Chapter XII

Use of Implicit Function and Fixed Point Theorems

Many different problems in the theory of differential equations are solved by the use of implicit function theory—either of the classical type or of a more general type involving fixed point theorems and/or functional analysis. This will be illustrated in this chapter. Part I deals with the existence of periodic solutions of linear and nonlinear differential equations. Part II deals with solutions of certain second order boundary value problems. In Part III, a general abstract theory is formulated. Use of this general theory is illustrated by an application to a problem of asymptotic integration.

Although Parts I and II are applications of the general theory of Part III, there are several reasons for giving them separate treatments. The first reason is the importance and comparative simplicity of the situations involved. The second reason is that Parts I and II serve as motivation for the somewhat abstract theory of Part III. The third and most important reason is the fact that, as usual, a general theory in the theory of differential equations only provides a guide for the procedure to be followed. Its use in a particular situation generally involves important problems of obtaining appropriate estimates in order to establish the applicability of the general theory.

Two general theorems will be used. The first is a very simple fact:

Theorem 0.1. *Let $\mathfrak{D}$ be a Banach space of elements $x, y, \ldots$ with norms $|x|, |y|, \ldots$. Let T_0 be a map of the ball $|x| \leqq \rho$ in $\mathfrak{D}$ into $\mathfrak{D}$ satisfying $|T_0[x] - T_0[y]| \leqq \theta\,|x - y|$ for some θ, $0 < \theta < 1$. Let $m = |T_0[0]|$ and $m \leqq \rho(1 - \theta)$. Then there exists a unique fixed point x_0 of T_0, i.e., a unique point x_0 satisfying $T_0[x_0] = x_0$. In fact, x_0 can be obtained as the limit of successive approximations $x_1 = T_0[0]$, $x_2 = T_0[x_1]$, $x_3 = T_0[x_2], \ldots$.*

Remark. If T_0 maps the ball $|x| \leqq \rho$ into itself, then the condition $m \leqq \rho(1 - \theta)$ can be omitted.

Exercise 0.1. Verify this theorem and the Remark.

A much more sophisticated fixed point theorem is the following:

Theorem 0.2 (Tychonov). *Let $\mathfrak{D}$ be a linear, locally convex, topological space. Let S be a compact, convex subset of $\mathfrak{D}$ and T_0 a continuous map of S into itself. Then T_0 has a fixed point $x_0 \in S$, i.e., $T_0[x_0] = x_0$.*

The following corollary of this will be used subsequently.

Corollary 0.1. *Let $\mathfrak{D}$ be a linear, locally convex, topological, complete Hausdorff space (e.g., let $\mathfrak{D}$ be a Banach or a Fréchet space). Let S be a closed, convex subset of $\mathfrak{D}$ and T_0 a continuous map of S into itself such that the image T_0S of S has a compact closure. Then T_0 has a fixed point $x_0 \in S$.*

Theorem 0.2 was first proved by Schauder under the assumption that $\mathfrak{D}$ is a Banach space and this case of the theorem is usually called "Schauder's fixed point theorem." For a proof of Theorem 0.2, see Tychonov [1].

Parts I and II will use the cases of Corollary 0.1 when $\mathfrak{D}$ is the Banach space C^0, C^1. Part III will use the case when $\mathfrak{D}$ is a simple Fréchet space, namely, the space of continuous functions on $J : 0 \leqq t < \omega \,(\leqq \infty)$ with the topology of uniform convergence on closed intervals in J.

Corollary 0.1 is obtained from Theorem 0.2 in the following way: Let $\mathfrak{D}$, S, T_0 be as in Corollary 0.1. Let S_1 be the closure of T_0S, so that S_1 is compact. Also $S_1 \subset S$ since S is closed. Under the assumptions on $\mathfrak{D}$, the convex closure of S_1 (i.e., the smallest closed convex set containing S_1) is compact since S_1 is. (This is an immediate consequence of Arzela's theorem in the applications below; cf., e.g., the Remark following the proof of Theorem 2.2.) Let S^0 denote this convex closure of S_1. Since S is convex $S^0 \subset S$. Thus T_0 is a continuous map of the convex compact S^0 into itself (in fact, $T_0S^0 \subset T_0S \subset S_1 \subset S^0$) and the corollary follows from Theorem 0.2.

Part III will depend on the "open mapping theorem" in functional analysis. This theorem will be used in the following form:

Theorem 0.3 (Open Mapping Theorem). Let X_1, X_2 *be Banach spaces and T_0 a linear operator from X_1 onto X_2 with a domain $\mathscr{D}(T_0)$, which is necessarily a linear manifold in X_1, and range $\mathscr{R}(T_0) = X_2$. Let T_0 be a closed operator, i.e., let the graph of T_0, $\mathscr{G}(T_0) = \{(x_1, T_0x_1) : x_1 \in \mathscr{D}(T_0)\}$ be a closed set in the Banach space $X_1 \times X_2 = \{(x_1, x_2) : x_1 \in X_1, x_2 \in X_2\}$ with norm $|(x_1, x_2)| = \max(|x_1|, |x_2|)$. Then there exists a constant K with the property that, for every $x_2 \in X_2$, there is at least one $x_1 \in \mathscr{D}(T_0)$ such that $T_0x_1 = x_2$ and $|x_1| \leqq K\,|x_2|$. [In particular, when T_0 is one-to-one, so that x_1 is unique, then $|x_1| \leqq K\,|T_0x_1|$ holds for all $x_1 \in \mathscr{D}(T_0)$.]*

