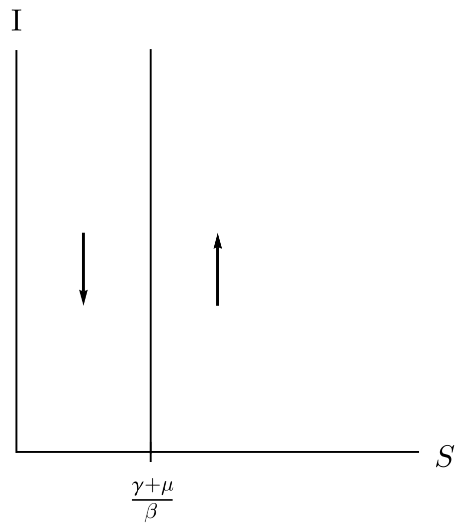


Figure 10.44. The I-nullclines $I = 0$ and $S = \frac{\gamma+\mu}{\beta}$.

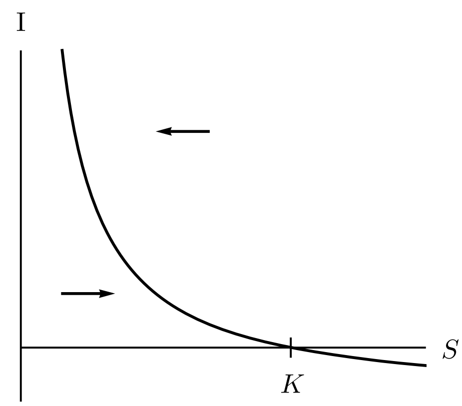


Figure 10.45. The S-nullcline $I = \frac{\mu K}{\beta}\frac{1}{S} - \frac{\mu}{\beta}$.

For the rest of this example, we make the following assumption:

$$K > \frac{\gamma + \mu}{\beta}. \tag{10.54}$$

With this assumption, the S- and I-nullclines intersect in the first quadrant as shown in Fig. 10.46.

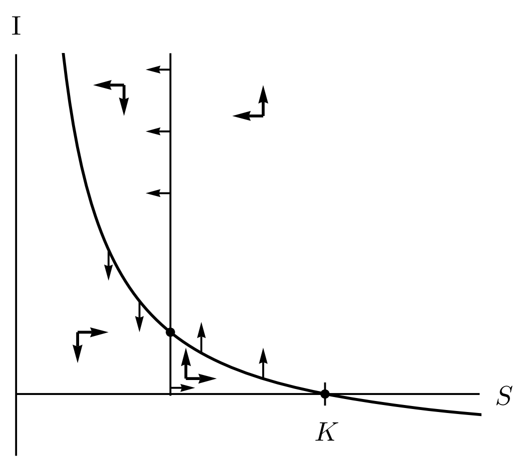


Figure 10.46. Nullcline-and-arrow diagram for first two equations in (10.53) if $\beta K > \gamma + \mu$.

Equilibrium points. Still assuming $\beta K > \gamma + \mu$, we see that there are two biologically relevant equilibrium points, obtained as the intersection of the S-nullcline with an I-nullcline. One is the "disease-free" equilibrium point $(K, 0)$. This should make perfect sense: If there are no infectives and $S = K$, then S should remain K, the total population size, for all times. The second equilibrium point is

$$\left(\frac{\gamma + \mu}{\beta}, \frac{\mu}{\beta(\gamma + \mu)}(\beta K - \gamma - \mu)\right). \tag{10.55}$$

To analyze these equilibrium points we compute the Jacobian matrix for the first two equations in (10.53), obtaining

$$\mathbf{J} = \begin{pmatrix} -\beta I - \mu & -\beta S \\ \beta I & \beta S - \gamma - \mu \end{pmatrix}.$$

At the equilibrium point $(K, 0)$ this is

$$\begin{pmatrix} -\mu & -\beta K \\ 0 & \beta K - \gamma - \mu \end{pmatrix}.$$

This matrix has eigenvalues $-\mu$ and $\beta K - \gamma - \mu$. Under assumption equation (10.54), one eigenvalue is positive and one is negative. By Theorem 10.2.3, we have an (unstable) saddle point at $(K, 0)$.

Computing the Jacobian at the second equilibrium point in equation (10.55) looks tedious, but we can extract what we need to know by observing that the 2-2 entry of the Jacobian at this point is 0: $\beta S - \gamma - \mu = 0$ when $S = (\gamma + \mu)/\beta$. From this it is immediate that the trace of the Jacobian is negative and the determinant, which is $\beta^2 SI$, is positive. This tells us that the second equilibrium point is stable (either a node or spiral).

Because we know that the equilibrium point (10.55) is stable, we can predict that the disease will become *endemic* in the population, with $I(t)$ tending to a constant, positive value as $t \to \infty$. An endemic disease is one that is consistently present in the population. Exercise 1 asks you to predict the long-term behavior if (10.54) does not hold.

The condition of (10.54) can be written as $\beta K/(\gamma + \mu) > 1$, and the qualitative long-term behavior depends on whether the quantity

$$R_0 = \frac{\beta K}{\gamma + \mu} \tag{10.56}$$

is greater than or less than 1. This is an example of a **threshold** quantity; R_0 is sometimes called the **basic reproduction number**.

The effect of vaccination. How can vaccination change the course of the disease? Assume that a fixed fraction ρ of all newborns is vaccinated against the disease, so they are born into the recovered compartment. Since the birth rate is μ, this adds a "rate in" term of $\rho\mu K$ to the equation for $\frac{dR}{dt}$ and modifies the "rate in" term for $\frac{dS}{dt}$ to be $(1-\rho)\mu K$. Our equations are now

$$\frac{dS}{dt} = (1-\rho)\mu K - \beta SI - \mu S, \quad \frac{dI}{dt} = \beta SI - \mu I - \gamma I, \quad \frac{dR}{dt} = \gamma I - \mu R + \rho\mu K. \tag{10.57}$$

Again we can focus on just the first two equations in (10.57). Our previous analysis still applies, except where we had "K" before, we now have $(1-\rho)K$. What determines whether we have one equilibrium point or two (in the first quadrant $I \geq 0, S \geq 0$) is whether $\beta(1-\rho)K > \gamma + \mu$ or not; this is (10.54) with K replaced by $(1-\rho)K$.

What fraction of newborns needs to be vaccinated so that instead of having a stable endemic equilibrium value, we have only a (stable) disease-free equilibrium? By our analysis and Exercise 1 we know this will happen if $\beta(1-\rho)K \leq \gamma + \mu$, or equivalently, if the new basic reproduction number

$$\widetilde{R_0} = \frac{\beta(1-\rho)K}{\gamma + \mu}$$

is less than 1. (The case of equality is ignored, as exact equality would not be expected to occur in practice.) Some algebra shows $\widetilde{R_0} < 1$ precisely if

$$\rho > 1 - \frac{\gamma + \mu}{\beta K}, \quad \text{or equivalently, if } \rho > 1 - \frac{1}{R_0} \tag{10.58}$$

where R_0 is the original basic reproduction number in (10.56). For different diseases, the value of R_0 can be estimated from epidemiological data. For example, measles has a high value, estimated at about 12–18 in urban areas. By equation (10.58) this translates into a need of vaccinating about 91.6–94.4 percent of the newborn population to ensure that measles cannot become endemic. Exercise 2 asks you to verify this and to similarly determine what fraction of newborns needs to be vaccinated to control several other diseases with different basic reproduction numbers. If enough of the population has been vaccinated to prevent the disease from becoming endemic, the population is said to have "herd immunity".

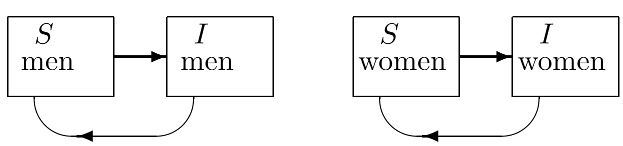
Example 10.5.2. Modeling the spread of a sexually transmitted disease. We start with a fixed population of N at-risk (sexually active) people and assume that men can only be infected by women and women can only be infected by men. Let M be the total number of men, and W the total number of women, so that $M + W = N$; M, W, and N are constants. As is typical of many bacterial STDs, we assume that an infection does not confer immunity, so after a person is infected and recovers, he or she is immediately susceptible to reinfection. At any point in time there will be $x(t)$ infected men and $M - x(t)$ susceptible men, and $y(t)$ infected women and $W - y(t)$ susceptible women. Our model will give differential equations for $\frac{dx}{dt}$ and $\frac{dy}{dt}$. To derive the form of these equations we make several basic assumptions:

- Men are infected at a rate jointly proportional to the number of susceptible men and the number of infected women.
- Women are infected at a rate jointly proportional to the number of susceptible women and the number of infected men.
- Infected men recover at a rate proportional to the number $x(t)$ of infected men.
- Infected women recover at a rate proportional to the number $y(t)$ of infected women.

These assumptions lead to the equations

$$\frac{dx}{dt} = a_1(M - x)y - b_1 x, \qquad \frac{dy}{dt} = a_2(W - y)x - b_2 y \tag{10.59}$$

for some positive constants a_1, a_2, b_1, and b_2. You should be able to identify which terms in these equations correspond to which "in" and "out" arrows in the schematic diagram below.



For a number of important STDs, like gonorrhea and chlamydia, women typically show no or few symptoms and thus do not seek (antibiotic) treatment. This fact is reflected in the value of b_2 being relatively small. If men do typically show symptoms and seek treatment (for example, this is the case with gonorrhea), then we would expect b_1 to be significantly larger than b_2. The reciprocals $1/b_1$ and $1/b_2$ can be interpreted as the average time a man or woman remains infective.

Equilibrium points. The system (10.59) has an equilibrium point at $(0, 0)$. There is a possibility of a second biologically meaningful equilibrium point. Solving the equations

$$a_1(M - x)y - b_1 x = 0 \quad \text{and} \quad a_2(W - y)x - b_2 y = 0,$$

we get a solution point

$$\left(\frac{MWa_1a_2 - b_1b_2}{b_1a_2 + Wa_1a_2}, \frac{MWa_1a_2 - b_1b_2}{b_2a_1 + Ma_1a_2}\right). \tag{10.60}$$

This will be of interest if it lies in the first quadrant of the xy-plane; this happens if

$$MWa_1a_2 - b_1b_2 > 0. \tag{10.61}$$

Before going further, let's rewrite the coordinates of the point in (10.60) by dividing each numerator and denominator by a_1a_2 to obtain

$$\frac{MWa_1a_2 - b_1b_2}{b_1a_2 + Wa_1a_2} = \frac{MW - \rho_1\rho_2}{W + \rho_1}, \qquad \frac{MWa_1a_2 - b_1b_2}{b_2a_1 + Ma_1a_2} = \frac{MW - \rho_1\rho_2}{M + \rho_2}$$

where $\rho_1 = b_1/a_1$ and $\rho_2 = b_2/a_2$. The equilibrium point

$$\left(\frac{MW - \rho_1\rho_2}{W + \rho_1}, \frac{MW - \rho_1\rho_2}{M + \rho_2}\right)$$

lies in the first quadrant if

$$MW - \rho_1\rho_2 > 0. \tag{10.62}$$

From this point on we assume that (10.62), or equivalently (10.61), holds. The Jacobian matrix for the linear approximation of (10.59) near $(0,0)$ is

$$\mathbf{J}_1 = \mathbf{J}(0,0) = \begin{pmatrix} -b_1 & Ma_1 \\ Wa_2 & -b_2 \end{pmatrix}.$$

Under our assumption (10.61), this is a matrix with negative determinant, so according to Section 8.9 and Theorem 10.2.3, $(0,0)$ is a saddle point for the system (10.59).

The Jacobian matrix at the second equilibrium point is

$$\mathbf{J}_2 = \begin{pmatrix} -a_1y_0 - b_1 & a_1M - a_1x_0 \\ a_2W - a_2y_0 & -a_2x_0 - b_2 \end{pmatrix},$$

where $x_0 = (MW - \rho_1\rho_2)/(W + \rho_1)$ and $y_0 = (MW - \rho_1\rho_2)(M + \rho_2)$ are the coordinates of the nonzero equilibrium point. This is a little unpleasant to deal with, but we can make two observations with minimal calculation:

- The trace of $\mathbf{J}_2$ is negative, since both entries on the main diagonal are negative.
- The determinant of $\mathbf{J}_2$ is positive. To see this, compute

  $$a_1M - a_1x_0 = a_1\left(M - \frac{MW - \rho_1\rho_2}{W + \rho_1}\right) = a_1\rho_1\frac{M + \rho_2}{W + \rho_1}$$

  and

  $$a_2W - a_2y_0 = a_2\left(W - \frac{MW - \rho_1\rho_2}{M + \rho_2}\right) = a_2\rho_2\frac{W + \rho_1}{M + \rho_2}.$$

  Using the definitions of ρ_1 and ρ_2 we see that the determinant of $\mathbf{J}_2$ is

  $$(a_1y_0 + b_1)(a_2x_0 + b_2) - a_1a_2\rho_1\rho_2 = a_1a_2y_0x_0 + b_1a_2x_0 + b_2a_1y_0$$

  which is clearly positive.

A negative trace and positive determinant tell us that the equilibrium point is stable, either a node or a spiral. We'll return to this shortly.

Nullcline-and-arrow diagrams. What are the nullclines for our system? We are only interested in sketching the phase portrait in the rectangle $0 \le x \le M, 0 \le y \le W$, since this is the only portion of the xy-plane that is biologically relevant. The restrictions $x \geq 0, y \geq 0$ are clear, and the restrictions $x \leq M, y \leq W$ follow since the number of infected men (women) cannot exceed the total number of men (women) in the population. The x-nullcline is the curve with equation

$$y = \frac{b_1 x}{a_1(M - x)}; \tag{10.63}$$

let's call the right-hand side of (10.63) $F(x)$. To sketch the graph, it's helpful to use a little calculus. From (10.63) we compute

$$F'(x) = \frac{Ma_1b_1}{[a_1(M-x)]^2} \quad \text{and} \quad F''(x) = \frac{2Mb_1}{a_1(M-x)^3}.$$

Since F' is always positive, the graph of F is increasing. We've observed that $x \leq M$, so the graph of F is concave up for $0 < x < M$. Finally, $F(0) = 0$ and $\lim_{x \to M^-} F(x) = \infty$. The x-nullcline is sketched in Fig. 10.47, where we have also shown the correct right/left arrows on either side of this nullcline. We've only shown the portion of the nullcline in the biologically relevant rectangle.

A similar analysis is possible for the y-nullcline, which has equation

$$y = \frac{Wa_2x}{b_2 + a_2x}.$$

The graph of this is an increasing, concave down curve, passing through $(0, 0)$ and which has limit W as $x \to \infty$. You are asked to verify these facts in Exercise 5. Fig. 10.48 shows the y-nullcline and the corresponding up/down arrows in the biologically relevant rectangle.

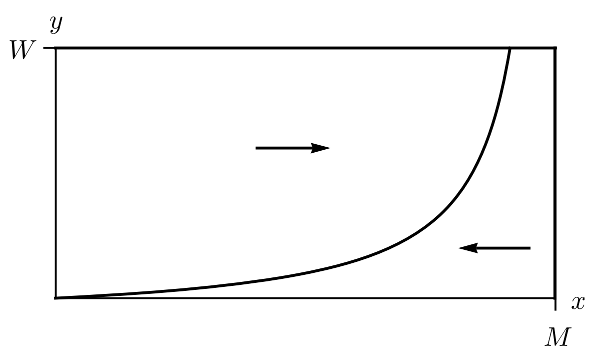


Figure 10.47. The x-nullcline.

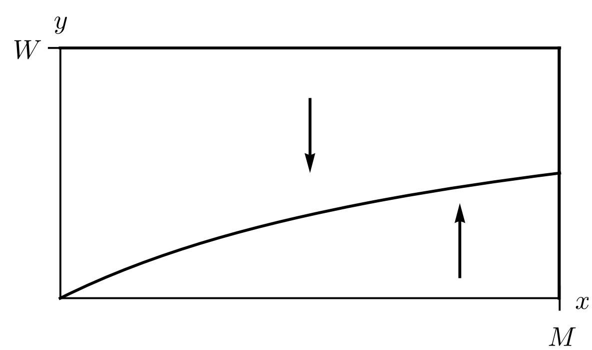
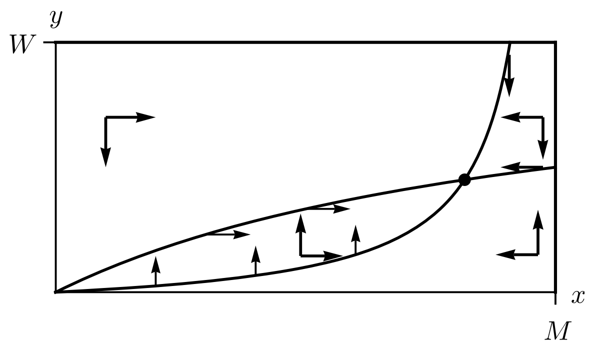


Figure 10.48. The y-nullcline.

We want to put Figs. 10.47 and 10.48 together. There are two possibilities here: Either the x- and y-nullclines intersect in two points, $(0, 0)$ and a second equilibrium point, or they only intersect in $(0, 0)$. From our earlier discussion of the equilibrium points, we know the first case occurs exactly when $MWa_1a_2 - b_1b_2 > 0$. (Exercise 5(c) gives another perspective on joining Fig. 10.47 and Fig. 10.48.) We focus on this case, leaving the case $MWa_1a_2 - b_1b_2 \leq 0$ for Exercise 5.

The nullcline-and-arrow picture looks like Fig. 10.49. Remembering that $(0, 0)$ is a saddle point, we sketch some trajectories. We hadn't classified the second equilibrium point, since the algebra involved in doing so was a bit off-putting, but we did note it was stable, and we can see from the nullcline-and-arrow diagram that it must be a node: A trajectory cannot spiral about the first quadrant equilibrium point because once it enters the region between the nullclines and to the left of the equilibrium point, it cannot leave. In that basic region trajectories rise, but they cannot rise above the y-nullcline because in so doing they would violate the downward direction-indicator arrow in the region above the y-nullcline.



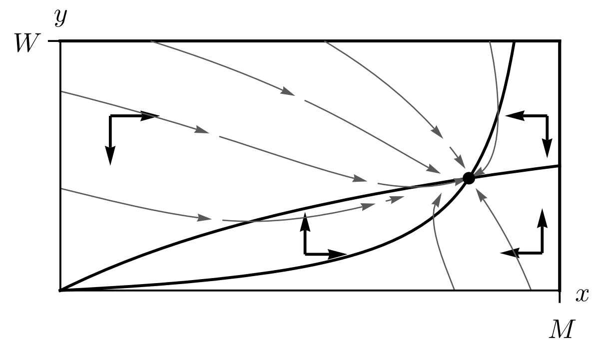


Figure 10.49. Nullcline-and-arrow diagram; $MWa_1a_2 - b_1b_2 > 0$.

Figure 10.50. Phase portrait; $MWa_1a_2 - b_1b_2 > 0$.

What does this predict for the long-term behavior of $x(t)$ and $y(t)$? As $t \to \infty$,

$$x(t) \to \frac{MW - \rho_1\rho_2}{W + \rho_1} \quad \text{and} \quad y(t) \to \frac{MW - \rho_1\rho_2}{M + \rho_2}.$$

Since these limiting values are nonzero, the disease will continue at an endemic level in the population.