For a proof of the open mapping theorem in the form that "if P is a continuous, linear map from a Banach space X to another Banach space X_2 with domain $\mathscr{D}(P) = X$ and range $\mathscr{R}(P) = X_2$, then P maps open sets into open sets," see Banach [1, pp. 38–40]. Theorem 0.3 results by applying this to the projection map $P : \mathcal{G}(T_0) \to X_2$, where $P(x_1, T_0x_1) = T_0x_1$ and noting that a sphere about the origin in $\mathcal{G}(T_0)$ has a P-image which contains a sphere about the origin in X_2.

As a motivation for the procedures to be followed consider the problem of finding a solution of the differential equation

$$y' = f^0(t, y) \tag{0.1}$$

in a certain set S of functions $y(t)$. Write this differential equation as

$$y' = A(t)y + f(t, y), \qquad \text{where} \quad f(t, y) = f^0(t, y) - A(t)y,$$

for some choice of $A(t)$. Suppose that for every $x(t) \in S$, the equation

$$y' = A(t)y + f(t, x(t)) \tag{0.2}$$

has a solution $y(t) \in S$. Define an operator $T_0 : S \to S$ by putting $y(t) = T_0[x(t)]$, where $y(t) \in S$ is a suitably selected solution of (0.2). It is clear that a fixed point $y_0(t)$ of T_0 [i.e., $T_0[y_0(t)] = y_0(t)$] is a solution of (0.1) in S.

For the applicability of the theorems just stated, it will be assumed that S is a subset of a suitable topological vector space $\mathfrak{D}$. It will generally be convenient to introduce another space $\mathfrak{B}$ and two operators L and T_1. The operator L is the linear differential operator $L[y] = y' - A(t)y$, so that $g(t) = L[y(t)]$ if

$$y' = A(t)y + g(t). \tag{0.3}$$

It will also be assumed that if $x(t) \in S$, then $g(t) = f(t, x(t))$ is in $\mathfrak{B}$ and $T_1 : S \to \mathfrak{B}$ is defined by $g(t) = T_1[x(t)]$. Investigations of T_0 are then reduced to examinations of the linear differential operator L and of the nonlinear operator T_1.

The applicability of Theorem 0.1 can arise in the following type of situation: Suppose that $\mathfrak{B}$, $\mathfrak{D}$, are Banach spaces and that $|g|_{\mathfrak{B}}$, $|y|_{\mathfrak{D}}$ denote the norms of elements $g \in \mathfrak{B}$, $y \in \mathfrak{D}$, respectively. Assume that for every $g(t) \in \mathfrak{B}$, the equation (0.3) (i.e., $L[y] = g$) has a unique solution $y(t) \in S \subset \mathfrak{D}$, that $y(t)$ depends linearly on $g(t)$, and that there exists a constant K such that $|y|_{\mathfrak{D}} \leqq K|g|_{\mathfrak{B}}$. Suppose that, for the map $T_1 : S \to \mathfrak{B}$ there is a constant θ such that $|T_1[x_1] - T_1[x_2]|_{\mathfrak{B}} \leqq \theta\,|x_1 - x_2|_{\mathfrak{D}}$ for $x_1, x_2 \in S$. Then T_0 satisfies $|T_0[x_1(t)] - T_0[x_2(t)]|_{\mathfrak{D}} \leqq \theta K\,|x_1 - x_2|_{\mathfrak{D}}$. According to Theorem 0.1, the sequence of successive approximations

$$x_1, x_2 = T_0[x_1], x_3 = T_0[x_2], \ldots$$

will converge to a fixed point of T_0 (under suitable conditions on S, x_1, and θK).

In some situations, the equation $L[y] = g$ may have solutions y satisfying $|y|_{\mathfrak{D}} \leqq K|g|_{\mathfrak{B}}$ although y is not unique; cf., e.g., Theorem 0.3. In this case, y need not depend linearly on g but it might be possible to form convergent successive approximations in the following way: For a given x_1, let $x_2 = y$ be a solution of $L[y] = T_1[x_1(t)]$. If $x_1, x_2, \ldots, x_{n-1}$ have been defined for $n > 2$, determine an x_n from the equation $L[x_n - x_{n-1}] = T_1[x_{n-1}] - T_1[x_{n-2}]$ and the inequality $|x_n - x_{n-1}|_{\mathfrak{D}} \leqq K|T_1[x_{n-1}] - T_1[x_{n-2}]|_{\mathfrak{B}}$. This situation will not arise below.