Example 10.5.3. Modeling smoking. In this example, we model cigarette smoking among high school students. We assume that peer pressure plays some role in recruiting nonsmokers to become smokers. A similar system of equations could be used to model drug or alcohol use. We consider the student population to be divided into three compartments: nonsmokers $x(t)$ (whom we think of as "potential smokers"), smokers $y(t)$, and students who have quit smoking $z(t)$. We will assume that the total population size is fixed. Thus while students graduate, an equal number of new students matriculate, so that $x(t) + y(t) + z(t) = N$, where N is the constant total size of the school. The nonsmoker compartment plays the role of the "susceptible" compartment in the disease models, and we assume that the rate at which nonsmokers become smokers is jointly proportional to the number of smokers and the number of nonsmokers. This is the quantification of the role of peer pressure, and it will give rise to a term $-(\text{constant})xy$ in the differential equation for $\frac{dx}{dt}$ and a corresponding term, with a positive sign, in the differential equation for $\frac{dy}{dt}$. Remembering that N is constant, we will actually write this term in the form

$$-bx\frac{y}{N},$$

where the factor $\frac{y}{N}$ is the *proportion* of smokers in the student body and b is a positive constant that reflects, for example, the overall amount of social interaction and the likelihood of a nonsmoker being influenced to start smoking by the presence of smokers. We will also assume that the graduation rates are the same from each of the three compartments and that all newly matriculated students are initially nonsmokers. Finally we assume that smokers "recover"—that is, move into the "quitters" compartment at a rate proportional to y, the number of smokers. Our first model doesn't consider the possibility of "relapse" for someone who has quit smoking. With these assumptions, we have the differential equation system

$$\frac{dx}{dt} = gN - bx\frac{y}{N} - gx \qquad \text{(nonsmokers)}, \tag{10.64}$$

$$\frac{dy}{dt} = bx\frac{y}{N} - cy - gy \qquad \text{(smokers)}, \tag{10.65}$$

$$\frac{dz}{dt} = cy - gz \qquad \text{(quitters)} \tag{10.66}$$

for some positive constants g, b, and c. The terms $-gx$, $-gy$, and $-gz$ represent the graduation rates from the three compartments; these are exactly balanced by the term $gN = g(x + y + z)$ in the first equation, representing the addition of new students to the school, all initially in the nonsmoker category. In the equation for $\frac{dy}{dt}$ the term $-cy$ represents smokers quitting. The units on g and c would be, for example, $\frac{1}{\text{years}}$, and their reciprocals $\frac{1}{g}$ and $\frac{1}{c}$ have the meaning of "average time in school" and "average time as a smoker". Thus if the average time that an student spends in high school is 4 years, we have $g = \frac{1}{4}$/year. If we take 12 years as the average number of years that a smoker continues to smoke, then $c = \frac{1}{12}$/year.

You may notice that the *form* of equations (10.64)–(10.66) is exactly the same as in the SIR disease model in Example 10.5.1. So the analysis we did there applies here as well. However, we are going to proceed a little differently and begin by using the relation $x(t) + y(t) + z(t) = N$, to reduce the system to a planar system, with dependent variables y and z. Making the substitution $x = N - y - z$ in equation (10.65), we obtain the equations

$$\frac{dy}{dt} = b\frac{y(N - y - z)}{N} - (g + c)y,$$

$$\frac{dz}{dt} = cy - gz.$$

Next we will make the substitutions $S = \frac{y}{N}$ and $Q = \frac{z}{N}$, so that S and Q are the *proportions* of smokers and of smokers who have quit, respectively. These are "dimensionless" variables; they have no units. Since $y = NS$ and $z = NQ$, where N is constant, this gives

$$N\frac{dS}{dt} = b\frac{NS(N - NS - NQ)}{N} - (g + c)NS,$$

$$N\frac{dQ}{dt} = cNS - gNQ,$$

or simply

$$\frac{dS}{dt} = bS(1 - S - Q) - (g + c)S, \qquad \frac{dQ}{dt} = cS - gQ. \tag{10.67}$$

Because $\frac{x(t)}{N} + \frac{y(t)}{N} + \frac{z(t)}{N} = 1$ for all t, the only relevant part of the SQ-plane is the triangular region $S \geq 0, Q \geq 0, S + Q \leq 1$ shown in Fig. 10.51.

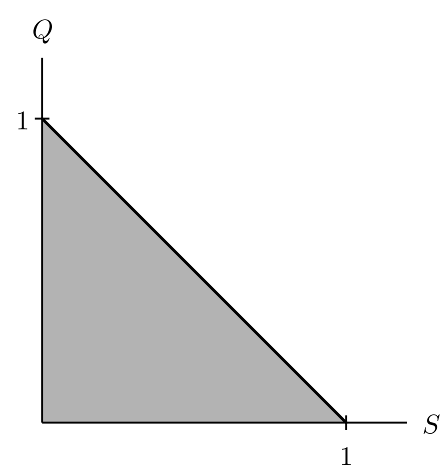


Figure 10.51. The biorelevant region $S \geq 0, Q \geq 0, S + Q \leq 1$.

Note that the Jacobian matrix for the system in equation (10.67) is

$$\begin{pmatrix} b-2bS-bQ-(g+c) & -bS \\ c & -g \end{pmatrix}.$$

To proceed with our analysis of this system, we will assign values to the constants b, g, and c. As previously discussed, we will use $g=\frac{1}{4}$ and $c=\frac{1}{12}$, and we will set $b=\frac{1}{2}$. With these values our system becomes

$$\frac{dS}{dt}=\frac{1}{2}S(1-S-Q)-\frac{1}{3}S, \qquad \frac{dQ}{dt}=\frac{1}{12}S-\frac{1}{4}Q. \tag{10.68}$$

Nullclines and arrows. Since $\frac{dS}{dt}=S(\frac{1}{6}-\frac{1}{2}S-\frac{1}{2}Q)$, the S-nullclines are the pair of lines $S=0$ and $S+Q=\frac{1}{3}$. The Q-nullcline is the line $Q=\frac{1}{3}S$. Fig. 10.52 shows the nullcline-and-arrow picture in the biologically relevant triangle.

Equilibrium points. Equilibrium points for the system (10.68) appear as the intersection of an S-nullcline with a Q-nullcline. One equilibrium point is $S=0, Q=0$, which corresponds in the original system (10.64)–(10.66) to $x=N, y=0, z=0$. A second equilibrium point is $S=\frac{1}{4}, Q=\frac{1}{12}$. To classify these equilibrium points we use the Jacobian matrix

$$\begin{pmatrix} \frac{1}{6}-S-\frac{1}{2}Q & -\frac{1}{2}S \\ \frac{1}{12} & -\frac{1}{4} \end{pmatrix}.$$

Evaluating the Jacobian matrix at the equilibrium point $(0,0)$ gives

$$\mathbf{J}_1=\begin{pmatrix} \frac{1}{6} & 0 \\ \frac{1}{12} & -\frac{1}{4} \end{pmatrix},$$

and the Jacobian matrix at the equilibrium point $(\frac{1}{4},\frac{1}{12})$ is

$$\mathbf{J}_2=\begin{pmatrix} -\frac{1}{8} & -\frac{1}{8} \\ \frac{1}{12} & -\frac{1}{4} \end{pmatrix}.$$

Since $\mathbf{J}_1$ has one positive eigenvalue $(\frac{1}{6})$ and one negative eigenvalue $(-\frac{1}{4})$, we conclude from Theorem 10.2.3 that $(0,0)$ is a saddle point and thus unstable. Since $\mathbf{J}_2$ has trace $\tau=-\frac{3}{8}<0$ and determinant $\Delta=\frac{1}{24}>0$ with $\tau^2-4\Delta=\frac{9}{64}-\frac{1}{6}<0$, Table 8.2 from Section 8.9 together with Theorem 10.2.3 tell us that $(\frac{1}{4},\frac{1}{12})$ is a stable spiral point. Figs. 10.52 and 10.53 show the nullcline-and-arrow diagram and a phase portrait. Trajectories approach the equilibrium point $(\frac{1}{4},\frac{1}{12})$, and long term we expect $S(t)\to\frac{1}{4}$ and $Q(t)\to\frac{1}{12}$.

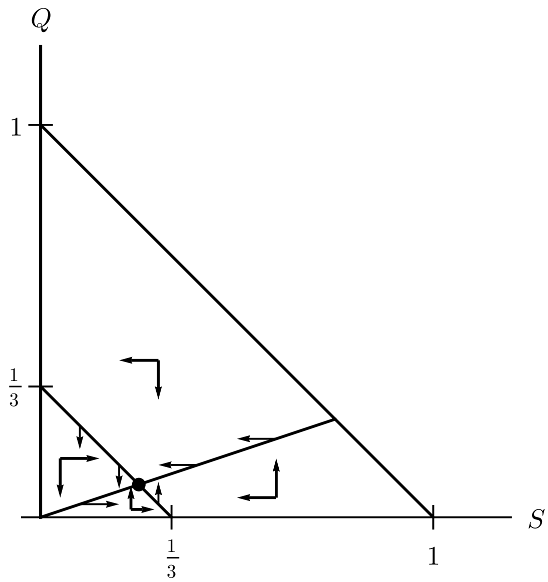


Figure 10.52. Nullcline-and-arrow diagram.

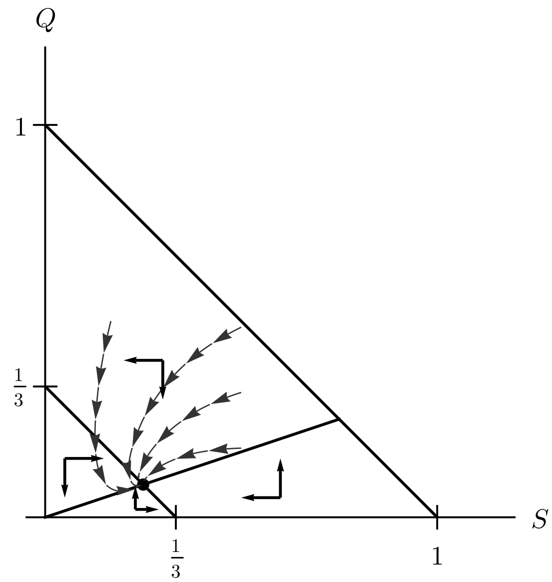
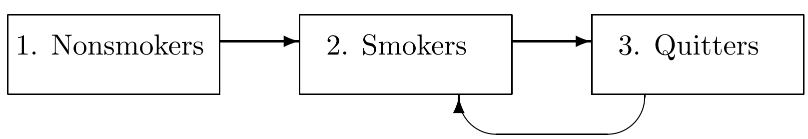


Figure 10.53. A phase portrait.

Allowing for relapse. Let's make this model more realistic. Smokers who have quit may very well start smoking again; this adds the arrow from compartment 3 back to compartment 2 in the diagram below.



We assume that the rate at which quitters relapse is jointly proportional to z, the number of students in compartment 3, and y/N, the proportion of smokers in the school population. So we modify equations (10.64)–(10.66) to

$$\frac{dx}{dt} = gN - bx\frac{y}{N} - gx, \quad \frac{dy}{dt} = bx\frac{y}{N} - (g+c)y + rz\frac{y}{N}, \quad \frac{dz}{dt} = cy - gz - rz\frac{y}{N}. \tag{10.69}$$

As before, we focus on the second two equations (substituting $x = N - y - z$) and make the change of variable $S = y/N$, $Q = z/N$ to obtain (see Exercise 8)

$$\frac{dS}{dt} = bS(1 - S - Q) - (g+c)S + rQS, \quad \frac{dQ}{dt} = cS - gQ - rQS. \tag{10.70}$$

The Jacobian matrix for this new system is

$$\begin{pmatrix} b - 2bS - bQ - (g+c) + rQ & -bS + rS \\ c - rQ & -g - rS \end{pmatrix}.$$

Analyzing this model with values for the parameters. As before, we will continue our analysis using specific values for the parameters. Now we will use the values $b = 0.62$, $g = 0.25$,

$c = 0.06$, and $r = 1.4$.[12] With these values the S-nullclines are the pair of lines

$$S = 0 \quad \text{and} \quad Q = \frac{0.62}{0.78}S - \frac{0.31}{0.78}$$

and the Q-nullcline is the curve with equation

$$Q = \frac{0.06S}{0.25 + 1.4S}.$$

This is a curve passing through $(0, 0)$ which is increasing and concave down for S in the relevant range $0 \leq S \leq 1$. These nullclines are shown separately in Figs. 10.54–10.55. (Note the scale on the Q-axis in Fig. 10.55.) In Exercise 11 you are asked to show the appropriate left/right and up/down arrows to complete this nullcline-and-arrow diagram and to finish the analysis of the phase portrait for these particular parameters.

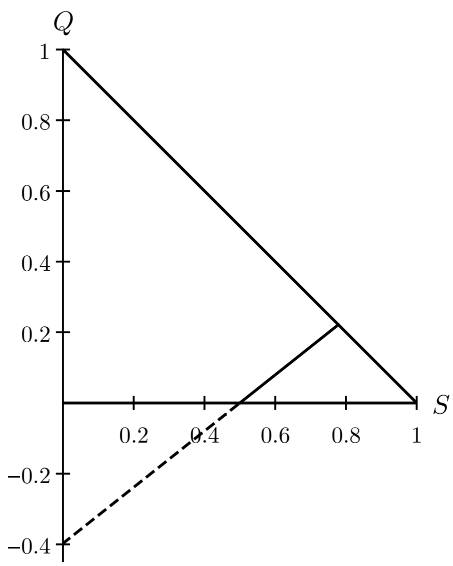


Figure 10.54. The S-nullcline in the biorelevant triangle.

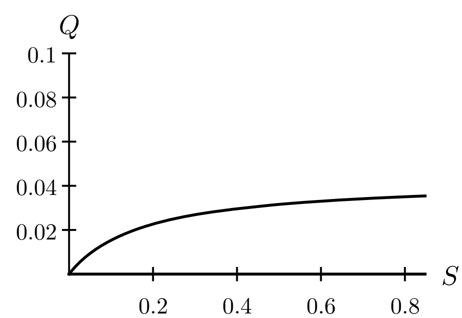


Figure 10.55. The Q-nullcline in a portion of the biorelevant triangle.

10.5.1. Exercises.

1. Suppose that in Example 10.5.1 we assume that $\beta K < \gamma + \mu$, so that the only biologically relevant equilibrium point is $(K, 0)$.
   (a) Show that this equilibrium point is a stable node.
   (b) Give the nullcline-and-arrow sketch for this case, showing several trajectories.
   (c) What happens to $S(t)$ and $I(t)$ as $t \to \infty$?
2. Use the following estimates for the basic reproduction number for the listed diseases (all are diseases where recovery confers immunity) to estimate what percentage of newborns need to be successfully immunized to prevent the disease from becoming endemic.

[12]The values of b and r are taken from *Mathematical models for the dynamics of tobacco use, recovery, and relapse* by C. Castillo-Garsow, G. Jordan-Salivia, and A. Rodriguez-Herrera, Biometrics Unit Technical Reports, Number BU-1505-M, Cornell University, Ithaca, 1997.

Table 10.3. Comparing R_0.	
Disease	Basic reproduction number R_0
measles	12–18
mumps	5–7
diphtheria	6–7
pertussis	12–17
smallpox	5–7

Only smallpox has been eliminated on a worldwide basis, by intensive vaccination efforts. The last known case was in 1977 in Somalia. Now it exists only in laboratories, and routine smallpox vaccination is no longer done. The deliberate release of smallpox as a terrorist act has been a recent concern, especially since 9/11/2001. Mathematical models are an important tool in planning for a response to such a bioterrorism attack. One recent such model assumes the release of smallpox in the New York City subway.[13] While the model is more complicated than that considered in Example 10.5.1 (in particular it separates "subway users" from "nonsubway users", and it considers behavioral changes people might make after such an attack), many of the basic features of our model are still present.

3. For endemic diseases that confer immunity, there is a rule of thumb for estimating the basic reproduction number R_0:

$$R_0 \approx 1 + \frac{L}{A}$$

where L is the average life span and A is the average age of contracting the disease. Suppose that in 1955 the average life span in the US was 70 and the average age of contracting polio was 17.9. What percentage of the population would have to be vaccinated for herd immunity?

4. This problem outlines Daniel Bernoulli's work on smallpox in 1760. We start with a group of people all born at the same time $t = 0$.
   - Let $x(t)$ denote the number of this original group who are still alive t years later.
   - Let $y(t)$ be the number who are still alive at time t and have not yet had smallpox.

   The y-population is thus the susceptibles who are alive at time t. There are two "rate out" terms for the y-population, corresponding to the fact that some members contract smallpox and others die from other causes. Bernoulli assumes that susceptibles contract smallpox at a rate proportional to the y-population and deaths occur from nonsmallpox causes at a rate which depends on time t but is the same for both the y- and x-populations. These assumptions give the equation

$$\frac{dy}{dt} = -\beta y - d(t)y \tag{10.71}$$

   where β is a positive constant. The x-population changes for two reasons: Some people die from smallpox, and some die from other causes. The "die from other causes" factor gives rise to a term $-d(t)x$ in the differential equation for $\frac{dx}{dt}$. The people who die from smallpox are some fraction δ of those who contract the disease. Thus the differential equation for the x-population is

$$\frac{dx}{dt} = -\delta\beta y - d(t)x. \tag{10.72}$$

[13] G. Chowell, A. Cintron-Arias, S. Del Valle, F. Sanchez, B. Song, J. Hyman, H. Hethcote, and C. Castillo-Chavez, *Mathematical applications associated with the deliberate release of infectious agents*, in Mathematical Studies on Human Disease Dynamics, Contemporary Mathematics, Vol. 410, A. Gumel, Editor-in-Chief, American Mathematical Society, 2006, pp. 51–71.

Since t appears explicitly in the equations for $\frac{dx}{dt}$ and $\frac{dy}{dt}$ and we don't know what the function $d(t)$ is, we will need to do something clever to solve the system (10.71)–(10.72).

(a) Set $z = y/x$ and show that $\frac{dz}{dt} = -\beta z + \delta\beta z^2$.

(b) Solve the equation in (a) for z, with the initial condition $z(0) = 1$. The rationale for this initial condition is that at a very young age, no survivors have smallpox, since it is almost always fatal to infants. So $z(t) = \frac{y(t)}{x(t)} = 1$ as $t \to 0^+$.

(c) In your answer to (b), use the values $\beta = \delta = \frac{1}{8}$, estimated by Bernoulli, and compute $z(10)$ and $z(20)$. What percentage of 10-year-olds have not had smallpox? What percentage of 20-year-olds have not?

5. (a) In Example 10.5.2, suppose t is measured in days. What are the units on $a_1, b_1, a_2, b_2, \rho_1$, and ρ_2?

(b) Sketch the graph of

$$y = \frac{Wa_2x}{b_2 + a_2x}$$

showing only the portion that lies in the rectangle $0 \le x \le M, 0 \le y \le W$.

(c) For the functions

$$F_1(x) = \frac{b_1x}{a_1(M-x)} \quad \text{and} \quad F_2(x) = \frac{Wa_2x}{b_2 + a_2x}$$

compute $F_1'(0)$ and $F_2'(0)$. Under what condition is $F_1'(0) > F_2'(0)$? How does this help explain how to fit together the two pictures in Figs 10.47 and 10.48?