When the inequality $|T_1[x_1(t)] - T_1[x_2(t)]|_{\mathfrak{B}} \leqq \theta|x_1 - x_2|_{\mathfrak{D}}$ is not available, Theorem 0.2 may still be applicable to assure the existence of a fixed point of T_0.

PART I. PERIODIC SOLUTIONS

1. Linear Equations

In this section, unless otherwise specified, the components of the d-dimensional vectors y, z are real- or complex-valued. Let $p > 0$ be fixed. Consider an inhomogeneous system of linear equations

$$y' = A(t)y + g(t) \tag{1.1}$$

and the corresponding homogeneous system

$$y' = A(t)y, \tag{1.2}$$

where $A(t)$ is a continuous $d \times d$ matrix and $g(t)$ a continuous vector-valued function for $0 \leqq t \leqq p$. In addition, consider a set of boundary conditions

$$My(0) - Ny(p) = 0, \tag{1.3}$$

where M, N are constant $d \times d$ matrices. For example, if $M = N = I$ and $A(t)$, $g(t)$ are periodic of period p, then a solution $y(t)$ of (1.1) or (1.2) satisfying (1.3) is of period p.

Lemma 1.1. *Let $A(t)$ be continuous for $0 \leqq t \leqq p$ and M, N constant $d \times d$ matrices. Let $Y(t)$ be a fundamental matrix for (1.2). Then a necessary and sufficient condition for (1.2) to have a nontrivial ($\not\equiv 0$) solution satisfying (1.3) is that $MY(0) - NY(p)$ be singular. In fact, the number k, $0 \leqq k \leqq d$, of linearly independent solutions of (1.2), (1.3) is the number of linearly independent vectors c satisfying*

$$[MY(0) - NY(p)]c = 0; \tag{1.4}$$

i.e., $d - k = \text{rank } [MY(0) - NY(p)]$.

This is clear since the general solution of (1.2) is $y = Y(t)c$.

Exercise 1.1. Let $A(t)$ be periodic of period p and

$$Y(t) = Z(t)e^{Rt}, \qquad \text{where} \quad Z(t+p) = Z(t) \tag{1.5}$$

and R is a constant matrix; cf. the Floquet theory in § IV 6. Then (1.2) has a nontrivial ($\not\equiv 0$) solution of period p if and only if $\lambda = 1$ is a characteristic root of (1.2); i.e., $e^{Rp} - I$ is singular. In fact, the number of linearly independent solutions of period p is the number of linearly independent solutions c of

$$[Y(0) - Y(p)]c = 0, \qquad \text{i.e.,} \quad (e^{Rp} - I)c = 0. \tag{1.6}$$

For algebraic linear equations, the inhomogeneous system $Cy = g$ has a solution y for every g if and only if the only solution of $Cy = 0$ is $y = 0$. The analogous situation is valid here.

Theorem 1.1. *Let $A(t)$ be continuous for $0 \leqq t \leqq p$; M, N constant $d \times d$ matrices such that the $d \times 2d$ matrix (M, N) is of rank d. Then* (1.1) *has a solution $y(t)$ satisfying* (1.3) *for every continuous $g(t)$ if and only if* (1.2), (1.3) *has no nontrivial* ($\not\equiv 0$) *solution; in which case $y(t)$ is unique and there exists a constant K, independent of $g(t)$, such that*

$$\|y(t)\| \leqq K\int_0^p \|g(s)\|\, ds \qquad \text{for} \quad 0 \leqq t \leqq p. \tag{1.7}$$

Proof. The general solution of (1.1) is given by

$$y(t) = Y(t)\left\{c + \int_0^t Y^{-1}(s)g(s)\, ds\right\}; \tag{1.8}$$

Corollary IV 2.1. This solution satisfies (1.3) if and only if

$$[MY(0) - NY(p)]c = NY(p)\int_0^p Y^{-1}(s)g(s)\, ds. \tag{1.9}$$

Assume that (1.2), (1.3) has no nontrivial solution. Then, by Lemma 1.1, the matrix $V = MY(0) - NY(p)$ is nonsingular, thus (1.9) has a unique solution. Substituting this value of c in (1.8) gives the unique solution of (1.1), (1.3):

$$y(t) = Y(t)\left\{V^{-1}N\int_0^p Y^{-1}(s)g(s)\, ds + \int_0^t Y^{-1}(s)g(s)\, ds\right\}. \tag{1.10}$$

It is clear that there exists a constant K satisfying (1.7) for $0 \leqq t \leqq p$.

This proves one-half of Theorem 1.1 (and this part did not use the assumption that rank $(M, N) = d$). The converse follows from Theorem 1.2.

Exercise 1.2. What is the Green's function $G(t, s)$ in the last part of Theorem 1.1, i.e., what is the function $G(t, s)$, $0 \leqq s, t \leqq p$, such that

$$y(t) = \int_0^p G(t, s)g(s)\, ds$$

is the unique solution (1.10) of (1.1), (1.3)?