(d) Give the nullcline-and-arrow diagram for equation (10.59) under the assumption $MWa_1a_2 - b_1b_2 < 0$. Predict the course of the disease in this case.

6. Suppose we model the spread of gonorrhea in a population of 1,000 sexually active women and 1,000 sexually active men by the system in equation (10.59), with $b_1 = 1/50$ days and $b_2 = 1/10$ days (this corresponds to an average infective period of 50 days for women and 10 days for men). Also suppose that $a_1 = 1/5{,}000$ and $a_2 = 1/25{,}000$.

(a) Find the equilibrium point that has both coordinates positive. What are the total number of infected persons (men + women) corresponding to this equilibrium solution?

(b) Suppose there is a public health policy that would halve the number of at-risk women. What is the new first quadrant equilibrium point, and is the total number of infected persons corresponding to this equilibrium?

(c) Next suppose there is a different public health policy that would instead halve the number of at-risk men. Find the equilibrium point in the first quadrant and the total number of infected persons corresponding to this equilibrium.

(d) Of the two public health policies just discussed, which is more effective in reducing the number of infected individuals at the equilibrium solution?

7. In equation (10.59), show that the substitutions $X = \frac{x}{M}$ and $Y = \frac{y}{W}$ lead to the system

$$\frac{dX}{dt} = a_1(1-X)WY - b_1X \quad \text{and} \quad \frac{dY}{dt} = a_2(1-Y)MX - b_2Y. \tag{10.73}$$

Notice that X represents the proportion of the male population that is infected and Y represents the proportion of the female population that is infected.

8. Show the details of obtaining the equations (10.70) from the second two equations in (10.69).

9. Show that for the system in equation (10.67), the equilibrium point $(0,0)$ is a saddle point if $b > g + c$ and a stable node if $b < g + c$.

10. In the model for "smoking with relapse" do you think it is more likely for a "quitter" to start smoking again or for a nonsmoker to start smoking? Based on your answer, would you have $b > r$ or $b < r$?
11. In this problem you will further analyze the smoking with relapse model in equation (10.70) with the values $b = 0.62, g = 0.25, c = 0.06$, and $r = 1.4$.
   (a) Using Figs. 10.54–10.55 give the nullcline-and-arrow diagram in the biologically relevant region $S \geq 0$, $Q \geq 0$, $S + Q \leq 1$.
   (b) Find (approximately) all biorelevant equilibrium points and classify them according to type and stability.
   (c) Sketch a phase portrait using (a) and (b). What do you predict for the long-term behavior of $S(t)$ and $Q(t)$?
12. Consider the initial value problem consisting of the system in equation (10.70), with the values for b, g, c, and r as given in the preceding exercise and initial condition $S(0) = 0.2$, $Q(0) = 0.01$.
   (a) Sketch the solution curve of this initial value problem based on your nullcline-and-arrow diagram from Exercise 11.
   (b) (CAS) Using a computer algebra system, solve this initial value problem numerically, and plot the corresponding trajectory over the time interval $[0, 18]$.
   (c) Using your numerical solution, compute $S(6)$, $S(12)$, and $S(18)$, as well as $Q(6)$, $Q(12)$, and $Q(18)$.
13. Suppose in Example 10.5.3 we model relapse in a different way, by assuming that quitters become smokers again at a rate proportional to the number of people who have quit.
   (a) What system of equations does this assumption give?
   (b) Convert your model in (a) to a planar system for $S = y/N$ and $Q = z/N$. When is the equilibrium point $S = 0$, $Q = 0$ stable? Unstable?
14. Norovirus ("stomach flu") is a common and unpleasant illness caused by a virus.[14] College students, especially those living in dorms, are frequent victims, as are cruise ship passengers. In this problem we will look at a model for the spread of this disease which divides the population into 5 compartments: (S) susceptibles, (E) exposed (these are people who do not yet show symptoms but soon will), (I) infected individuals showing symptoms, (A) asymptotic infected people (these are people who do not have symptoms but are still shedding the virus), and (R), the recovered class of people with temporary immunity to norovirus. Movement between these compartments follows these rules:
   (1) All births go into the susceptible category with constant births of B per day.
   (2) Susceptibles move into the exposed compartment at a rate proportional to the product of the number of suspectibles ($S(t)$) and the number of symptomatic infectives ($I(t)$), with proportionality constant β. Notice this means we are assuming only symptomatic infected people are infectious, not the "presymptomatic" exposed or "postsymptomatic" asymptomatic infected people.
   (3) Exposed individuals move into the symptomatic infected compartment at a rate proportional to the number of exposed ($E(t)$), with proportionality constant $1/\mu_s$, where μ_s is the average length of the incubation period in days.
   (4) Symptomatic individuals move into the asymptomatic infected compartment at a rate proportional to the number of (symptomatic) infecteds ($I(t)$), with proportionality constant $1/\mu_a$, where μ_a is the average length of symptoms in days.
   (5) Asymptomatic infected individuals move into the temporarily immune category (R compartment) at a rate proportional to the number of asymptomatic infected people ($A(t)$),

[14]The model discussed in this exercise comes from the article *Duration of Immunity to Norovirus Gastroenteritis* by K. Simmons, M. Gambhir, J. Leon, and B. Lopman, Emerging Infectious Diseases, Aug. 2013, 19(8), 1260–1267.

with proportionality constant $1/\rho$, where ρ is average number of days a person sheds the virus without showing symptoms.

(6) People leave the R compartment in one of two ways: Either they lose immunity entirely, moving into the S (susceptible) compartment, or they "lose immunity to infection without losing immunity to disease"; this means they move back into the A compartment, shedding the virus but not showing any symptoms. The first case occurs at a rate proportional to R with proportionality constant $1/\theta$ where θ is the average duration of temporary immunity. The second case occurs at a rate jointly proportional to $R(t)$ and $I(t)$, with proportionality constant β as in (2). Moving from R to A has the effect of temporarily boosting immunity again, and these individuals eventually move back into the R compartment.

(7) Deaths occur from all compartments, at a rate proportional to the compartment size with proportionality constant δ, which we assume is the same for all compartments.

(a) Draw a compartment diagram, labeled S, E, I, A, and R, with arrows showing what transfers we have between compartments. You may include a sixth compartment, D, for dead, or you may omit this.

(b) Using your answer to (a), give a system of differential equations for $\frac{dS}{dt}$, $\frac{dE}{dt}$, $\frac{dI}{dt}$, $\frac{dA}{dt}$, and $\frac{dR}{dt}$, using the following facts: The average incubation period for norovirus is 1 day, the average duration of symptoms is 2 days, the average duration of asymptotic virus shedding is 10 days, and the average duration of temporary immunity is 5 *years*. Your equations will include the parameter β from (2) above; we do not give a numerical value for this. Whether or not you included a "dead" compartment in (a), your equations should include births of B people per day and deaths as described in (7) above. We do not give a numerical value for δ.

(c) Another model allows for susceptibles to become infected by people in the exposed and asymptomatic compartments, but at a lower rate than by symptomatic infected persons. This changes the "rate in" terms for $\frac{dE}{dt}$ to be $\beta_1 S(t)E(t) + \beta_2 S(t)I(t) + \beta_3 S(t)A(t)$ with β_1 and β_3 to be smaller than β_2. What are the corresponding changes in $\frac{dS}{dt}$? In $\frac{dR}{dt}$? In $\frac{dA}{dt}$?

10.6. Hamiltonians, gradient systems, and Lyapunov functions

Conserved quantities. In Section 4.8 we used the second-order equation $my'' + ky = 0$ to model an undamped mass-spring system, where m is the mass, k is the spring constant, and $y(t)$ is the displacement of the mass from equilibrium at time t. We can convert this to a first-order system, with dependent variables y and v, using the substitution $y' = v$:

$$y' = v, \qquad v' = -\frac{k}{m}y. \tag{10.74}$$

This linear system has a center at $(0, 0)$. The orbits are ellipses in the yv-plane with equations determined by

$$\frac{dv}{dy} = \frac{dv/dt}{dy/dt} = -\frac{ky}{mv}.$$

Separating variables in this first-order equation gives $\frac{1}{2}mv^2 + \frac{1}{2}ky^2 = C$; for $C > 0$ these describe ellipses as shown in Fig. 10.56.

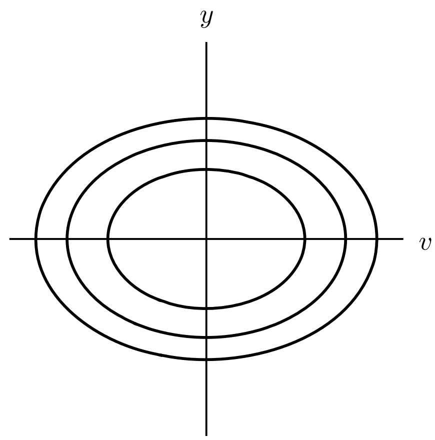


Figure 10.56. Ellipses $\frac{1}{2}mv^2 + \frac{1}{2}ky^2 = C$, $C > 0$.

The first term, $\frac{1}{2}mv^2$, is the kinetic energy of the system, while the second term, $\frac{1}{2}ky^2$, is the potential energy stored in the stretched or compressed spring. Their sum is the total energy, and we have just shown that the total energy is constant along any solution curve. We say that the total energy $E(y, v) = \frac{1}{2}mv^2 + \frac{1}{2}ky^2$ is a **conserved quantity**, and we call (10.74) a **conservative system**. In general, we have the following definition.

Definition 10.6.1. A conserved quantity for an autonomous system

$$\frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y) \tag{10.75}$$

on an open rectangle $\mathcal{R}$ is a differentiable function $E = E(x, y)$ such that E is constant along any trajectory of the system that lies in $\mathcal{R}$.

The Chain Rule shows that if $E(x, y) = K$ is an (implicit) solution of

$$\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)}$$

on a open rectangle $\mathcal{R}$, then E is a conserved quantity for the system (10.75) on $\mathcal{R}$. We will be interested in conserved quantities that are not constant on any nonempty open disks (or rectangles) in their domains.

Let's also think about a conserved quantity with reference to a three-dimensional picture. In the mass-spring system in (10.74) notice that the graph of the total energy $z = E(y, v) = \frac{1}{2}mv^2 + \frac{1}{2}ky^2$ is a paraboloid surface in 3-space as shown in Fig. 10.57. A **level curve** for the function $E(y, v)$ is the set of points in the yv-plane along which $E(y, v)$ is constant: $\frac{1}{2}mv^2 + \frac{1}{2}ky^2 = C$. These are our elliptical orbits from Fig. 10.56. Every point on the paraboloid which lies directly above the level curve $\frac{1}{2}mv^2 + \frac{1}{2}ky^2 = C$ is at height C, reflecting the total energy of the system at any point (v, y) on the curve.

By contrast, the *damped* mass-spring system with equation $my'' + cy' + ky = 0$ converts to the linear system

$$y' = v, \quad v' = -\frac{c}{m}v - \frac{k}{m}y,$$

which has a spiral point at the origin. If you revisit Exercise 21 in Section 4.1, you can see that what you showed in part (b) of that problem is that the total energy $E(y, v) = \frac{1}{2}mv^2 + \frac{1}{2}ky^2$ is not conserved for this system. The curve that lies on the surface of the paraboloid $z = \frac{1}{2}mv^2 + \frac{1}{2}ky^2$ and directly above a trajectory in the phase plane spirals down to $(0, 0, 0)$; see Fig. 10.58.

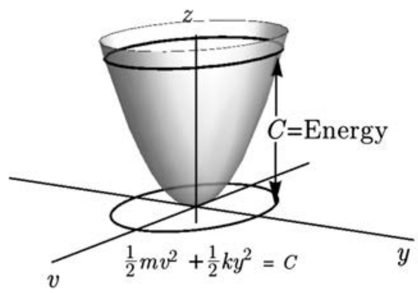


Figure 10.57. Graph of $z = \frac{1}{2}mv^2 + \frac{1}{2}ky^2$ and the energy C level curve.

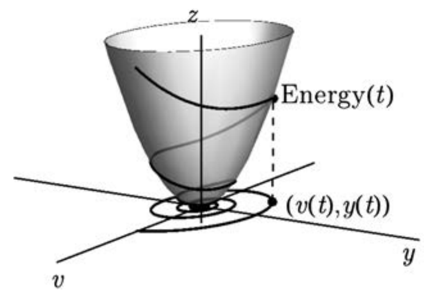


Figure 10.58. Energy is not conserved in the damped mass-spring system.

Example 10.6.2. We revisit the Lotka-Volterra system

$$\frac{dx}{dt} = ax - bxy, \qquad \frac{dy}{dt} = -dy + cxy$$

from Section 10.4. This has an equilibrium point $P = (\frac{d}{c}, \frac{a}{b})$. The linear approximation at P has a center; this gives no information on the stability of P. We saw in (10.4) that the orbits are described by the equations $a \ln y - by + d \ln x - cx = K$, for constant K, and we gave an ad hoc argument that these are closed curves encircling P.

In our new terminology, we say that the function $E(x, y) = a \ln y - by + d \ln x - cx$ is a conserved quantity for the Lotka-Volterra system on the open first quadrant $\{(x, y) : 0 < x < \infty, 0 < y < \infty\}$. We claim that the equilibrium point P cannot be asymptotically stable. If it were, then every orbit $(x(t), y(t))$ starting close enough to P would approach P as $t \to \infty$. By continuity of $E(x, y)$ (in the first quadrant), this would say that

$$E(x(t), y(t)) \to E\left(\frac{d}{c}, \frac{a}{b}\right) \quad \text{as } t \to \infty. \tag{10.76}$$

But $E(x(t), y(t))$ is constant; this is what it means for E to be a conserved quantity, and equation (10.76) says the constant value must be $E(\frac{d}{c}, \frac{a}{b})$. This applies to any orbit that starts sufficiently close to P and thus gives the conclusion that $E(x, y)$ is constant in some entire disk centered at P, which is not the case (note the only point at which both $\frac{\partial E}{\partial x}$ and $\frac{\partial E}{\partial y}$ are zero is $(d/c, a/b)$). So P cannot be asymptotically stable.

This argument says something quite general about any autonomous system

$$\frac{dx}{dt} = f(x, y), \qquad \frac{dy}{dt} = g(x, y): \tag{10.77}$$

If there is a conserved quantity $E = E(x, y)$ for (10.77) *on some open rectangle $\mathcal{R}$ of the plane such that E is not constant in $\mathcal{R}$, then no equilibrium point of* (10.77) *in $\mathcal{R}$ is asymptotically stable.*

Hamiltonian systems. Suppose that we start with a function $H(x, y)$ all of whose first- and second-order partial derivatives exist and are continuous functions; we say that $H(x, y)$ is **twice continuously differentiable**. Form the autonomous system

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}, \qquad \frac{dy}{dt} = -\frac{\partial H}{\partial x}. \tag{10.78}$$

Such a system is called a **Hamiltonian system**, and the function $H(x, y)$ is called a **Hamiltonian function**. The trajectories in the phase portrait for this system satisfy the first-order differential equation

$$\frac{dy}{dx} = \frac{-\frac{\partial H}{\partial x}}{\frac{\partial H}{\partial y}}.$$

Rewrite this as

$$\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y}\frac{dy}{dx} = 0. \tag{10.79}$$

This an exact equation, whose solutions are given implicitly by $H(x, y) = C$ for arbitrary constants C (see Section 2.8). In other words:

> Each trajectory for the system (10.78) is contained in a level curve $H(x, y) = C$ of the Hamiltonian function H, and $H(x, y)$ is a conserved quantity for the system.

Can we recognize when an autonomous system

$$\frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y) \tag{10.80}$$

is a Hamiltonian system? Is there a twice continuously differentiable function $H(x, y)$ with

$$\frac{\partial H}{\partial y} = f(x, y) \quad \text{and} \quad \frac{\partial H}{\partial x} = -g(x, y)?$$

This is a variant of a question we have considered before, in Section 2.8. If such a function $H(x, y)$ exists, then

$$\frac{\partial^2 H}{\partial x \partial y} = \frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial^2 H}{\partial y \partial x} = -\frac{\partial g}{\partial y}.$$

Since the mixed second-order partials of H must be the same, this forces

$$\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y} \tag{10.81}$$

if (10.80) is Hamiltonian. In other words, the relationship (10.81) is a *necessary* condition for the system to be Hamiltonian. The next theorem says that the converse holds as well, when (10.81) holds in the plane $\mathbb{R}^2$ or in some open rectangle $\mathcal{R}$ in $\mathbb{R}^2$.

Theorem 10.6.3. *Given an autonomous system*

$$\frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y),$$

where $f(x, y)$ and $g(x, y)$ are continuously differentiable functions in an open rectangle $\mathcal{R}$, the system is Hamiltonian if and only if

$$\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y}$$

for all (x, y) in $\mathcal{R}$.

Example 10.6.4. Show that the system

$$\frac{dx}{dt} = 2y + 3 + x\sin y, \quad \frac{dy}{dt} = x^2 + \cos y$$

is Hamiltonian, and find a Hamiltonian function. Since

$$\frac{\partial}{\partial x}(2y + 3 + x\sin y) = \sin y \quad \text{and} \quad \frac{\partial}{\partial y}(x^2 + \cos y) = -\sin y,$$

Theorem 10.6.3 applies. A Hamiltonian function $H(x,y)$ must satisfy

$$\frac{\partial H}{\partial y} = 2y + 3 + x\sin y \tag{10.82}$$

and

$$\frac{\partial H}{\partial x} = -x^2 - \cos y. \tag{10.83}$$

From (10.82) we see that

$$H(x,y) = y^2 + 3y - x\cos y + \varphi(x)$$

for some function $\varphi(x)$. Then

$$\frac{\partial H}{\partial x} = -\cos y + \varphi'(x)$$

and we want this to equal $-x^2 - \cos y$. Choosing $\varphi(x) = -\frac{1}{3}x^3$ we see that

$$H(x,y) = y^2 + 3y - x\cos y - \frac{1}{3}x^3$$

is a Hamiltonian function.

10.6.1. Lyapunov functions. For a conservative system, there is a nonconstant function which is constant on trajectories. In this section, we will focus on systems which are not necessarily conservative, but for which we can find a function which *decreases* along the trajectories. This will provide a method for determining the stability of some equilibrium points whose stability cannot be determined from linear approximation (i.e., Theorem 10.2.3 does not apply).

Consider an autonomous system

$$\frac{dx}{dt} = f(x,y), \qquad \frac{dy}{dt} = g(x,y) \tag{10.84}$$

with an isolated equilibrium point at (x_e, y_e). Suppose we have a continuously differentiable function $E(x,y)$ in a neighborhood U of (x_e,y_e) with $E(x_e,y_e) = 0$. By a neighborhood of (x_e,y_e) we mean some disk centered at (x_e,y_e) with positive radius; that is, all points whose distance to (x_e,y_e) is less than this radius. If $x = x(t), y = y(t)$ is a trajectory of the system (10.84), then $E(x(t),y(t))$ gives the value of E along this trajectory at time t. So we are thinking now of $E(x(t),y(t))$ as a function of the single variable t and we write

$$\frac{dE}{dt} = \frac{d}{dt}\left[E(x(t),y(t))\right]$$

for the time rate of change of E along the trajectory at the point $(x(t),y(t))$. By the Multivariable Chain Rule

$$\begin{aligned}
\frac{dE}{dt} &= \frac{d}{dt}\left[E(x(t),y(t))\right] \\
&= \frac{\partial E}{\partial x}(x(t),y(t))\frac{dx}{dt} + \frac{\partial E}{\partial y}(x(t),y(t))\frac{dy}{dt} \\
&= \frac{\partial E}{\partial x}(x(t),y(t))f(x(t),y(t)) + \frac{\partial E}{\partial y}(x(t),y(t))g(x(t),y(t)),
\end{aligned} \tag{10.85}$$

where we have used (10.84) to obtain the last equality. This leads us to *define*:

$$\boxed{\dot{E}(x,y) = \frac{\partial E}{\partial x}(x,y)f(x,y) + \frac{\partial E}{\partial y}(x,y)g(x,y),} \tag{10.86}$$

which is purely a function of x and y, with no reference to t. Even so, $\dot{E}$ is sometimes called the **time derivative** of E, or the **derivative of E along trajectories**, because by equation (10.85)

$$\frac{dE}{dt} = \dot{E}(x(t), y(t)).$$

It tells you how the function $E(x, y)$ is changing along the solution curves of (10.84):

- If $\dot{E}(x, y) > 0$, then E is increasing along the trajectory that passes through the point (x, y).
- If $\dot{E}(x, y) < 0$, then E is decreasing along the trajectory that passes through the point (x, y).

Notice that we can compute $\dot{E}(x, y)$ (using equation (10.86)) without actually knowing the solution curve passing through (x, y).

These observations about the time derivative $\dot{E}(x, y)$ lead to the important notion of a Lyapunov function, which we define next.

Definition 10.6.5. A function $E = E(x, y)$ is a **Lyapunov function** of the system (10.84) corresponding to its equilibrium point (x_e, y_e) provided E is continuously differentiable on a neighborhood U of (x_e, y_e) and satisfies

(i) $E(x_e, y_e) = 0$,

(ii) $E(x, y) > 0$ for all (x, y) in U for which $(x, y) \neq (x_e, y_e)$, and

(iii) $\dot{E}(x, y) \leq 0$ for all (x, y) in U, where $\dot{E}$ is given by (10.86).

Example 10.6.6. For the system

$$y' = f(y, v) = v, \qquad v' = g(y, v) = -\frac{c}{m}v - \frac{k}{m}y,$$

arising from the damped mass-spring equation $my'' + cy' + ky = 0$, set

$$E(y, v) = \frac{1}{2}mv^2 + \frac{1}{2}ky^2.$$

At the equilibrium point $(0, 0)$ we have $E(0, 0) = 0$ and $E(x, y) > 0$ for all $(x, y) \neq (0, 0)$. Computing,

$$\dot{E}(y, v) = \frac{\partial E}{\partial y}f(y, v) + \frac{\partial E}{\partial v}g(y, v) = kyv + mv\left(-\frac{c}{m}v - \frac{k}{m}y\right) = -cv^2 \leq 0.$$

Thus $E(y, v)$ is a Lyapunov function for this system (we may choose $U = \mathbb{R}^2$ for our neighborhood of $(0, 0)$).

The next result shows that finding a Lyapunov function can help determine the stability of the equilibrium point. It is helpful when the eigenvalues of the linear approximation Jacobian matrix are purely imaginary or one is 0.

Theorem 10.6.7. *Suppose (x_e, y_e) is an isolated equilibrium point of the autonomous system* (10.84), *and suppose further that we can find a corresponding Lyapunov function, that is, a continuously differentiable function $E(x, y)$ with the following properties:*

(a) *$E(x_e, y_e) = 0$ and $E(x, y) > 0$ for all $(x, y) \neq (x_e, y_e)$ in some neighborhood U of (x_e, y_e).*

(b) *$\dot{E}(x, y) \leq 0$ for all (x, y) in U.*

Then (x_e, y_e) is a stable equilibrium. If we have the strict inequality $\dot{E}(x, y) < 0$ for all $(x, y) \neq (x_e, y_e)$ in U, then (x_e, y_e) is asymptotically stable.

When E is as in (a) and $\dot{E}$ is strictly less than 0 at all points $(x, y) \neq (x_e, y_e)$ in U, we say E is a **strong Lyapunov function**.

In Example 10.6.6 we have $\dot{E} = -cv^2 \leq 0$ in $\mathbb{R}^2$. According to Theorem 10.6.7, $(0, 0)$ is stable, but we cannot use this theorem to make the stronger conclusion (which we already know!) that $(0, 0)$ is asymptotically stable, since $\dot{E} = 0$ whenever $v = 0$. However, perhaps a different Lyapunov function can be found which will give the stronger conclusion; see Exercise 19.

Theorem 10.6.7 doesn't tell us anything about how to *find* a Lyapunov function. For systems that arise from mechanical (or electrical) models with a notion of energy, total energy can often serve as a Lyapunov function. In the case of the Lotka-Volterra model, we can modify the conserved quantity E identified in Example 10.6.2 to obtain a Lyapunov function for the model corresponding to its first-quadrant equilibrium point. In other cases, we are forced to rely on trial and error. For an equilibrium point at $(0, 0)$, functions of the form $ax^2 + bxy + cy^2$ are often considered as possible Lyapunov functions; such functions are said to be of *quadratic form*[15].

Example 10.6.8. The system

$$\frac{dx}{dt} = -y - x^3 - xy^2, \qquad \frac{dy}{dt} = x - y^3 - x^2 y$$

has $(0, 0)$ as its only equilibrium point. The function $E(x, y) = x^2 + y^2$ is a strong Lyapunov function, since $E(0, 0) = 0$, $E(x, y) > 0$ when $(x, y) \neq (0, 0)$ and

$$\begin{aligned} \dot{E}(x, y) &= \frac{\partial E}{\partial x} f(x, y) + \frac{\partial E}{\partial y} g(x, y) \\ &= 2x(-y - x^3 - xy^2) + 2y(x - y^3 - x^2 y) \\ &= -2x^4 - 4x^2 y^2 - 2y^4, \end{aligned}$$

which is 0 at $(0, 0)$ and is strictly less than 0 for all $(x, y) \neq 0$. By Theorem 10.6.7, $(0, 0)$ is an asymptotically stable equilibrium point. We looked at the same system in Example 10.3.2 of Section 10.3, arriving at the same conclusion by a rather different argument.

A Lyapunov function for the Lotka-Volterra model For the Lotka-Volterra system

$$\frac{dx}{dt} = ax - bxy, \qquad \frac{dy}{dt} = -dy + cxy,$$

set $(x_e, y_e) = (d/c, a/b)$, so that (x_e, y_e) is the (only) first-quadrant equilibrium point of the system. Let

$$E(x, y) = c\left(x - x_e - x_e \ln\left(\frac{x}{x_e}\right)\right) + b\left(y - y_e - y_e \ln\left(\frac{y}{y_e}\right)\right).$$

It's clear that $E(x_e, y_e) = 0$. That $E(x, y) > 0$ for all x, y in the first quadrant for which $(x, y) \neq (x_e, y_e)$ follows from the observation that for every positive number a, the function $f(x) = x - a - a\ln(x/a)$ is strictly decreasing on $(0, a)$, strictly increasing on (a, ∞), and $f(a) = 0$. Finally, the reader should check that $\dot{E}(x, y) = 0$ for all (x, y) in the first quadrant. Thus, E is a Lyapunov function of the Lotka-Volterra system corresponding to its equilibrium point (x_e, y_e). Hence, by Theorem 10.6.7, (x_e, y_e) is stable. From Example 10.6.2 we know that (x_e, y_e) is not asymptotically stable. These observations together with Theorem 10.2.4 tell us that (x_e, y_e) is a center or hybrid center/spiral. The hybrid center/spiral case cannot occur since $f(x, y) = ax - bxy$ and $g(x, y) = -dy + cxy$ are polynomials in the variables x and y (the hybrid center/spiral case is also ruled out by the ad hoc argument in Section 10.4) and (x_e, y_e) must be a center.

[15]More generally, for an equilibrium point (x_e, y_e) one can look for a Lyapunov function of the quadratic form $a(x - x_e)^2 + b(x - x_e)(y - y_e) + c(y - y_e)^2$.

Visualizing Theorem 10.6.7 geometrically. For simplicity we assume that the curves $E(x,y) = k$ encircle the isolated equilibrium point and if $k_1 < k_2$, then the curve $E(x,y) = k_1$ lies inside the curve $E(x,y) = k_2$ (see, for example, Fig. 10.59). Notice that we can write the time derivative $\dot{E}(x,y)$ as the dot product of the vectors

$$\left(\frac{\partial E}{\partial x}, \frac{\partial E}{\partial y}\right) \quad \text{and} \quad (f(x,y), g(x,y)),$$

since

$$\left(\frac{\partial E}{\partial x}, \frac{\partial E}{\partial y}\right) \cdot (f(x,y), g(x,y)) = \frac{\partial E}{\partial x} f(x,y) + \frac{\partial E}{\partial y} g(x,y) = \dot{E}(x,y).$$

The vector

$$\text{grad } E = \left(\frac{\partial E}{\partial x}, \frac{\partial E}{\partial y}\right)$$

is called the gradient vector of the function $E(x,y)$. The gradient is also written as ∇E.

The gradient vector has an important geometric meaning: At any (x,y) where it is nonzero, grad $E(x,y)$ is a vector which is perpendicular to the level curve of E passing through that point; in other words, grad $E(x,y)$ is a **normal vector** to this level curve. It points in the direction in which E is increasing most rapidly. Fig. 10.59 illustrates this for the function $E(x,y) = x^2 + 2y^2$, which has gradient grad $E(x,y) = (2x, 4y)$. The level curve of E passing through the point $(a,b) \neq (0,0)$ is an ellipse with equation $x^2 + 2y^2 = k$, where $k = a^2 + 2b^2$. At the point (a,b), the gradient vector $(2a, 4b)$ is perpendicular to this ellipse and points "outward", in the direction in which E is increasing most rapidly.

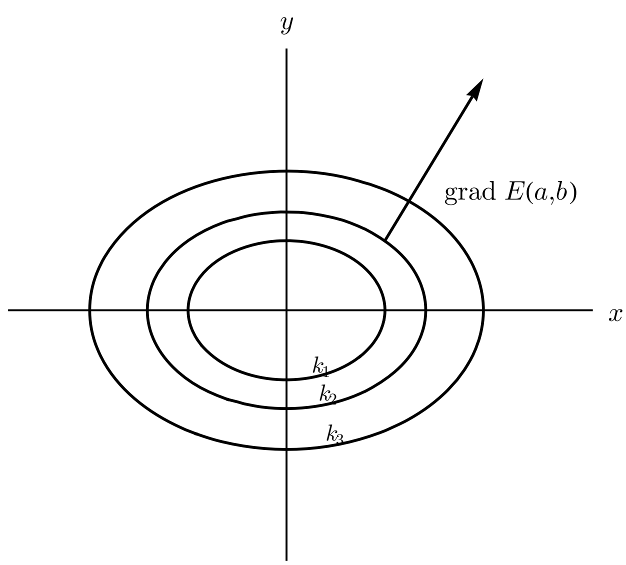


Figure 10.59. $x^2 + 2y^2 = k_j, k_1 < k_2 < k_3$.

Returning to $\dot{E}(x,y)$ as the dot product of grad $E(x,y)$ and $(f(x,y), g(x,y))$, recall that the dot product of two vectors is negative if and only if the angle between the vectors is greater than $\frac{\pi}{2}$, i.e., the angle between them is obtuse.

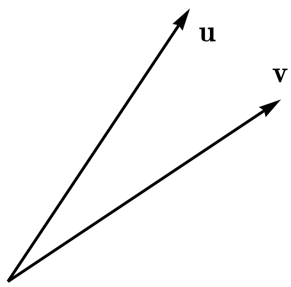


Figure 10.60. $\mathbf{u} \cdot \mathbf{v} > 0$.

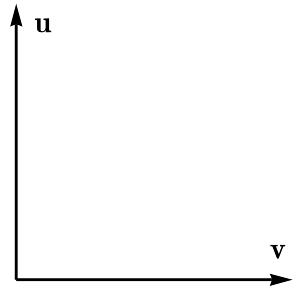


Figure 10.61. $\mathbf{u} \cdot \mathbf{v} = 0$.

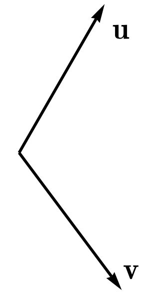


Figure 10.62. $\mathbf{u} \cdot \mathbf{v} < 0$.

Now $(f(x,y), g(x,y))$ is a vector in the vector field for our system. At each nonequilibrium point, it gives the tangent vector to the trajectory passing through that point, and it points along the direction of motion for the trajectory. Suppose we are in a neighborhood of an equilibrium point (x_e, y_e) with $\dot{E}$ negative at points different from (x_e, y_e) in this neighborhood. Pick any point $(x,y) \neq (x_e, y_e)$ and find the trajectory passing through that point and also the level curve $E(x,y) = k$ of E passing through the point. The hypothesis that $\dot{E}(x,y) < 0$ says that the angle between the vectors grad $E(x,y)$ and $(f(x,y), g(x,y)$ is obtuse, i.e., in the range $(\frac{\pi}{2}, \frac{3\pi}{2})$. This tells us the picture looks like Fig. 10.63, rather than like Fig. 10.64. The trajectory is driven *inside* the level curve $E(x,y) = k$ surrounding the equilibrium point by the requirement that $\dot{E}(x,y) < 0$ for (x,y) near (x_e, y_e), as asserted by Theorem 10.6.7.

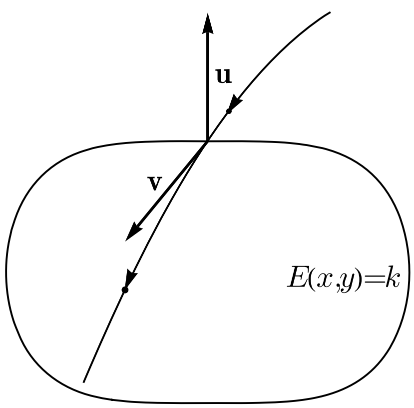


Figure 10.63. Level curve of $E(x,y)$, $\mathbf{u} = \text{grad } E(x,y)$, and $\mathbf{v} = (\frac{dx}{dt}, \frac{dy}{dt})$.

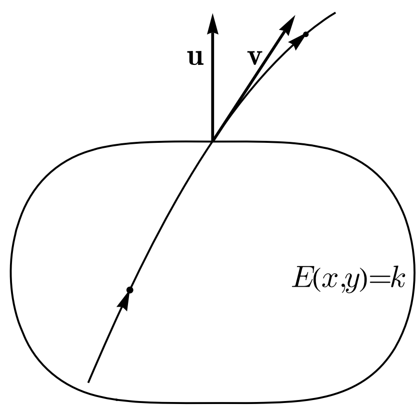


Figure 10.64. Configuration not possible if $\mathbf{u} \cdot \mathbf{v} < 0$.

The values of a Lyapunov function decrease (or at least can't increase) as you move along a trajectory in the phase portrait in the direction of increasing time. So if a trajectory crosses several level curves $E(x,y) = k$ for different values of k as t increases, it must do so in order, from the largest value of k to the smallest. These ideas are illustrated in the next example.

Example 10.6.9. The system

$$\frac{dx}{dt} = -y - x^3, \qquad \frac{dy}{dt} = x - 2y^3 \tag{10.87}$$

has equilibrium point $(0,0)$. The linear approximation near $(0,0)$ has a center. If $E(x,y) = x^2 + y^2$, then

$$\dot{E}(x,y) = 2x(-y - x^3) + 2y(x - 2y^3) = -2x^4 - 4y^4,$$

and $E(x, y)$ is a strong Lyapanov function. Theorem 10.6.7 tells us that $(0, 0)$ is asymptotically stable. The level curves of $E(x, y)$ are circles $x^2 + y^2 = k$ centered at the origin. The larger the value of k, the larger the radius of a circle. Fig. 10.65 shows a trajectory in the phase portrait (satisfying the initial condition $x(0) = 1, y(0) = 1$). Note that it crosses these circles from larger to smaller with increasing time, and, as the trajectory crosses one of the circles, the angle between the tangent vector to the trajectory and the "outward pointing" normal vector to the circle at the point of intersection is more than 90° (as shown at the point P).

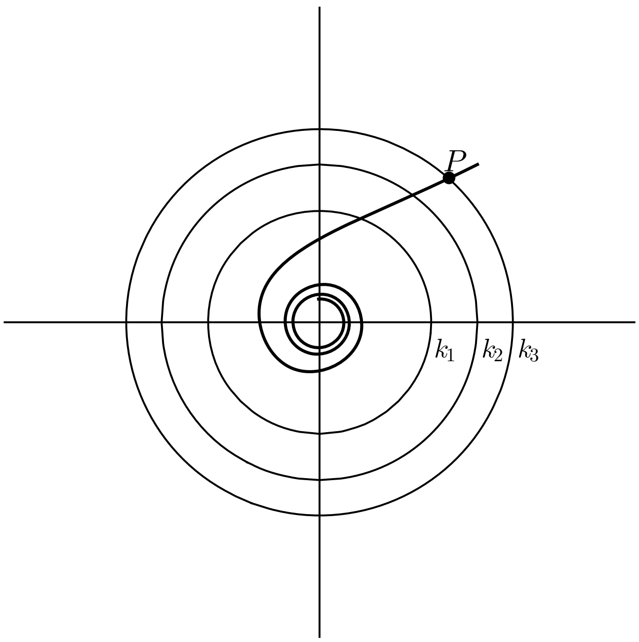


Figure 10.65. A trajectory for (10.87) and level curves $x^2 + y^2 = k_j$, $k_1 < k_2 < k_3$.

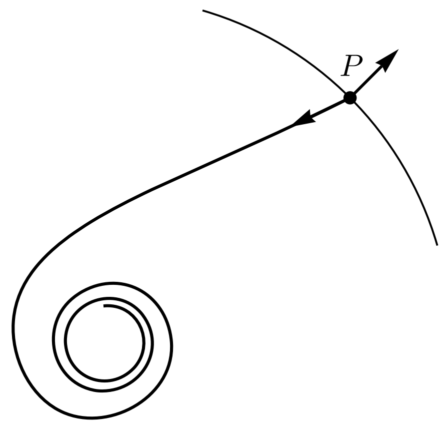


Figure 10.66. Detail showing tangent vector to trajectory and outward normal vector to circle at point P.

There is some terminology that goes along with the hypotheses in Theorem 10.6.7.

- If $E(x_e, y_e) = 0$ and $E(x, y) > 0$ on a neighborhood U of (x_e, y_e), except at (x_e, y_e), then E is said to be **positive definite** on U. Notice that these conditions say that the graph of the surface $z = E(x, y)$ over U will be roughly like a bowl turned right-side up, with its lowest point at $(x_e, y_e, 0)$; see Fig. 10.67

- If $\dot{E}(x, y) \leq 0$ on U, we say that $\dot{E}$ is **negative semidefinite**, and if we have strict inequality $\dot{E}(x, y) < 0$, for all $(x, y) \neq (x_e, y_e)$ in U, we say that $\dot{E}$ is **negative definite**. In the latter case, we see that the curve on the surface of Fig. 10.67 that lies directly above a trajectory in the phase portrait for (10.84) is moving down, towards $(x_e, y_e, 0)$.