Consider the equations adjoint to (1.1), (1.2)

$$z' + A^*(t)z + h(t) = 0, \tag{1.11}$$

$$z' + A^*(t)z = 0, \tag{1.12}$$

where A^* is the complex conjugate transpose of A; cf. § IV 7. Consider also a set of boundary conditions

$$Pz(0) - Qz(p) = 0, \tag{1.13}$$

where P, Q are constant $d \times d$ matrices. If $y(t)$ is a solution of (1.1) and $z(t)$ a solution (1.11), the Green formula (IV 7.3) is

$$\int_0^p [g(s) \cdot z(s) - y(s) \cdot h(s)]\, ds = [y(t) \cdot z(t)]_0^p. \tag{1.14}$$

When do the boundary conditions (1.3) and (1.13) imply that

$$y(p) \cdot z(p) - y(0) \cdot z(0) = 0, \tag{1.15}$$

i.e., that the right side of (1.14) is 0? Note that if M, Q are nonsingular, then this is the case if and only if $0 = y(p) \cdot Q^{-1}Pz(0) - M^{-1}Ny(p) \cdot z(0) = (P^*Q^{*-1} - M^{-1}N)y(p) \cdot z(0) = [M^{-1}(MP^* - NQ^*)Q^{*-1}]y(p) \cdot z(0)$. In this case, necessary and sufficient for (1.3), (1.13) to imply (1.15) is that

$$MP^* - NQ^* = 0. \tag{1.16}$$

Lemma 1.2. *Let M, N be constant $d \times d$ matrices such that* rank$(M, N) = d$. *Then there exist $d \times d$ matrices P, Q* satisfying rank $(P, Q) = d$, (1.16), *and having the property that the relations* (1.3), (1.13) *imply* (1.15). *The pairs of vectors $z(0)$, $z(p)$ satisfying* (1.13) *are independent of the choice of P, Q.*

Proof. Since rank $(M, N) = d$, there exist $d \times d$ matrices M_1, N_1 such that the $2d \times 2d$ matrix

$$W = \begin{pmatrix} M & -N \\ M_1 & N_1 \end{pmatrix} \tag{1.17}$$

is nonsingular. Write the inverse of W as

$$W^{-1} = \begin{pmatrix} P_1^* & P^* \\ Q_1^* & Q^* \end{pmatrix} \quad \text{or} \quad W^{*-1} = \begin{pmatrix} P_1 & Q_1 \\ P & Q \end{pmatrix},$$

so that (1.16) holds and rank $(P, Q) = d$.

Let y_1, y_2, z_1, z_2 be d-dimensional vectors and $\eta = (y_1, y_2)$, $\zeta = (z_1, z_2)$ be corresponding $2d$-dimensional vectors. Then

$$\eta \cdot \zeta = W^{-1}W\eta \cdot \zeta = W\eta \cdot W^{*-1}\zeta; \tag{1.18}$$

thus

$$My_1 - Ny_2 = 0, \qquad Pz_1 + Qz_2 = 0 \qquad \text{imply that} \quad \eta \cdot \zeta = 0. \tag{1.19}$$

The choices $y_1 = y(0)$, $y_2 = y(p)$, $z_1 = z(0)$, $z_2 = -z(p)$ show that (1.3), (1.13) imply (1.16). This completes the existence proof.

The formulation (1.19) of the implication (1.3), (1.13) $\Rightarrow$ (1.16) makes the last part of the lemma clear. For if $\eta = (y_1, y_2) \neq 0$ satisfies $My_1 - Ny_2 = 0$, then $M_1y_1 + N_1y_2 \neq 0$. In fact, since rank $(P, Q) = d$, the set of vectors $\zeta = (z(0), -z(p))$ satisfying $Pz(0) - Qz(p) = 0$ is the set of vectors satisfying $\eta \cdot \zeta = 0$ for all $\eta = (y_1, y_2)$ such that $My_1 - Ny_2 = 0$. Since this set of vectors $\zeta = (z(0), -z(p))$ is determined by M, N, the proof of the theorem is complete.

Boundary conditions (1.13) satisfying the conditions of Lemma 1.2 will be called the *adjoint boundary conditions of* (1.3). Correspondingly, the problems (1.2)–(1.3) and (1.12)–(1.13) will be called "*adjoint problems*." (Note that the adjoint of the "periodic boundary conditions" $y(p) = y(0)$, i.e., $M = N = I$, are equivalent to the "periodic conditions" $z(p) = z(0)$, i.e., $P = Q = I$.)

There is an analogue of the algebraic fact that if C is a $d \times d$ matrix, then the number of linearly independent solutions of $Cy = 0$ and of the "adjoint" equation $C^*z = 0$ is the same:

Lemma 1.3. *Let $A(t)$ be continuous for $0 \leqq t \leqq p$; M, N constant $d \times d$ matrices such that rank $(M, N) = d$; and* (1.13) *boundary conditions adjoint to* (1.3). *Then* (1.2)–(1.3) *and* (1.12)–(1.13) *have the same number of linearly independent solutions.*

Proof. Since the relationship between (1.2)–(1.3) and (1.12)–(1.13) is symmetric, it suffices to show that if (1.12)–(1.13) has k linearly independent solutions, where $0 \leqq k \leqq d$, then (1.2)–(1.3) has at least k linearly independent solutions.