In this language, a Lyapunov function for (10.84) at an equilibrium point (x_e, y_e) is a function $E(x, y)$ that is positive definite on a neighborhood U of (x_e, y_e) and for which the time derivative $\dot{E}$ is negative semidefinite. It is a strong Lyapunov function if $\dot{E}$ is negative definite in U.

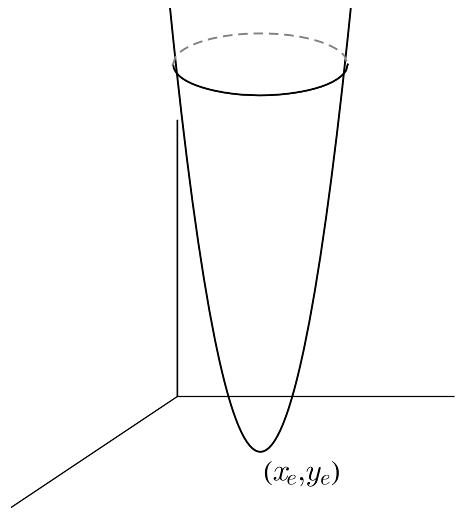


Figure 10.67. The surface $z = E(x, y)$.

To look for a Lyapunov function in the class $ax^2 + bxy + cy^2$ it is helpful to know, as a first step, when a quadratic form is positive definite.

Theorem 10.6.10. *The function $E(x, y) = ax^2 + bxy + cy^2$, having $E(0, 0) = 0$, is positive definite on $\mathbb{R}^2$ if and only if $a > 0$ and $b^2 < 4ac$.*

A proof is outlined in Exercise 15.

10.6.2. Gradient systems.

Recall that the gradient of a differentiable function $F(x, y)$ is

$$\text{grad } F(x, y) = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right).$$

A point (x_0, y_0) at which grad $F(x_0, y_0) = (0, 0)$ is called a **critical point** of F, and other points at which grad $F(x, y) \neq (0, 0)$ are called **regular points**. In $\mathbb{R}^2$, a **gradient system** has the form

$$\frac{dx}{dt} = -\frac{\partial F}{\partial x}, \qquad \frac{dy}{dt} = -\frac{\partial F}{\partial y} \tag{10.88}$$

for some twice continuously differentiable function $F(x, y)$. An equilibrium point for this system is a point (x_e, y_e) such that grad $F(x_e, y_e) = (0, 0)$; in other words the equilibrium points are exactly the critical points of the $F(x, y)$.

If we draw the vector field for the gradient system (10.88), at every regular point of F we are drawing the vector

$$\left(-\frac{\partial F}{\partial x}, -\frac{\partial F}{\partial y}\right) = -\text{grad } F(x, y).$$

Since grad $F(x, y)$, and thus also $-$grad $F(x, y)$, is perpendicular to the level curves of F, this tells us that:

> The trajectories of the gradient system (10.88) are perpendicular to the level curves of $F(x, y)$.

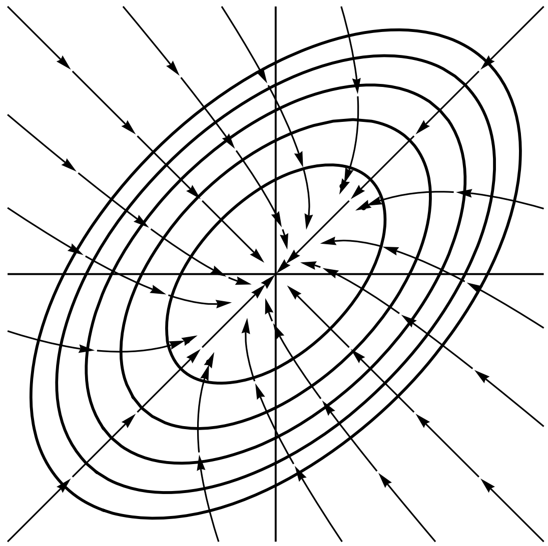


Figure 10.68. Trajectories of the gradient system for $F(x, y) = x^2 - xy + y^2$ are perpendicular to the level curves of F.

By comparison, the trajectories of a Hamiltonian system lie **in** the level curves of the Hamiltonian function.

For the gradient system (10.88), we have

$$\begin{aligned}\dot{F}(x,y) &= \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}\\ &= \frac{\partial F}{\partial x}\left(-\frac{\partial F}{\partial x}\right) + \frac{\partial F}{\partial y}\left(-\frac{\partial F}{\partial y}\right)\\ &= -\left(\frac{\partial F}{\partial x}\right)^2 - \left(\frac{\partial F}{\partial y}\right)^2\end{aligned}$$

which is 0 at the equilibrium points of (10.88) and strictly less than 0 elsewhere. This means that if (x_e, y_e) is an isolated equilibrium point for (10.88) at which F has a strict minimum, then $F(x, y) - F(x_e, y_e)$ is a strong Lyapunov function, and Theorem 10.6.7 tells us that (x_e, y_e) is asymptotically stable.

The Jacobian matrix for the linear approximation at (x_e, y_e) is

$$\begin{pmatrix} -\frac{\partial^2 F}{\partial x^2} & -\frac{\partial^2 F}{\partial y \partial x} \\ -\frac{\partial^2 F}{\partial x \partial y} & -\frac{\partial^2 F}{\partial y^2}\end{pmatrix}$$

where each derivative is evaluated at (x_e, y_e). Because the mixed second-order partial derivatives of F are equal, this matrix has the form

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

and we say it is a **symmetric** matrix (the first column and first row are equal, and the second column and second row are equal). In Exercise 13 you are asked to show that eigenvalues of any 2×2 symmetric matrix with real entries are real numbers. Thus (x_e, y_e) could be a node or a saddle, but not a spiral point or a center.

10.6.3. Exercises.

1. Show that a linear system

$$x'(t) = ax + by, \qquad y'(t) = cx + dy$$

is a Hamiltonian system if and only if $d = -a$. Find a Hamiltonian function if $d = -a$.

2. Write the second-order equation $x'' - x + 2x^3 = 0$ as a system of two first-order equations by means of the substitution $x' = y$. Is your system Hamiltonian? If so, describe a Hamiltonian function for it.

3. Write the second-order equation $x'' + f(x) = 0$ as a system of two first-order equations by means of the substitution $x' = y$. Is your system Hamiltonian? If so, describe a Hamiltonian function for it.

4. Show that any system of the form

$$\frac{dx}{dt} = f(y), \qquad \frac{dy}{dt} = g(x),$$

where f and g are continuous, is a Hamiltonian system. What is a Hamiltonian function?

5. This problem continues the differential equation models of Romeo and Juliet, first introduced in Exercise 8 of Section 8.1. The functions $R(t)$ and $J(t)$ measure Romeo's feelings for Juliet and Juliet's feelings for Romeo, respectively. A positive value denotes love, and a negative value, dislike. Romeo's emotions are complicated. When Juliet is somewhat enthusiastic about Romeo, his feelings for her increase. But if Juliet gets too enamored of Romeo, he gets cold feet and his feelings for her start to wane. However, if Juliet dislikes Romeo, then Romeo's feelings for Juliet decrease sharply. To quantify this, suppose the rate of change of Romeo's feelings are given by

$$\frac{dR}{dt} = J - J^2. \tag{10.89}$$

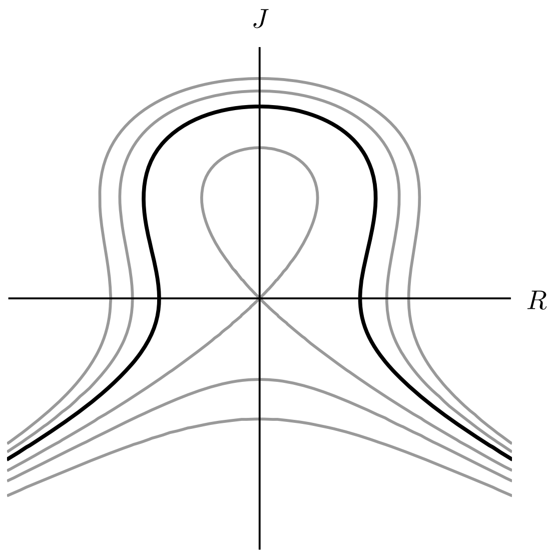
Notice that $\frac{dR}{dt} > 0$ when $0 < J < 1$ but $\frac{dR}{dt} < 0$ if $J > 1$ or $J < 0$. Juliet's feelings are more straightforward. The more Romeo loves her, the more her feelings for him grow, and the more he dislikes her, the more her feelings for him decrease. We quantify this by the differential equation

$$\frac{dJ}{dt} = R. \tag{10.90}$$

(a) What are the equilibrium points of the system given by equations (10.89) and (10.90)?
(b) Show that this system is Hamiltonian and that a Hamiltonian function is

$$H(R, J) = \frac{J^2}{2} - \frac{J^3}{3} - \frac{R^2}{2}.$$

(c) Some level curves for the function $\frac{J^2}{2} - \frac{J^3}{3} - \frac{R^2}{2}$ are shown below. We know that the trajectories of the solutions to our system lie along level curves of the Hamiltonian function. The level curve shown in boldface, which passes through the point $R = 1, J = 0$ is indeed a (portion of) a trajectory in the phase portrait. Indicate the direction on this trajectory. If at time $t = 0$, $R = 1$ and $J = 0$, what is the fate of the relationship as $t \to \infty$? Mutual dislike? Romeo in love but Juliet not? Something else?



6. **Equilibrium points of a Hamiltonian system.** If
$$\frac{dx}{dt} = \frac{\partial H}{\partial y}, \qquad \frac{dy}{dt} = -\frac{\partial H}{\partial x},$$
where $H(x, y)$ is twice continuously differentiable, describe the Jacobian matrix at an equilibrium point. What possibilities exist for the eigenvalues of the Jacobian? What does this mean about the possible types of equilibrium points for a Hamiltonian system? You may assume the eigenvalues are nonzero.

7. In this problem we compare two systems:
$$\frac{dx}{dt} = yx^3, \qquad \frac{dy}{dt} = x + 2$$
and
$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = \frac{1}{x^2} + \frac{2}{x^3}.$$
   (a) Show that the second system is Hamiltonian, but the first is not.
   (b) Find a conserved quantity for the second system. Is it also conserved for the first? Hint: A function $E(x, y)$ is conserved for a planar differential equation system if $E(x, y)$ is constant along any solution $x = x(t), y = y(t)$ of the system. The Multivariable Chain Rule says
$$\frac{d}{dt}\left[E(x(t), y(t))\right] = \frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt}.$$
   (c) How does the direction field for the first system compare to that for the second system? What does this imply about the phase portraits for the two systems?

8. Show that $E(x, y) = x^2 + \frac{1}{2}y^2$ is a Lyapunov function for
$$\frac{dx}{dt} = -x^3, \qquad \frac{dy}{dt} = -2x^2y - \frac{1}{2}y^3,$$
corresponding to its equilibrium point $(0, 0)$. Determine the stability of this equilibrium point.

9. Find a Lyapunov function for
$$\frac{dx}{dt} = -y - x^3, \qquad \frac{dy}{dt} = x - y^3,$$
corresponding to its equilibrium point $(0, 0)$, and use it to determine the stability of $(0, 0)$.

10. Find values of a and b so that $E(x,y) = ax^2 + by^4$ is a Lyapunov function for
$$\frac{dx}{dt} = -2y^3, \quad \frac{dy}{dt} = x - 3y^3,$$
corresponding to its equilibrium point $(0,0)$.

11. Find values of a and b so that $E(x,y) = ax^2 + by^2$ is a Lyapunov function for
$$\frac{dx}{dt} = y - xf(x,y), \quad \frac{dy}{dt} = -x - yf(x,y),$$
corresponding to its equilibrium point $(0,0)$, where $f(x,y)$ is continuously differentiable with $f(x,y) > 0$ in $\mathbb{R}^2$. Determine the stability of $(0,0)$.

12. Consider the system
$$\frac{dx}{dt} = y - f(x,y), \quad \frac{dy}{dt} = -x - g(x,y),$$
where we have the following information about f and g:
    - f and g are continuously differentiable, with $f(0,0) = 0$, $g(0,0) = 0$,
    - when $x > 0, f(x,y) > 0$ and when $x < 0, f(x,y) < 0$, and
    - when $y > 0, g(x,y) > 0$ and when $y < 0, g(x,y) < 0$.

    Show that $E(x,y) = x^2 + y^2$ is a Lyapunov function and that $(0,0)$ is asymptotically stable.

13. Show that a 2×2 symmetric matrix with real entries must have real eigenvalues.

14. (a) Show that
$$\frac{dx}{dt} = -2x + 2y, \quad \frac{dy}{dt} = 2x - 4y^3 - 2y$$
is a gradient system.
    (b) Determine the stability of the equilibrium point $(0,0)$ by finding a Lyapunov function.
    (c) Can you use Theorems 10.2.3 or 10.2.4 to determine the stability of $(0,0)$?

15. Follow the outline to show that $E(x,y) = ax^2 + bxy + cy^2$ is positive definite on $\mathbb{R}^2$ if and only if $a > 0$ and $b^2 < 4ac$.
    (a) Suppose $E(x,y) > 0$ for all $(x,y) \neq (0,0)$. Show that $a > 0$.
    (b) Show that if $E(x,y) > 0$ for all $(x,y) \neq (0,0)$, then $b^2 < 4ac$. Hint: If $y \neq 0$, write $ax^2 + bxy + cy^2 = y^2(av^2 + bv + c)$ where $v = x/y$. By (a) you know $a > 0$.
    (c) Conversely, assume $a > 0$ and $b^2 < 4ac$. Show that $E(x,y) > 0$ for all $(x,y) \neq (0,0)$. Hint: As in (b), write $ax^2 + bxy + cy^2 = y^2(av^2 + bv + c)$ where $v = x/y$.

16. Give statements, analogous to that in Theorem 10.6.10, that characterize when the quadratic form $E(x,y) = ax^2 + bxy + cy^2$ (which has $E(0,0) = 0$) is
    (i) positive semidefinite (meaning $E(x,y) \geq 0$ for all (x,y)),
    (ii) negative definite (meaning $E(x,y) < 0$ for all $(x, y \neq (0,0))$),
    (iii) negative semidefinite (meaning $E(x,y) \leq 0$ for all (x,y)).

17. Show that the function $E(x,y) = x^3 + x^2y + xy^2 + y^3$ is neither positive definite nor negative definite.

18. Consider the system
$$\frac{dx}{dt} = -x - 2y, \quad \frac{dy}{dt} = 2x.$$
    (a) Show that with the choice $E(x,y) = x^2 + y^2$, $\dot{E}$ is negative semidefinite, but not negative definite.
    (b) Show that with the choice $E(x,y) = 9x^2 + 4xy + 10y^2$, $E(x,y)$ is positive definite and $\dot{E}$ is negative definite.

(c) Let's see how the function $E(x,y)$ of (b) was found. Suppose we seek E in the form $E(x,y) = ax^2 + bxy + cy^2$. From Theorem 10.6.10 we know $E(x,y)$ is positive definite if $a > 0$ and $b^2 < 4ac$. A computation shows that

$$\dot{E} = (2ax + by)(-x - 2y) + (bx + 2cy)(2x).$$

Show that one way to ensure that $\dot{E}$ is negative definite is to choose a, b, and c so that $b > 0$, $b - a < 0$, and $4c - 4a - b = 0$. The choices $a = 9$, $b = 4$, and $c = 10$ meet all five of the desired conditions. Find a different choice of a, b, and c which also meets the desired conditions, and hence provides a different Lyapunov function.

19. Consider the damped mass-spring system

$$\frac{dy}{dt} = v, \qquad \frac{dv}{dt} = -2v - y.$$

(a) Show that $E(y,v) = y^2 + v^2$ is a Lyapunov function for this system, but not a strong Lyapunov function.
(b) Find a strong Lyapunov function of the form $E(y,v) = ay^2 + byv + cv^2$. Hint: Use the idea of part (c) of Exercise 18.

20. (a) Show that the function $E(x,y) = x^2 + y^2$ is a Lyapunov function for the system

$$\frac{dx}{dt} = -xy - y^2, \qquad \frac{dy}{dt} = -y + x^2$$

on the open disk of radius 1 centered at $(0,0)$, but not on the whole plane $\mathbb{R}^2$.
(b) Can you use Theorem 10.6.7 to classify the stability of the equilibrium point $(0,0)$?
(c) Why can't Theorems 10.2.3 or 10.2.4 be applied to classify the stability of $(0,0)$?

21. Suppose $H(x,y) = 2x^2 + y^2$.
(a) Sketch the level curves of $H(x,y) = c$ for $c = 1, 2, 4$, and 8.
(b) In the same picture as (a), sketch some trajectories of the gradient system

$$\frac{dx}{dt} = -4x, \qquad \frac{dy}{dt} = -2y.$$

What geometric relationship should you see in your picture?

In Exercises 22–25 show that the given system is either Hamiltonian or a gradient system (or possibly both). Sketch a phase portrait for each, near the specified equilibrium point, showing the particular features the trajectories have by virtue of the system being a Hamiltonian or gradient system.

22.
$$\frac{dx}{dt} = -2y, \qquad \frac{dy}{dt} = -2x, \ \text{near}(0,0).$$

23.
$$\frac{dx}{dt} = \sin x, \qquad \frac{dy}{dt} = -y\cos x, \ \text{near}(0,0).$$

24.
$$\frac{dx}{dt} = -y\cos x, \qquad \frac{dy}{dt} = -\sin x, \ \text{near}(0,0).$$

25.
$$\frac{dx}{dt} = 2y - 4, \qquad \frac{dy}{dt} = 2x - 2, \ \text{near}(1,2).$$

26. Show that if

$$\frac{dx}{dt} = f(x,y), \qquad \frac{dy}{dt} = g(x,y) \tag{10.91}$$

is a Hamiltonian system, then

$$\frac{dx}{dt} = g(x, y), \qquad \frac{dy}{dt} = -f(x, y) \tag{10.92}$$

is a gradient system. Show that both systems have the same equilibrium points. How are the trajectories of (10.91) related to the trajectories of (10.92)?

27. The purpose of this problem is to prove the "stable equilibrium" conclusion of Theorem 10.6.7. To simplify notation, we assume the isolated equilibrium point is $(0, 0)$. By the hypothesis we have $E(0, 0) = 0$, $E(x, y) > 0$ for all $(x, y) \neq (0, 0)$ in a neighborhood U of $(0, 0)$ and $\dot{E}(x, y) \leq 0$ for all (x, y) in U. Our goal is to show that $(0, 0)$ is a stable equilibrium point.

    Pick any positive number r such that the closed disk of radius r and centered at $(0, 0)$ is contained in U. Set $f(t) = E(r\cos t, r\sin t)$. As t ranges over $[0, 2\pi]$, $f(t)$ gives the values of E at points on the circle of radius r centered at $(0, 0)$.
    (a) By citing the appropriate Calculus I result, explain why f attains a minimum value m at some point t_0 in the interval $[0, 2\pi]$, so that $m = f(t_0) \leq f(t)$ for all $0 \leq t \leq 2\pi$. Explain why $m > 0$.
    (b) Explain why we can find a positive number δ so that $E(x, y) < m$ on the closed disk D of radius δ centered at $(0, 0)$. Show that $\delta < r$.
    (c) Suppose that $x = x(t), y = y(t)$ is a trajectory of the system with $(x(0), y(0))$ in the disk D from (b). Explain why, for $t \geq 0$, this trajectory cannot intersect with the circle of radius r centered at $(0, 0)$.
    (d) By appealing to the definition of "stable equilibrium" explain why $(0, 0)$ must be a stable equilibrium.