Let $Y(t)$ be a fundamental matrix of (1.2), then $Y^{*-1}(t)$ is a fundamental solution of (1.12) by Lemma IV 7.1. In terms of (1.17), define a constant $2d \times 2d$ matrix

$$U = W \operatorname{diag}[Y(0), Y(p)] = \begin{pmatrix} MY(0) & -NY(p) \\ M_1Y(0) & N_1Y(p) \end{pmatrix}; \tag{1.20}$$

so that U is nonsingular and

$$U^{*-1} = W^{*-1} \operatorname{diag}[Y^{*-1}(0), Y^{*-1}(p)] = \begin{pmatrix} P_1Y^{*-1}(0) & Q_1Y^{*-1}(p) \\ PY^{*-1}(0) & QY^{*-1}(p) \end{pmatrix}.$$

Thus, if c_0 is a constant d-dimensional vector such that $z(t) = Y^{*-1}(t)c_0$ is a solution of (1.12)–(1.13), then $U^{*-1}(c_0, -c_0) = (b, 0)$. Here b is a d-dimensional vector, and if c_0 varies over a set of k linearly independent vectors, then b varies over a set of k linearly independent vectors, since U^{*-1} is nonsingular. From (1.20), it is easy to see that the equation $(c_0, -c_0) = U^*(b, 0)$ gives

$$c_0 = Y^*(0)M^*b = Y^*(p)N^*b, \tag{1.21}$$

so that

$$[Y^*(0)M^* - Y^*(p)N^*]b = 0. \tag{1.22}$$

Hence the matrix $Y^*(0)M^* - Y^*(p)N^*$ annihilates k linearly independent vectors b; therefore, the same is true of its complex conjugate transpose $MY(0) - NY(p)$. In view of Lemma 1.1, this proves Lemma 1.3.

Remark. For the purpose of the next proof, note that the lemma just proved implies that (1.22) holds if and only if the vector c_0 in (1.21) is such that the solution $z = Y^{*-1}(t)c_0$ of (1.12) satisfies (1.13).

Another algebraic fact is that if C is a singular matrix, then $Cy = g$ has a solution y if and only if g is orthogonal (i.e., $g \cdot z = 0$) to all solutions z of the homogeneous "adjoint" system $C^*z = 0$. Again an analogous situation is valid here:

Theorem 1.2. *Let $A(t)$ be continuous for $0 \leqq t \leqq p$, M and N constant $d \times d$ matrices such that* rank $(M, N) = d$, *and let* (1.2)–(1.3) *and* (1.12)–(1.13) *be adjoint problems. Suppose that* (1.2)–(1.3) *has exactly k linearly independent solutions $y_1(t), \ldots, y_k(t)$ and let $z_1(t), \ldots, z_k(t)$ be linearly independent solutions of* (1.12)–(1.13). *Let $g(t)$ be continuous for $0 \leqq t \leqq p$. Then* (1.1) *has a solution $y_0(t)$ satisfying* (1.3) *if and only if*

$$\int_0^p g(s) \cdot z_j(s)\, ds = 0 \qquad \text{for} \quad j = 1, \ldots, k. \tag{1.23}$$

In this case, the solutions of (1.1), (1.3) *are given by $y_0(t) + \alpha_1 y_1(t) + \cdots + \alpha_k y_k(t)$, where $\alpha_1, \ldots, \alpha_k$ are arbitrary constants.*

Proof. Note that, by the proof of Theorem 1.1, the problem (1.1), (1.3) has a solution if and only if (1.9) has a solution c. This is the case if and only if

$$\left(NY(p)\int_0^p Y^{-1}(s)g(s)\, ds\right) \cdot b = 0$$

for all solutions b of (1.22). In view of (1.21), this is equivalent to the condition that

$$0 = \int_0^p [g(s) \cdot Y^{*-1}(s)Y^*(p)N^*b]\, ds = \int_0^p g(s) \cdot z(s)\, ds$$

for all solutions $z = Y^{*-1}(s)c_0$ of (1.12)–(1.13), i.e., that (1.23) holds. This proves the theorem.

The next theorem is a rather particular result for the case that $A(t)$, $g(t)$ are of period p.

Theorem 1.3. *Let $A(t)$ be continuous and of period p. Then, for a fixed continuous $g(t)$ of period p,* (1.1) *has a solution of period p if and only if* (1.1) *has at least one bounded solution for $t \geqq 0$.*

Proof. The necessity of the existence of a bounded solution is clear. In order to prove the converse, assume that (1.1) has a solution $y(t)$ bounded for $t \geqq 0$. Let $Y(t)$ be the fundamental matrix of (1.2) satisfying $Y(0) = I$. Then (1.1) has a solution of period p if and only if the equation $c = Y(p)c + b$, where

$$b = Y(p)\int_0^p Y^{-1}(s)g(s)\,ds,$$

has a solution c; cf. (1.9) in the proof of Theorem 1.1.

If $c = y(0)$ in (1.8), then $y(p) = Y(p)y(0) + b$ holds for every solution $y(t)$ of (1.1). Since $y(t + p)$ is also a solution, $y(2p) = Y(p)y(p) + b = Y^2(p)y(0) + Y(p)b + b$, or more generally,

$$y(np) = Y^n(p)y(0) + \left(\sum_{k=0}^{n-1} Y^k(p)\right)b.$$

Suppose, if possible, that $[I - Y(p)]c = b$ has no solution. Then $[Y(p) - I]^*$ is singular and there exists a vector c_0 such that $[Y(p) - I]^*c_0 = 0$ and $b \cdot c_0 \neq 0$. Thus $c_0 = Y^*(p)c_0$ and $c_0 = (Y^k(p))^*c_0$ for $k = 0, 1, \ldots$. Multiply the equation in the last formula line scalarly by c_0 to obtain

$$y(np) \cdot c_0 = y(0) \cdot c_0 + n(b \cdot c_0),$$

since $Y^k(p)y(0) \cdot c_0 = y(0) \cdot (Y^k(p))^*c_0$. As $b \cdot c_0 \neq 0$ and the sequence $y(p)$, $y(2p), \ldots$ is bounded, a contradiction results. This proves the theorem.

2. Nonlinear Problems

This section deals with the existence of periodic solutions for nonlinear systems. With very minor changes, the methods and results are applicable to the situation when the requirement of "periodicity" is replaced by boundary conditions of the type (1.3). The results depend on those of the last section for linear equations and, in particular, on the "a priori bound" for certain solutions of (1.1) given by (1.7). The first two theorems concern a nonlinear system of the form

$$y' = A(t)y + f(t, y) \tag{2.1}$$

in which y is a vector with real- or complex-valued components.

Theorem 2.1. *Let $A(t)$ be continuous and periodic of period p and such that* (1.2) *has no nontrivial solution of period p. Let K be as in* (1.7) *in Theorem* 1.1, *where $M = N = I$. Let $f(t, y)$ be continuous for all (t, y), of period p in t for fixed y, and satisfy a Lipschitz condition of the form*

$$\|f(t, y_1) - f(t, y_2)\| \leqq \theta \, \|y_1 - y_2\| \tag{2.2}$$

for all t, y_1, y_2 with a Lipschitz constant θ so small that $K\theta p < 1$. Then (2.1) *has a unique solution of period p.*

Actually, it is not necessary that $f(t, y)$ be defined for all y. If $m = \max \|f(t, 0)\|$, it is sufficient to require that $f(t, y)$ be defined for $\|y\| \leqq r$, where

$$\frac{Kpm}{1 - K\theta p} \leqq r. \tag{2.3}$$

Proof. Introduce the Banach space $\mathfrak{D}$ of continuous periodic functions $g(t)$ of period p with the norm $|g| = \max \|g(t)\|$. Thus convergence of $g_1(t), g_2(t), \ldots$ in $\mathfrak{D}$ is equivalent to the usual uniform convergence over $0 \leqq t \leqq p$.

Let $g(t)$ be a continuous function of period p satisfying $\|g(t)\| \leqq r$. Thus by Theorem 1.1 the equation

$$y' - A(t)y = f(t, g(t)) \tag{2.4}$$

has a unique solution $y(t)$ of period p. Define an operator T_0 on the set of all such $g(t)$ by putting $y(t) = T_0[g]$. Note that (1.7), (2.4) and (2.2) show that if $z(t) = T_0[h]$, then

$$|y - z| \leqq Kp\theta \, |g - h|; \qquad \text{i.e.,} \quad |T_0[g] - T_0[h]| \leqq Kp\theta \, |g - h|, \tag{2.5}$$

where $|y| = \max \|y(t)\|$ for $0 \leqq t \leqq p$. In addition, if $m = \max \|f(t, 0)\|$, then $|T_0[0]| \leqq Kpm$.

Thus Theorem 2.1 follows from Theorem 0.1, for $y_0(t)$ is a fixed point of T_0, $T_0[y_0] = y_0$, if and only if $y_0(t)$ is a solution of (2.1) of period p; cf. (2.4) where $y = T_0[g]$.

In Theorem 2.1, we can omit assumption (2.2) when $\|f(t, y)\|$ is "small," at the cost of losing "uniqueness."

Theorem 2.2. *Let $A(t)$, K be as in Theorem* 2.1. *Let $f(t, y)$ be continuous for all t and $\|y\| \leqq r$, of period p in t for fixed y, and satisfy*

$$Kp \, \|f(t, y)\| \leqq r \qquad \text{for} \quad 0 \leqq t \leqq p, \quad \|y\| \leqq r. \tag{2.6}$$

Then (2.1) *has at least one periodic solution of period p.*

Proof. As in the last proof, define $y(t) = T_0[g]$ as the unique solution of (2.4) of period p, where $g(t)$ is of period p and $|g| \leqq r$. In order to prove the theorem, it suffices to show that T_0 has a fixed point y_0, $T_0[y_0] = y_0$. This will be proved by an appeal to Corollary 0.1 of Tychonov's theorem.