10.7. Pendulums

A pendulum consists of a rigid rod (of negligible weight) with length L, fixed to a pivot point at one end and with mass m (called a bob) at the other end. The rod is free to move in a plane, so that the mass moves along a circular path. We describe the motion by giving the angle θ as shown in Fig. 10.69, so that θ is 0 when the pendulum is in its rest position. The force due to gravity of the bob is mg, where g is the acceleration due to gravity. We think of this as a vector pointing down, and we resolve this vector into two components as shown in Fig. 10.70, with one component tangent to the circle on which the bob moves.

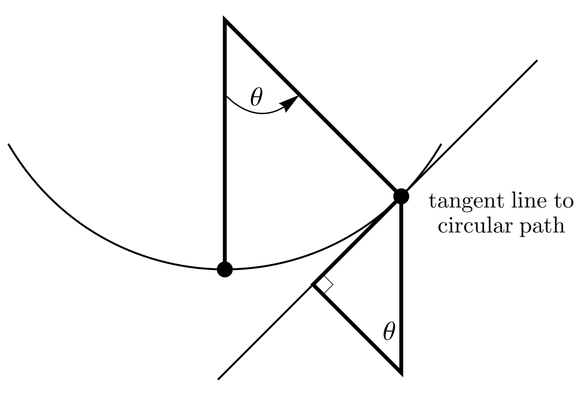


Figure 10.69. Pendulum.

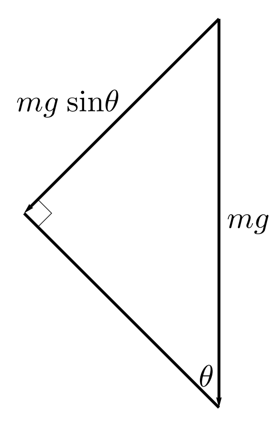


Figure 10.70. Resolving the force vector into components.

Only the tangential force affects the motion of the pendulum. We also allow for a damping force and make the familiar assumption that it is proportional to the velocity and acts opposite to the direction of motion. When the bob moves through an angle θ, it travels a distance $L\theta$ and has velocity $L\frac{d\theta}{dt}$ and acceleration $L\frac{d^2\theta}{dt^2}$. From Newton's second law we have

$$mL\frac{d^2\theta}{dt^2} = -cL\frac{d\theta}{dt} - mg\sin\theta$$

for some positive constant c. The first term on the right-hand side represents damping, and the minus sign reflects our assumption that the damping force acts opposite to the direction of motion. The minus sign with the second term indicates that the tangential component of the gravitational force always pushes θ towards zero when $-\pi < \theta < \pi$. Dividing by mL gives

$$\frac{d^2\theta}{dt^2} + \frac{c}{m}\frac{d\theta}{dt} + \frac{g}{L}\sin\theta = 0.$$

Convert this to an autonomous system by the substitution $v = \frac{d\theta}{dt}$:

$$\frac{d\theta}{dt} = v, \qquad \frac{dv}{dt} = -\frac{c}{m}v - \frac{g}{L}\sin\theta. \tag{10.93}$$

The undamped pendulum. To ignore damping, we set $c = 0$ and obtain the equations

$$\frac{d\theta}{dt} = v, \quad \frac{dv}{dt} = -\frac{g}{L}\sin\theta. \tag{10.94}$$

This system has equilibrium points $(n\pi, 0)$ for any integer n. For even integers $n = 0, \pm 2, \pm 4, \ldots$ these correspond to the pendulum hanging straight down. Odd values of n correspond to the pendulum being in the straight up position. You can probably guess at the stability of these equilibrium points based on your physical intuition.

At the equilibrium point $(n\pi, 0)$ the linear approximation to this system is $\mathbf{X}' = \mathbf{J}\mathbf{X}$ for

$$\mathbf{X} = \begin{pmatrix} \theta \\ v \end{pmatrix} \quad \text{and} \quad \mathbf{J} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{L}\cos(n\pi) & 0 \end{pmatrix}.$$

For odd values of n, the eigenvalues of $\mathbf{J}$ are $\pm\sqrt{g/L}$ and thus the equilibrium points $(n\pi, 0)$ for n odd are saddles. When n is an even integer, $\mathbf{J}$ has eigenvalues $\pm i\sqrt{g/L}$ and the linear approximation has a center. Since this is a borderline case, linear approximation by itself doesn't give a classification of the equilibrium points $(n\pi, 0)$, n even, or a determination of their stability, for the nonlinear pendulum. We proceed further by recognizing that (10.94) is a Hamiltonian system with Hamiltonian function

$$H(\theta, v) = \frac{1}{2}v^2 + \frac{g}{L}(1 - \cos\theta).$$

In fact, the function

$$E(\theta, v) = mL^2H(\theta, v) = \frac{1}{2}mL^2v^2 + mgL(1 - \cos\theta)$$

is the total energy function: The term $\frac{1}{2}mL^2v^2$ is the bob's kinetic energy and the second term $mgh = mgL(1 - \cos\theta)$ is its potential energy due to the gravitational force (see Exercise 10; we assume the bob has zero potential energy when $\theta = 0$). The orbits lie in the level curves of $H(\theta, v)$, or what is the same thing, in the level curves of the energy function $E(\theta, v)$.

For a sufficiently small value of K, the level curve

$$\frac{1}{2}v^2 + \frac{g}{L}(1 - \cos\theta) = K \tag{10.95}$$

is a collection of closed curves encircling the equilibrium points $(n\pi, 0)$ for n even, as illustrated in Fig. 10.71; see Exercise 5. The corresponding motion of the pendulum is the familiar back and forth oscillation. When θ is in the range $-\pi < \theta < \pi$, the value of θ oscillates back and forth between a maximum $M < \pi$ and a minimum $-M$.

The nullcline-and-arrow diagram for (10.94), with θ-nullcline $v = 0$ and v-nullclines $\theta = n\pi$ for n any integer, is shown in Fig. 10.72. This gives the clockwise direction on the closed loop orbits in Fig. 10.71.

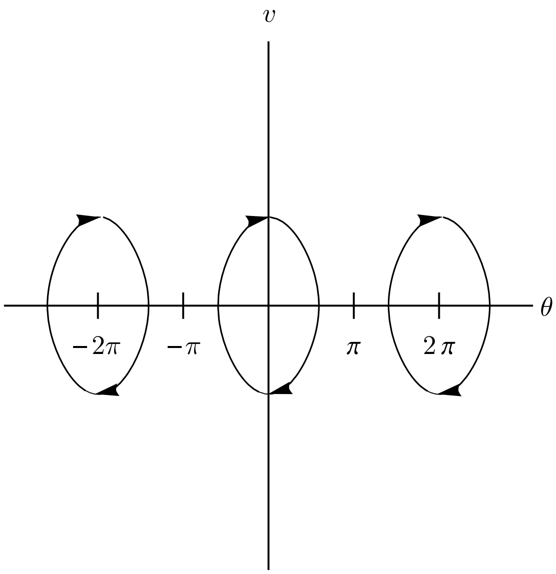


Figure 10.71. $\frac{1}{2}v^2 + \frac{g}{L}(1 - \cos\theta) = K$, $K < \frac{2g}{L}$.

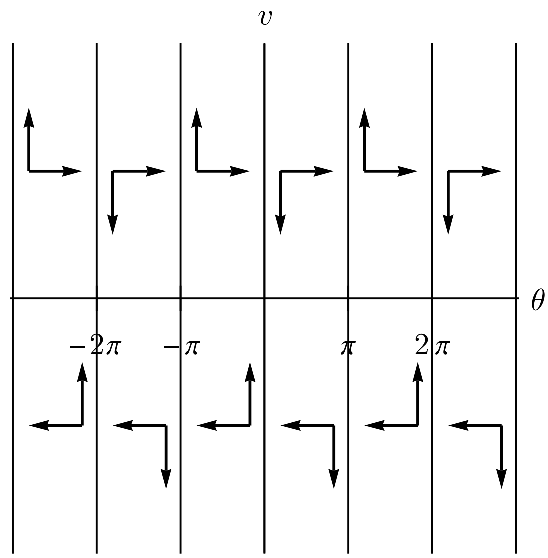


Figure 10.72. Nullcline-and-arrow diagram for undamped pendulum.

For large values of K, the level curves look like those shown in Fig. 10.73. Again, the nullcline-and-arrow diagram in Fig. 10.72 gives the direction on these "high energy" orbits, which correspond to the pendulum whirling around and around in complete counterclockwise circles (top) or clockwise circles (bottom).

The value of K that separates the two very different types of motion shown in Figs. 10.71 and 10.73 is $K = \frac{2g}{L}$. With this choice of K we have the level curve

$$\frac{1}{2}v^2 = \frac{g}{L}(1 + \cos\theta),$$

or

$$v = \pm\sqrt{\frac{2g}{L}(1 + \cos\theta)}.$$

Included in this level curve are the unstable equilibrium points $(n\pi, 0)$ (n an odd integer) and the separatrices which join these saddle points; see Fig. 10.74 and Exercise 3. This corresponds to motion we would never observe in real life—for example, the pendulum swinging up towards its straight up position, never actually reaching that position but approaching it as $t \to \infty$.

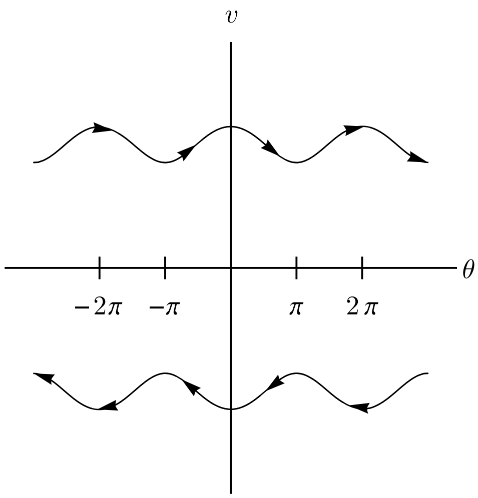


Figure 10.73. $\frac{1}{2}v^2 + \frac{g}{L}(1 - \cos\theta) = K$, $K > \frac{2g}{L}$.

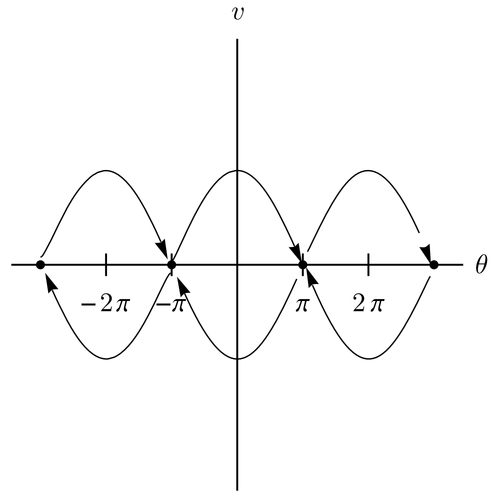


Figure 10.74. $\frac{1}{2}v^2 = \frac{g}{L}(1 + \cos\theta)$.

Putting together Figs. 10.71, 10.73, and 10.74 we get a sketch of the phase portrait in Fig. 10.75.

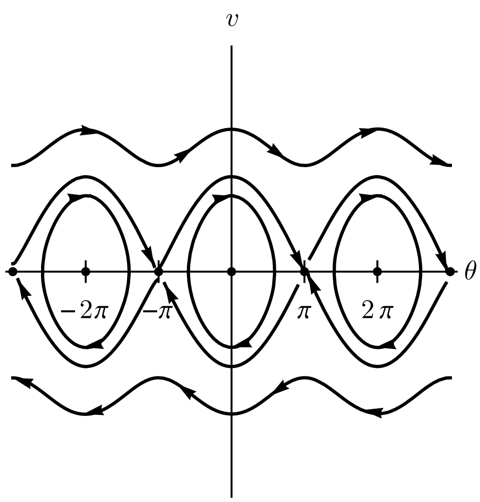


Figure 10.75. Phase portrait for undamped pendulum.

The damped pendulum. Suppose that $c \neq 0$ in the system (10.93). The equilibrium points are still $(n\pi, 0)$ for any integer n. Exercise 9 asks you to determine the linear approximation at each of these points and show that for odd values of n, the equilibrium point is a saddle, while for even values of n, the point $(n\pi, 0)$ is a stable spiral for small positive values of c but becomes a stable node for larger values of c. We'll focus here on the small values of $c > 0$, assembling information we get from various perspectives about the phase portrait and pendulum motion.

When $c \neq 0$ the energy function

$$E(\theta, v) = \frac{1}{2}mL^2v^2 + mgL(1 - \cos\theta)$$

is no longer a conserved quantity, and in fact

$$\dot{E}(\theta, v) = -cL^2v^2$$

(see Exercise 1). Moreover, $E(\theta, v) \geq 0$ and $E(\theta, v) = 0$ if and only if $v = 0$ and $\theta = n\pi$ for n an even integer. Thus the function $E(\theta, v)$ is a Lyapunov function for (10.93) in a neighborhood of each of the equilibrium points $(n\pi, 0)$ for n even. Since $\dot{E}$ is negative semidefinite, but not negative definite ($\dot{E} = 0$ whenever $v = 0$), Theorem 10.6.7 only lets us conclude that we have stability at each equilibrium point $(n\pi, 0)$, n even. Moreover, the orbits at points with $v \neq 0$ must cross the level curves $E = k$ from larger to smaller values. So we can still use the level curves of E as guidelines for sketching the phase portrait, but instead of the orbits staying in a level curve, they will cross over the level curves from higher energy to lower energy. You should be able to see this in Fig. 10.77.

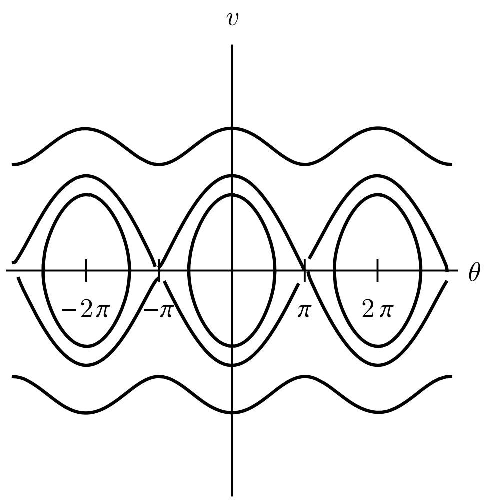


Figure 10.76. Level curves of $E(\theta, v)$.

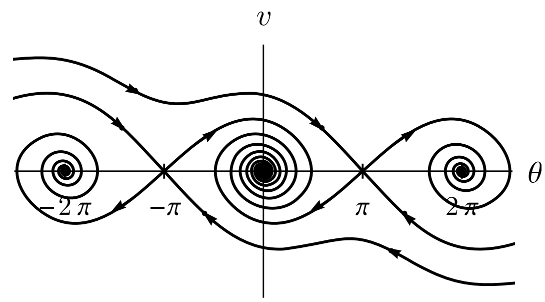


Figure 10.77. Phase portrait for the damped pendulum.

The nullclines of (10.93) are the horizontal line $v = 0$ (the θ-nullcline) and the (scaled) sine curve $v = -\frac{gm}{cL}\sin\theta$ (the v-nullcline). These are sketched in Fig. 10.78, along with the direction-indicator arrows. A sketch of the phase portrait, when c is small enough that we have spiral points at the stable equilibria, is shown in Fig. 10.77.

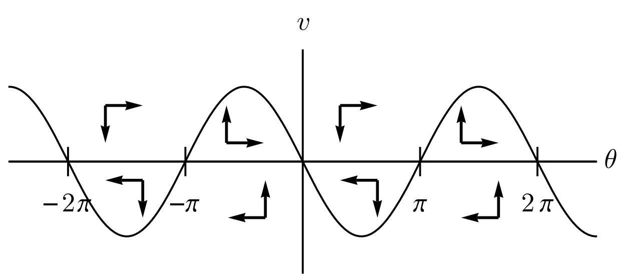


Figure 10.78. Nullcline-and-arrow diagram for damped pendulum.

Basin of attraction. We revisit the concept of "basin of attraction" (introduced at the end of Section 10.2) in the context of the damped pendulum. Recall, for example, that the point (a, b) is in the basin of attraction of the asymptotically stable equilibrium point $(0, 0)$ if the solution $(x(t), y(t))$ of (10.93) which satisfies $x(0) = a$, $y(0) = b$ will have $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} y(t) = 0$. Each of the asymptotically stable equilibrium points $(n\pi, 0)$, n even, for (10.93) has its own basin of attraction and the separatrices that enter each unstable equilibria ($(n\pi, 0)$ for n odd) serve to separate the basins of attraction of the stable equilibria. In Fig. 10.79, the basin of attraction for $(0, 0)$ is shown shaded.

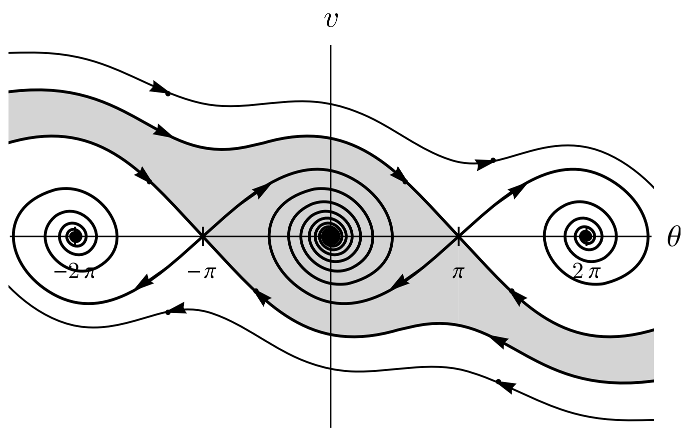
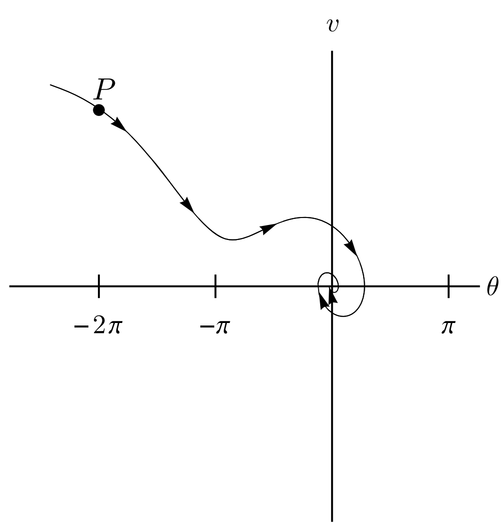


Figure 10.79. Basin of attraction for $(0, 0)$ in the damped pendulum model.

10.7.1. Exercises.

1. Compute $\dot{E}(\theta, v)$ for the system (10.93) if $E(\theta, v) = \frac{1}{2}mL^2v^2 + mgL(1 - \cos\theta)$.
2. The figure below shows a particular orbit for a damped pendulum. The point P on this orbit corresponds to $t = 0$, so that at time $t = 0$ the pendulum is in the straight down position and moving counterclockwise. Which of the following describes the motion for $t > 0$?
   (a) The pendulum never reaches the straight up position but just oscillates back and forth, with decreasing amplitude and tending to the straight down position as t goes to ∞.
   (b) The pendulum makes one complete revolution, before settling down to back and forth oscillations with decreasing amplitude and tending to the straight down position as t goes to ∞.
   (c) The pendulum makes two complete revolutions, before settling down to back and forth oscillations with decreasing amplitude and tending to the straight down position as t goes to ∞.
   (d) None of the above are correct.