It follows from (1.7) and (2.6) that $y = T_0[g]$ satisfies $|y| \leqq r$. In other words, if $\mathfrak{D}$ is the same Banach space as in the last proof, then T_0 maps the sphere $|g| \leqq r$ of $\mathfrak{D}$ into itself. Also, (1.7) gives

$$|T_0[g] - T_0[h]| \leqq K \int_0^p \|f(t, g(t)) - f(t, h(t))\| \, dt.$$

Since f is continuous, it is clear that if $|g - h| = \max \|g(t) - h(t)\| \to 0$, then $T_0[g] - T_0[h] \to 0$. Thus T_0 is a continuous map.

If $y = T_0[g]$, then $\|y(t)\| \leqq r$ and (2.4) show that there is a constant C, independent of g, such that $\|y'(t)\| \leqq C$. This implies that the set of functions $y(t) = T_0[g]$ in the range of T_0 is bounded and equicontinuous. Hence, by Arzela's theorem, it has a compact closure in $\mathfrak{D}$ (i.e., any sequence $y_1, y_2, \ldots$ has a uniformly convergent subsequence). Consequently, Corollary 0.1 implies that T_0 has a fixed point y_0. Clearly $y = y_0(t)$ is a periodic solution of period p. This proves the theorem.

Remark. In the deduction of Corollary 0.1 from the Tychonov Theorem 0.2, it is necessary to know that the convex closure of the range $\mathscr{R}(T_0)$ of T_0 is compact. This is clear in the proof just completed, for $y(t)$ in the range of T satisfies the conditions: (i) $y(t)$ is continuous of period p; (ii) $\|y(t)\| \leqq r$; and (iii) $\|y(t) - y(s)\| \leqq C\,|t - s|$. The convex hull of $\mathscr{R}(T_0)$ [i.e., the smallest convex set containing $\mathscr{R}(T_0)$] is the set of functions $y(t)$ representable in the form $\lambda_1 y_1(t) + \cdots + \lambda_n y_n(t)$, where $n = 1, 2, \ldots$; $\lambda_j \geqq 0$ and $\lambda_1 + \cdots + \lambda_n = 1$. It is clear that functions in this set satisfy (i)–(iii). The closure of this set of functions under the norm of $\mathfrak{D}$ (i.e., under uniform convergence over $0 \leqq t \leqq p$) gives a set of functions satisfying (i)–(iii). Thus the compactness of this set in $\mathfrak{D}$ is clear from Arzela's theorem. (A remark similar to this can be made for the other applications of Corollary 0.1 in this chapter; see Theorem 4.2 and Theorem 8.2.)

Consider now a system of nonlinear differential equations depending on a parameter μ,

$$x' = F(t, x, \mu), \tag{2.7}$$

where F is continuous, of period p in t for fixed (x, μ), and x, F are real d-dimensional vectors. Suppose that for $\mu = 0$, (2.7) has a periodic solution $x = g_0(t)$. Write $y = x - g_0(t)$; then (2.7) becomes

$$y' = F(t, y + g_0(t), \mu) - F(t, g_0(t), 0).$$

If F has continuous partial derivatives with respect to x and $A(t) = \partial_x F(t, g_0(t), 0)$, where $\partial_x F$ is the Jacobian matrix of F with respect to x, then the last equation is of the form (2.1), where

$$f(t, y) = F(t, y + g_0(t), \mu) - F(t, g_0(t), 0) - A(t)y$$

and $\|f(t, y)\|/\|y\| \to 0$ as $(y, \mu) \to 0$ uniformly in t for $0 \leqq t \leqq p$. In particular, when $|\mu|$ is small, (2.6) holds for small $r > 0$; in fact, (2.2) holds for small $\|y_1\|$, $\|y_2\|$ with arbitrarily small θ and $f(t, 0) \equiv 0$. It follows from Theorem 2.1 that if (1.2) has no nontrivial periodic solution of period p, then (2.7) has a unique solution $x(t) = x(t, \mu)$ of period p for each small $|\mu|$. The proof of Theorem 2.1 can also be used to show that if F depends smoothly on μ, then $x(t, \mu)$ depends smoothly on μ. All of these assertions can, however, be proved more directly by the use of the classical implicit function theorem.

Theorem 2.3. *Let x, F be real vectors. Let $F(t, x, \mu)$ be continuous for all t, small $|\mu|$, and x on some d-dimensional domain. Let F be of period p in t for fixed (x, μ) and have continuous partial derivatives with respect to the components of x. Let (2.7), where $\mu = 0$, have a solution $x \doteq g_0(t)$ of period p with the property that if $A(t) = \partial_x F(t, g_0(t), 0)$, then (1.2) has no nontrivial solution of period p. Then, for each small $|\mu|$, (2.7) has a unique solution $x = x(t, \mu)$ of period p with initial point $x(0, \mu)$ near $g_0(0)$; $x(t, \mu)$ is a continuous function of (t, μ), and $x(t, 0) = g_0(t)$. If, in addition, F has a continuous partial derivative with respect to μ, then $x(t, \mu)$ is of class C^1.*

It will be clear from the proof that if more smoothness is assumed for F (e.g., $F \in C^k$ or F analytic), then $x(t, \mu)$ is correspondingly smoother (e.g., $x(t, \mu) \in C^k$ or $x(t, \mu)$ analytic).