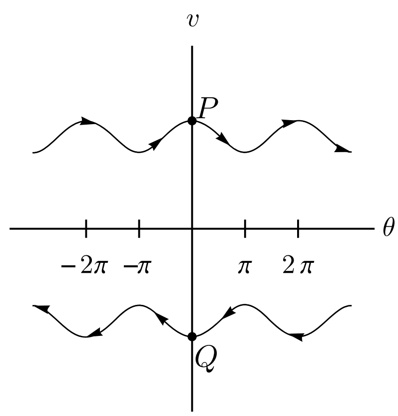


3. Working with Fig. 10.74, identify the stable curve S and unstable curve U corresponding to the saddle point $(\pi, 0)$.

4. The level curve $\frac{1}{2}v^2 + (1 - \cos\theta) = 4$ for the undamped pendulum

$$\frac{d\theta}{dt} = v, \qquad \frac{dv}{dt} = -\sin\theta$$

is shown below. Notice that if the point P corresponds to $t = 0$, the motion of the pendulum has the following description: At time 0 it is in the straight down position and moving counterclockwise. It continues to revolve counterclockwise around the pivot over and over. Similarly, give a description of the pendulum if the point Q corresponds to the time $t = 0$.



5. Show that the level curve

$$\frac{1}{2}v^2 + (1 - \cos\theta) = \frac{1}{2}$$

(a special case of equation (10.95), with $L = g$ and $K = \frac{1}{2}$) consists of closed curves, each encircling one point $(n\pi, 0)$ with n even. For the orbit encircling $(0, 0)$, how high does the pendulum go? What are the maximum and minimum values of θ along this orbit?

6. Why isn't $E(\theta, v) = \frac{1}{2}mL^2v^2 + mgL(1 - \cos\theta)$ a Lyapunov function for the damped pendulum near an equilibrium point $(n\pi, 0)$, n odd?

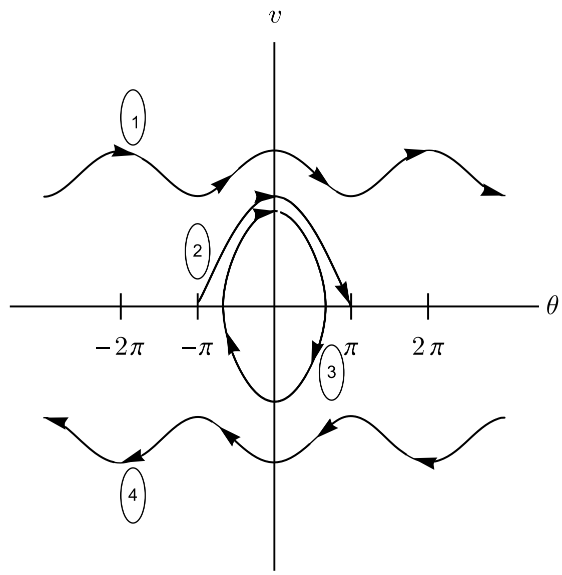
7. The figure below shows four orbits in the phase portrait of the undamped pendulum (equation (10.94)). Match these four orbits with the corresponding graph of θ vs. t for $t \geq 0$, assuming $\theta = 0$ at $t = 0$. The pendulum equations are

$$\frac{d\theta}{dt} = v, \qquad \frac{dv}{dt} = -\sin\theta$$

and the four orbits lie, respectively, in the level curves

$$\frac{1}{2}v^2 + (1 - \cos\theta) = k$$

for $k = 4$ (with $v > 0$), $k = 2$, $k = 1.5$, and $k = 4$ (with $v < 0$).



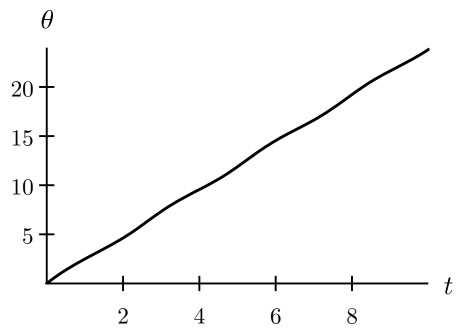


Figure 10.80. (A).

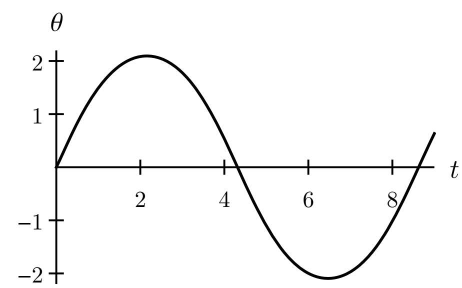


Figure 10.81. (B).

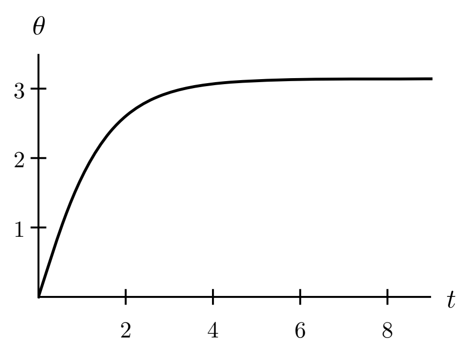


Figure 10.82. (C).

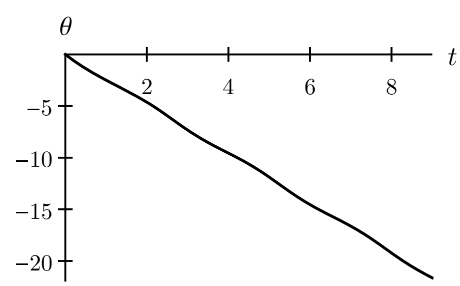
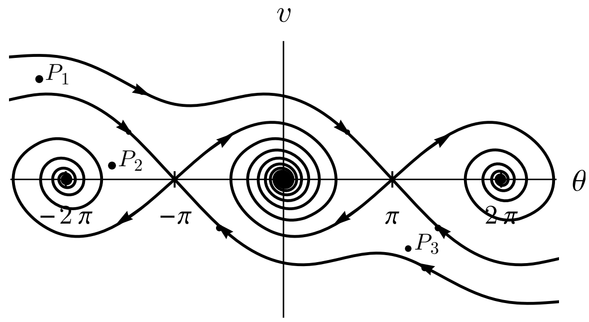


Figure 10.83. (D).

8. Separatrices for the phase portrait for the damped pendulum system

$$\frac{d\theta}{dt} = v, \qquad \frac{dv}{dt} = -0.2v - \sin\theta$$

are shown in the figure below.



(a) If the point P_1 corresponds to time $t = 0$, how many complete revolutions will the pendulum make (for $t > 0$) before settling into a back and forth oscillation?
(b) The point labeled P_1 lies in the basin of attraction for which equilibrium point? Answer the same question for the points P_2 and P_3.

9. Using linear approximation, show that the equilibrium points $(n\pi, 0)$ with n odd are saddle points of the damped pendulum described by equation (10.93) with $c \neq 0$. For n even, show that $(n\pi, 0)$ could be a spiral point or a node, but in either case it is stable. What condition on c, m, and L distinguishes nodes from spiral points?
10. The undamped pendulum equation

(10.96)
$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

can also be derived by starting with the conservation of energy principle

$$\text{kinetic energy} \; + \; \text{potential energy} \; = C$$

applied to the pendulum's bob. The kinetic energy is $\frac{1}{2}mV^2$ where m is the mass and V is the velocity. The potential energy is the product mgh where h is the height above the reference position which we take to be the rest position.

(a) Show that the kinetic energy is given by

$$\frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2,$$

where θ is as in Fig. 10.69 and L is the length.
(b) Show that the potential energy is given by $mgL(1 - \cos\theta)$.
(c) By differentiating the equation

$$\text{kinetic energy} \; + \; \text{potential energy} \; = C$$

with respect to t, obtain equation (10.96).

11. The motion of an undamped pendulum is described by the equation $\theta'' + \frac{g}{L}\sin\theta = 0$. Any equation of the form $\theta'' + f(\theta) = 0$ where f is a continuously differentiable function satisfying

$$f(0) = 0, \quad f > 0 \text{ on some interval } (0, \alpha), \quad \text{and} \quad f < 0 \text{ on } (-\alpha, 0)$$

is called a generalized pendulum equation. The substitution $v = d\theta/dt$ converts this to the system

(10.97)
$$\frac{d\theta}{dt} = v, \qquad \frac{dv}{dt} = -f(\theta).$$

(a) Show that if

$$F(\theta) = \int_0^\theta f(u)du,$$

then the energy function

$$E(\theta, v) = \frac{1}{2}v^2 + F(\theta)$$

is a conserved quantity for the system, and hence the orbits lie in the level curves of $E(\theta, v)$.

(b) Show that the equilibrium points of the system are the points $(\theta^*, 0)$ where θ^* is a critical point of F.

(c) Show that if θ^* is a critical point of F and $f'(\theta^*) < 0$, then $(\theta^*, 0)$ is a saddle point of (10.97).

(d) Show that the hypothesis on $f(\theta)$ says that $F(0) = 0$, $F(\theta)$ is strictly increasing for $0 < \theta < \alpha$ and $F(\theta)$ is strictly decreasing for $-\alpha < \theta < 0$. Thus F has a strict relative minimum at $\theta = 0$ and $f'(0) \geq 0$.

(e) Show that if $f'(0) > 0$, the linear approximation near $(0, 0)$ to the system in (10.97) has a center there (a borderline case).

(f) Use the following argument to show that the nonlinear system has a center at $(0, 0)$: Show that if k is a sufficiently small positive value, then we can find two values a, b with $a < 0 < b$, $F(a) = F(b) = k$, and $F(\theta) < k$ on (a, b). Explain why for each value $\tilde{\theta}$ between a and b the level curve $E(v, \theta) = k$ will contain exactly two points $(\tilde{\theta}, \tilde{v})$ and $(\tilde{\theta}, -\tilde{v})$, and as $\tilde{\theta}$ approaches either a or b, $\tilde{v}$ will approach 0. Thus the orbits of (10.97) near $(0, 0)$ are closed curves that encircle the origin, so that $(0, 0)$ is indeed a center.

10.8. Cycles and limit cycles

A periodic solution of an autonomous system

$$\frac{dx}{dt} = f(x, y), \qquad \frac{dy}{dt} = g(x, y) \tag{10.98}$$

is a solution $x = x(t), y = y(t)$ with the property that $x(t+T) = x(t)$ and $y(t+T) = y(t)$ for some $T > 0$ and all t. We also require that $x(t)$ and $y(t)$ be nonconstant functions. A periodic solution has an orbit which is a closed curve, traversed over and over as t varies. We'll call such an orbit a **cycle**. An important part of the global analysis of (10.98) is to determine if the system has any cycles. Throughout this section we assume that the functions $f(x, y)$ and $g(x, y)$ are defined and continuously differentiable at every point in the xy-plane.

We've seen numerous examples of systems with cycles. In any planar linear system $\mathbf{X}' = \mathbf{AX}$ where the eigenvalues of $\mathbf{A}$ are a pair of purely imaginary conjugates $\pm i\beta$, $\beta \neq 0$, all nonconstant orbits are cycles (in fact, they are ellipses, by Exercise 13 of Section 8.9). The Lotka-Volterra predator-prey system of Section 10.4 has orbits in the first quadrant that are cycles enclosing the equilibrium point $(\frac{d}{c}, \frac{a}{b})$.

A more exotic example of a cycle was seen in Example 10.3.3 of Section 10.3. We saw there that the system

$$\frac{dx}{dt} = -y + x(1 - x^2 - y^2), \qquad \frac{dy}{dt} = x + y(1 - x^2 - y^2)$$

has the periodic solution $x(t) = \cos t$, $y(t) = \sin t$ whose orbit is the unit circle $\mathbb{T}$. Moreover we saw that orbits that start inside the unit circle, as well as orbits that start outside $\mathbb{T}$, spiral towards $\mathbb{T}$ as $t \to \infty$. Recall that we called $\mathbb{T}$ a **limit cycle**, to capture the fact that it is approached by other nearby trajectories as $t \to \infty$. In fact, it is approached by *every* nearby trajectory as $t \to \infty$, and we will call it an **attracting limit cycle**. Limit cycles can also be **repelling**; this means that nearby trajectories approach it as $t \to -\infty$. The cycles in the Lotka-Volterra system are not limit cycles, since nearby trajectories are cycles themselves and so remain at a positive distance from the given cycle.

Cycles and equilibrium points. Cycles must always enclose one or more equilibrium points. This fact can sometimes be used to rule out the presence of cycles. For example, the competing species model

$$\frac{dx}{dt} = x(2 - x - y), \qquad \frac{dy}{dt} = y(1 - x - 4y)$$

cannot have any cycles in the biologically relevant first quadrant $x > 0$, $y > 0$, since its equilibrium points are $(0, 0)$, $(0, \frac{1}{4})$, $(2, 0)$, and $(\frac{7}{3}, -\frac{1}{3})$, none living in the first quadrant. The first quadrant is an example of a **simply connected region**, which roughly means a region with no holes. Other examples of simply connected regions include any rectangular region or any disk. A region that is not simply connected is the ring-shaped region between two concentric circles. A region R is simply connected if every **simple closed curve** in R encloses only points of R. A **simple closed curve** begins and ends at the same point but otherwise does not cross itself. Figs. 10.84–10.89 illustrate these ideas.

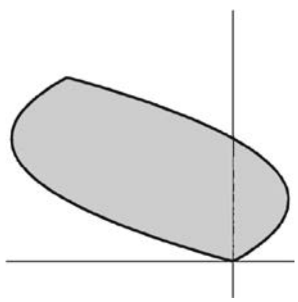


Figure 10.84. A simply connected region.

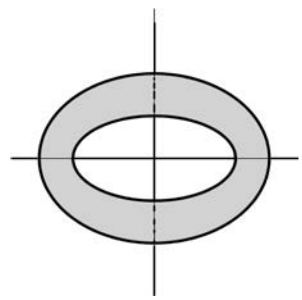


Figure 10.85. A region that is not simply connected.

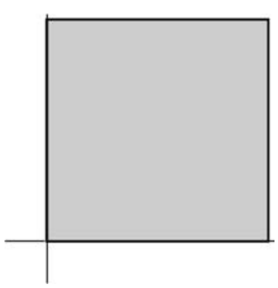


Figure 10.86. Another simply connected region—the first quadrant.

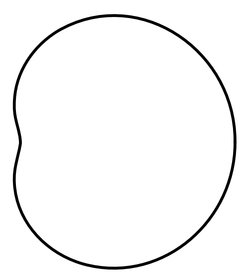


Figure 10.87. A simple closed curve.

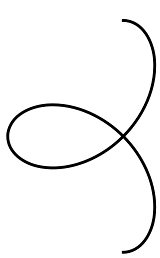


Figure 10.88. A curve that is neither simple nor closed.

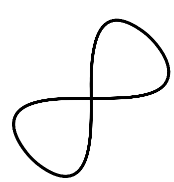


Figure 10.89. A closed curve that is not simple.

Cycles or no cycles? In a simply connected region, the absence of equilibrium points means no cycles:

> If a region R is simply connected and contains no equilibrium point of the system (10.98), then there is no cycle in R.

On the other hand, there are tools that sometimes allow us to know that a cycle exists in a certain region. The most important of these is the **Poincaré-Bendixson** Theorem, which we will describe next.

Our version of the Poincaré-Bendixson Theorem deals with the annular, or ring-shaped, region between two simple closed curves, one lying inside the other, as shown in Fig. 10.90.

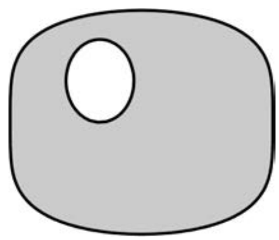


Figure 10.90. Annular region between two simple closed curves.

Theorem 10.8.1 (Poincaré-Bendixson)**.** *Let R be the region consisting of two simple closed curves C_o and C_i, where C_i is inside C_o, and the annular region between them. Suppose that R contains no equilibrium points of the system* (10.98)*, where f and g are continuously differentiable in $\mathbb{R}^2$, and suppose we have a trajectory $(x(t), y(t))$ which stays in R for all $t \geq 0$. Then either this trajectory is itself a cycle in R or it spirals towards some limit cycle in R as $t \to \infty$.*

Notice that this result tells you, in particular, that when the hypotheses are satisfied, *the region R must contain a cycle.*

The work in applying Theorem 10.8.1 is identifying curves C_o and C_i, and thus the region R, so that the hypotheses are satisfied. Roughly speaking, you want to look for a region such that the vector field for (10.98) points *into* the region at every point of the boundary curves C_o and C_i. This will guarantee that a trajectory that starts in R will stay in R for all later times.

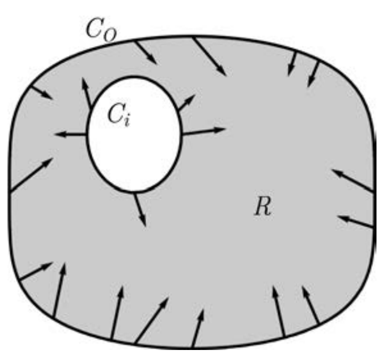


Figure 10.91. Vector field points into R at each point of C_o and C_i.

We'll illustrate Theorem 10.8.1 with some examples. The first one is a "warm-up".

Example 10.8.2. Consider again the system

$$\frac{dx}{dt} = -y + x(1 - x^2 - y^2), \qquad \frac{dy}{dt} = x + y(1 - x^2 - y^2)$$

of Example 10.3.3 in Section 10.3. A portion of its vector field, with the vectors scaled to all have the same length, is shown in Fig. 10.92. We aim to use the Poincaré-Bendixson Theorem to show this system has a cycle, which must enclose the only equilibrium point, located at $(0, 0)$. We'll use an idea suggested by our work on Lyapunov functions in the last section. If $E(x, y) = x^2 + y^2$, then the derivative of E following the motion of the trajectories is

$$\begin{aligned}\dot{E}(x,y) &= \frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt} \\ &= 2x(-y + x(1 - x^2 - y^2)) + 2y(x + y(1 - x^2 - y^2)) \\ &= 2(x^2 + y^2)(1 - (x^2 + y^2)).\end{aligned}$$

The level curves of E are circles centered at the origin. Since $\dot{E}$ is negative at points of the circle $x^2+y^2 = r$ for any $r > 1$, this tells us that trajectories crossing circles with radius greater than one do so moving in the direction of circles with smaller radii; i.e., they cross the circle $x^2+y^2 = r, r > 1$, pointing inwards. Similarly, since $\dot{E}$ is positive at points of the circle $x^2 + y^2 = r, r < 1$, as orbits cross this circle they are moving outwards, towards circles with larger radii. Thus if we choose C_i to be the circle $x^2 + y^2 = r_1$ where $r_1 < 1$, C_o to be the circle $x^2 + y^2 = r_2$ where $r_2 > 1$, and R to be the region between two circles C_i and C_o, together with the circles themselves, then an orbit that starts in R must stay in R. By the Poincaré-Bendixson Theorem, R must contain a cycle. This is true for any choice $r_1 < 1 < r_2$, which leaves us with the only possible conclusion that the unit circle is a cycle! We can verify this directly, by checking that $x(t) = \cos t, y(t) = \sin t$ is a solution; its orbit is included in Fig. 10.93.

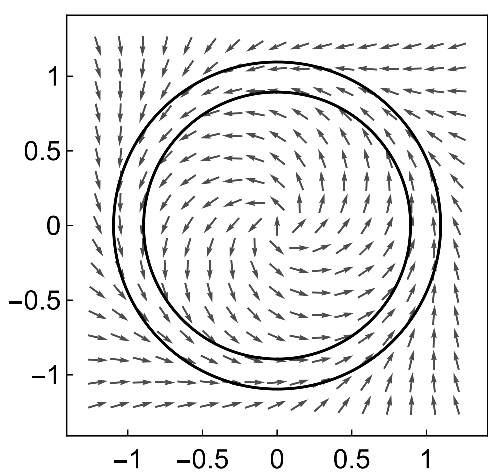


Figure 10.92. Vector field and circles $x^2 + y^2 = r_1$ and $x^2 + y^2 = r_2$ for $r_1 < 1 < r_2$.

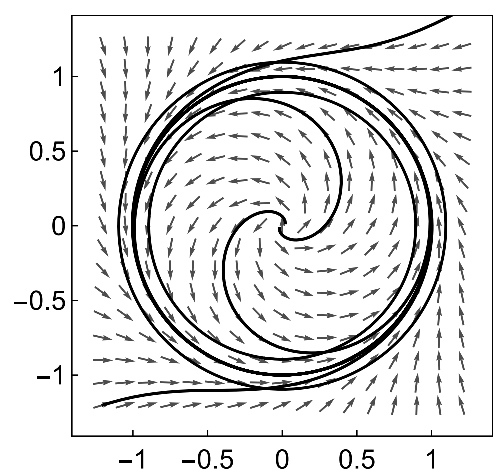


Figure 10.93. Some trajectories for Example 10.8.2.

Example 10.8.3. Consider the system

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -x - y \ln\left(9x^2 + 4y^2 + \frac{1}{2}\right).$$

We will show how to apply the Poincaré-Bendixson Theorem in the region R consisting of the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = \frac{1}{25}$ and the annular ring between them (see Fig. 10.94), to conclude this region contains a cycle. The only equilibrium point of our system is $(0, 0)$, and this lies outside of R.

Conversion to polar coordinates will make our calculations simpler. Note that in polar coordinates the boundary circles for R have equations $r = 1$ and $r = \frac{1}{5}$. Since $r^2 = x^2 + y^2$ and $x = r\cos\theta$, $y = r\sin\theta$, we have (as in equation (10.42)) $\frac{dr}{dt} = \frac{1}{r}(x\frac{dx}{dt} + y\frac{dy}{dt})$ and using the equations for the system,

$$\begin{aligned}\frac{dr}{dt} &= \frac{xy + y(-x - y\ln\left(9x^2 + 4y^2 + \frac{1}{2}\right))}{r} = \frac{-y^2\ln(9x^2 + 4y^2 + \frac{1}{2})}{r} \\ &= -r\sin^2\theta\ln\left(4r^2 + 5r^2\cos^2\theta + \frac{1}{2}\right).\end{aligned}$$

On the circle $r = 1$ we have

$$\frac{dr}{dt} = -\sin^2\theta\ln\left(4 + 5\cos^2\theta + \frac{1}{2}\right) \leq 0$$

for all θ, and r is nonincreasing. This tells us that no trajectory can leave the region R by crossing over the outer boundary $r = 1$ and moving farther away from the origin.

On the circle $r = \frac{1}{5}$ we have

$$\frac{dr}{dt} = -\frac{1}{5}\sin^2\theta\ln\left(\frac{4}{25} + \frac{5}{25}\cos^2\theta + \frac{1}{2}\right) \geq 0$$

for all θ, and r is nondecreasing. Thus no trajectory can leave the region R by crossing over the inner boundary $r = \frac{1}{5}$ and moving towards the origin. Thus a trajectory which is in R at $t = 0$ must stay in R for $t > 0$. By the Poincaré-Bendixson Theorem, there must be a cycle in R. Fig. 10.95 is a sketch of the phase portrait for this system.

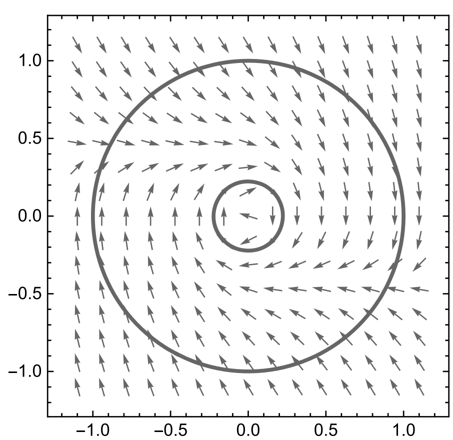


Figure 10.94. Scaled vector field for Example 10.8.3 and circles $x^2 + y^2 = 1$, $x^2 + y^2 = \frac{1}{25}$.

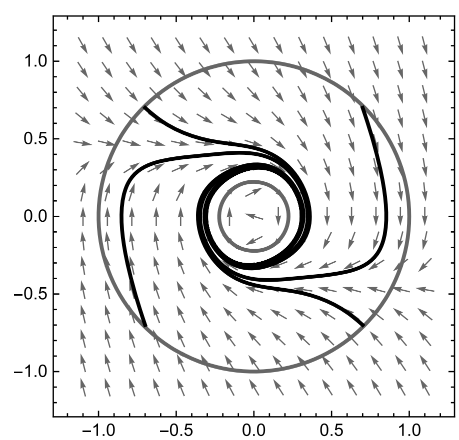


Figure 10.95. Phase portrait in the region R for Example 10.8.3.

Example 10.8.4. The nonlinear system

$$\frac{dx}{dt} = f(x, y) = y - x^3 + x, \qquad \frac{dy}{dt} = g(x, y) = -x \tag{10.99}$$

is an example of a van der Pol equation. Variations of the van der Pol equation are used in the study of nonlinear circuits, as well as in modeling the pumping action of the human heart. There is a single equilibrium point at the origin, and linear approximation shows that $(0, 0)$ is an unstable spiral point. We look for a cycle using the Poincaré-Bendixson Theorem. Fig. 10.96 suggests how this might be done. The vector field (with all vectors scaled to have the same length) is shown there. Focus on the region R, consisting of the six-sided polygon C_o with vertices at $(\frac{1}{2}, 3)$, $(3, 3)$, $(\frac{1}{2}, -2)$, $(-\frac{1}{2}, -3)$, $(-3, -3)$, and $(-\frac{1}{2}, 2)$, the circle C_i with equation $x^2 + y^2 = \frac{1}{4}$, and the annular region between them. We will show that any orbit which starts in R at $t = 0$ must remain in R for all later times $t > 0$. Theorem 10.8.1 then says that R must contain a cycle (which encircles the origin). Fig. 10.97, which shows several orbits of the van der Pol system, suggests correctly that there is a *unique* cycle, which attracts all nonconstant orbits.

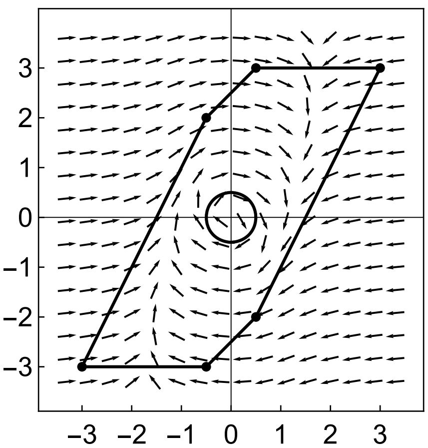


Figure 10.96. Scaled vector field for equation (10.99) with region R bounded by the circle and polygon.

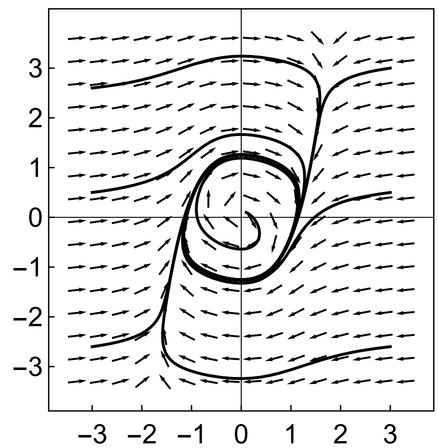


Figure 10.97. Phase portrait for Example 10.8.4.

To verify that no orbit that starts in R can leave R, either by escaping outward across C_o or inward across C_i, we will first show that at every point of C_o the vectors in the vector field point *into* the region R. Since C_o is a polygon, we will work on each of its six edges, as well as the six vertices, in turn. The basic principle we use is: *If the dot product of an outward pointing normal vector at a point of C_o and the vector field vector at that point is negative, then the vector field vector points into the region R at that point.* This follows since if the dot product is negative, the angle between the two vectors is greater than $\pi/2$; see Fig.10.98.

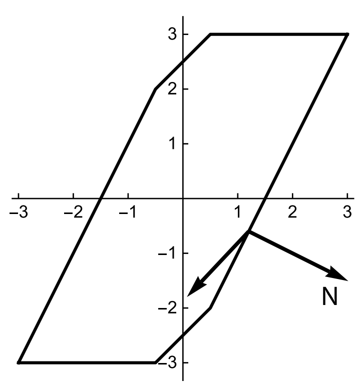


Figure 10.98. Outward normal vector and field vector at a point of C_o.

The top edge of C_o, between the vertices $(\frac{1}{2}, 3)$ and $(3, 3)$, is a horizontal line segment $y = 3$, $\frac{1}{2} < x < 3$. The vector $(0, 1)$ is perpendicular to this line segment and points outward from the region R; this serves as an outward normal vector at each point of this edge. The dot product of this outward normal with a vector in the vector field is

$$(y - x^3 + x, -x) \cdot (0, 1) = -x$$

and for all x with $\frac{1}{2} < x < 3$ this is negative, as desired. At the vertex $(\frac{1}{2}, 3)$ the field vector is $(3 + \frac{3}{8}, -\frac{1}{2})$, which clearly points into the region R. Similarly, at the vertex $(3, 3)$ the field vector is $(-21, -3)$, which points into R as well.

Next we consider the edge of the C_o joining $(3, 3)$ to $(\frac{1}{2}, -2)$. This is the portion of the line $y = 2x - 3$ between $x = \frac{1}{2}$ and $x = 3$. Since this line has direction vector $(1, 2)$, we can choose $N = (2, -1)$ as an outward normal vector at any point on this line. The dot product of this normal vector and a field vector is

$$(y - x^3 + x, -x) \cdot (2, -1) = 2y - 2x^3 + 3x = 2(2x - 3) - 2x^3 + 3x = 7x - 6 - 2x^3.$$

A calculus argument shows that the maximum value of $7x - 6 - 2x^3$ for $\frac{1}{2} < x < 3$ occurs at $x = \sqrt{7/6}$ and is approximately -0.96. Thus the dot product is negative for $-\frac{1}{2} < x < 3$ and the field vectors point into the region R along this segment. Checking the vertex $(\frac{1}{2}, -2)$ we compute the field vector there to be $(-\frac{13}{8}, -\frac{1}{2})$ and this is easily seen to point into the region R (notice that the segment from $(\frac{1}{2}, -2)$ to $(-\frac{1}{2}, -3)$ has slope 1).

The edge of C_o which joins the vertex $(\frac{1}{2}, -2)$ to the vertex $(-\frac{1}{2}, -3)$ has equation $y = x - \frac{5}{2}$ for $-\frac{1}{2} < x < \frac{1}{2}$. We can choose $(1, -1)$ as an outward normal vector at points along this segment. Computing the dot product

$$(y - x^3 + x, -x) \cdot (1, -1) = y - x^3 + 2x = x - \frac{5}{2} - x^3 + 2x = 3x - \frac{5}{2} - x^3.$$

The function $3x - \frac{5}{2} - x^3$ has no critical points in the interval $-\frac{1}{2} < x < \frac{1}{2}$ and is increasing in this interval, with value $-\frac{9}{8}$ at $x = \frac{1}{2}$. Thus $3x - \frac{5}{2} - x^3$ is negative for all $-\frac{1}{2} < x < \frac{1}{2}$, giving the desired conclusion that the field vectors point into R along this edge. We check the vertex $(-\frac{1}{2}, -3)$ separately, obtaining the field vector $(-\frac{27}{8}, \frac{1}{2})$ there; this clearly points into the region R.

The remaining three edges (and two vertices) of C_o are checked similarly. We leave the details to you in Exercise 7. Summarizing our work so far, we see that no orbit starting in R can exit R through C_o.

To finish, we show that no orbit which begins in R at $t = 0$ can exit R by crossing over the circle C_i. Set $E(x, y) = x^2 + y^2$ and compute the derivative of E along orbits:

$$\dot{E}(x, y) = 2x(y - x^3 + x) + 2y(-x) = 2x^2(1 - x^2).$$

Since this is nonnegative at every point (x, y) with $x^2 + y^2 < 1$, no orbit can cross the circle C_i moving towards the origin, giving the desired conclusion. Thus the Poincaré-Bendixson Theorem applies to show that R contains a cycle (pictured in Fig. 10.97) of the van der Pol system.

The Poincaré-Bendixson Theorem gives conditions under which you can conclude a region contains a cycle. The next result gives conditions under which a region is guaranteed to *not* contain a cycle.

Theorem 10.8.5. *Consider the system*

$$\frac{dx}{dt} = f(x, y), \qquad \frac{dy}{dt} = g(x, y),$$

where f and g are continuously differentiable. If there is a rectangle R in which the function

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$$

is either always strictly positive or always strictly negative, then there are no cycles in R.

The rectangle in this result is allowed to be the entire xy-plane (an infinite rectangle).[16] The next example shows such an application.

Example 10.8.6. We revisit the damped pendulum equations

$$\frac{d\theta}{dt} = v, \qquad \frac{dv}{dt} = -\frac{c}{m}v - \frac{g}{L}\sin\theta.$$

Here our dependent variables are θ and v instead of x and y. Adapting the statement of Theorem 10.8.5, we then consider

$$\frac{\partial}{\partial\theta}(v) + \frac{\partial}{\partial v}\left(-\frac{c}{m}v - \frac{g}{L}\sin\theta\right) = -\frac{c}{m}.$$

Since c and m are positive constants, this is negative at every point in the θv-plane. We conclude that the phase portrait for the damped pendulum system contains no cycles.

Theorem 10.8.5 can be proved in a few lines as an application of Green's Theorem from multivariable calculus; see Exercise 8.

10.8.1. Exercises.

1. Show that the system
$$\frac{dx}{dt} = -y + x(1-2x^2-3y^2), \qquad \frac{dy}{dt} = x + y(1-2x^2-3y^2)$$
has a cycle in the region between the circles $x^2+y^2=1$ and $x^2+y^2=\frac{1}{4}$.
2. Use the Poincaré-Bendixson Theorem to show that the system
$$\frac{dx}{dt} = 3x+3y-x(x^2+y^2), \qquad \frac{dy}{dt} = -3x+3y-y(x^2+y^2)$$
has a cycle in the region R between the two circles $x^2+y^2=2$ and $x^2+y^2=4$. Then verify this by finding values of a and b so that
$$x = -a\cos(bt), \qquad y = a\sin(bt)$$
is a solution whose orbit lies in R.
3. Find values r_1 and r_2 such that
$$\frac{dx}{dt} = 2x+y-xe^{x^2+y^2}, \qquad \frac{dy}{dt} = -x+2y-ye^{x^2+y^2}$$
has a cycle in the region between the two circles $x^2+y^2=r_1$ and $x^2+y^2=r_2$.
4. Modify the argument in Example 10.8.3 to show that there is a cycle in the region between the ellipses $36x^2+16y^2=1$ and $\frac{9}{2}x^2+2y^2=1$.
5. Show that the system
$$\frac{dx}{dt} = -xy+x, \qquad \frac{dy}{dt} = xy+y$$
has no cycles in the right half-plane $\{(x,y): x>0\}$.
6. Show that the system (10.59) used in Example 10.5.2 to model the spread of a sexually transmitted disease has no cycles in the open first quadrant $\{(x,y): x>0, y>0\}$.
7. In this problem you will finish the analysis of the polygon C_o in Example 10.8.4.
   (a) Find an outward normal vector at each point of the edge joining $(-\frac{1}{2},-3)$ to $(-3,-3)$ and verify that the dot product of this normal vector with the vector field is negative for $-3<x<-\frac{1}{2}$, so that the vector field points into R at points of this edge.
   (b) Show that the vector field at the vertex $(-3,-3)$ points into R.

[16] In fact, the rectangle R in Theorem 10.8.5 can be replaced by a simply connected region.

(c) Find an outward normal vector at each point of the edge joining $(-3,-3)$ to $(-\frac{1}{2},2)$ and verify that the dot product of this normal vector with the vector field is negative for $-3 < x < -\frac{1}{2}$.

(d) Show that the vector field at the vertex $(-\frac{1}{2},2)$ points into R.

(e) Find an outward normal vector at each point of the edge joining $(-\frac{1}{2},2)$ to $(\frac{1}{2},3)$ and verify that the dot product of this normal vector with the vector field is negative for $-\frac{1}{2} < x < \frac{1}{2}$.

8. For a simple closed curve C enclosing a region Ω as shown in Fig. 10.99, Green's Theorem says that

$$\int_C f(x,y)dy - g(x,y)dx = \int\int_\Omega \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dxdy.$$

The left side of this identify is a line integral over the curve C and the right side is a double integral over the region Ω bounded by C.

(a) Suppose that the hypothesis of Theorem 10.8.5 hold for some rectangle R and suppose that C is a cycle in R. Use Green's Theorem to explain why

$$\int_C f(x,y)dy - g(x,y)dx \neq 0.$$

(b) Directly compute the line integral

$$\int_C f(x,y)dy - g(x,y)dx$$

using the fact that C is a cycle and therefore parametrized by a solution $x = x(t), y = y(t)$ of the system featured in Theorem 10.8.5. You should have reached a contradiction to the result in (a); hence there can be no cycle in R.

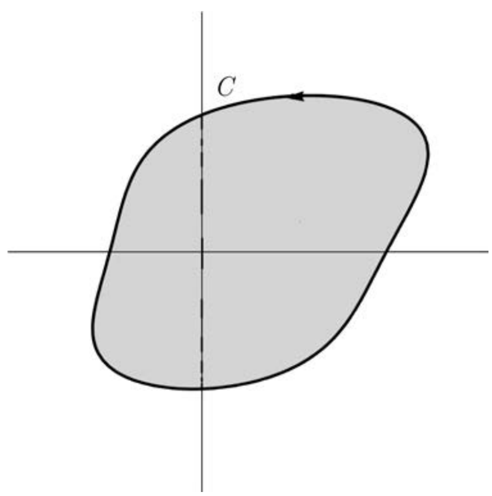


Figure 10.99. The region Ω bounded by the simple closed curve C.

9. (a) Write the second-order equation

$$x'' + p(x)x' + q(x) = 0$$

as a system by setting $y = x'$.

(b) Show that if $p(x) > 0$ for all x, then the system in (a) cannot have any periodic solution. Assume $p(x)$ and $q(x)$ are continuously differentiable.

(c) Explain why the damped pendulum is a special case of (a)–(b).