Proof. Let $x = \xi(t, x_0, \mu)$ be the unique solution of (2.7) satisfying the initial condition $x(0) = x_0$. Then $\xi(t, x_0, \mu)$ is continuous and has continuous partial derivatives with respect to t and the components of x_0; see Corollary V 3.3. Also, if x_0 is near to $g_0(0)$, then $\xi(t, x_0, \mu)$ exists on the interval $0 \leqq t \leqq p$; see Theorem V 2.1. The solution $x = \xi(t, x_0, \mu)$ is periodic of period p if and only if

$$\xi(p, x_0, \mu) - x_0 = 0. \tag{2.8}$$

Since $\xi(t, g_0(0), 0) = g_0(t)$, the equation (2.8) is satisfied if $(x_0, \mu) = (g_0(0), 0)$. Hence it can be solved for $x_0 = x_0(\mu)$ if the Jacobian matrix of the left side, $\partial_{x_0}\xi(p, x_0, \mu) - I$, is nonsingular at $(x_0, \mu) = (g_0(0), 0)$. The partial derivatives of $\xi(t, x_0, \mu)$ with respect to a component of x_0, when $(x_0, \mu) = (g_0(0), 0)$, is a solution of the equations of variation (1.2); see Theorem V 3.1. In fact, $Y(t) = \partial_{x_0}\xi(t, g_0(0), 0)$ is a fundamental matrix for (1.2) satisfying $Y(0) = I$. Hence the assumption that (1.2) has no periodic solution is equivalent to the assumption that $Y(p) - I$ is nonsingular; cf. Lemma 1.1, where $M = N = I$. Thus the implicit function theorem is applicable to (2.8) and gives a continuous function $x_0 = x_0(\mu)$. Correspondingly, $x = \xi(t, x_0(\mu), \mu)$ is a periodic solution of (2.7) of period p and the only such solution with initial point x_0 near $g_0(0)$. The other assertions of Theorem 2.3 also follow from the implicit function theorem.

The question of the existence of periodic solutions when

$$\det [Y(p) - I] = 0$$

has a vast literature and will not be pursued here.

Note that if F in (2.7) does not depend on t and $g_0(t) \not\equiv$ const., then the conditions of Theorem 2.3 cannot be satisfied since $x = g_0'(t)$ is a nontrivial periodic solution of the equations of variation (1.2). Here, however, we have the following analogue.

Theorem 2.4. *Let x, F be real vectors. Let $F(x, \mu)$ be continuous for small $|\mu|$ and for x on some d-dimensional domain and have continuous partial derivatives with respect to the components of x. When $\mu = 0$, let*

$$x' = F(x, \mu) \tag{2.9}$$

have a solution $x = g_0(t) \not\equiv$ const. of period $p_0 > 0$ such that if $A(t) = \partial_x F(g_0(t), 0)$, then exactly one of the characteristic roots of (1.2) is 1 [i.e., e^{Rp_0} has $\lambda = 1$ as a simple eigenvalue; cf. (1.5) where $p = p_0$]. Then, for small $|\mu|$, (2.9) has a unique periodic solution $x = x(t, \mu)$ with a period $p(\mu)$, depending on μ, such that $x(t, \mu)$ is near $g_0(t)$ and the period $p(\mu)$ is near p_0; furthermore $x(t, \mu)$, $p(\mu)$ are continuous, $x(t, 0) = g_0(t)$, and $p(0) = p_0$.

Remarks similar to those for Theorem 2.3 concerning the smoothness of F and corresponding smoothness of $x(t, \mu)$, $p(\mu)$ hold.

The geometrical considerations in the proof to follow are clarified by reference to Lemma IX 10.1, which shows that we obtain all solutions of (2.9) near $x = g_0(t)$ by considering solutions with initial points $x(0) = x_0$ near to $g_0(0)$ and x_0 restricted to be on the hyperplane π normal to $F(g_0(0), 0)$ and passing through $g_0(0)$.

Proof. Let $x = \xi(t, x_0, \mu)$ be the unique solution of (2.9) satisfying $x(0) = x_0$. This solution is of period p if and only if (2.8) holds. The equation (2.8) is satisfied when $(p, x_0, \mu) = (p_0, g_0(0), 0)$.

Since solutions of (2.9) are uniquely determined by initial conditions and $g_0(t) \not\equiv$ const., it follows that $F(g_0(t), 0) \neq 0$ for all t. Suppose that the coordinates in the x-space are chosen so that $g_0(0) = 0$ and $F(0, 0) = (0, \ldots, 0, \alpha)$, $\alpha \neq 0$, and let π denote the hyperplane $x^d = 0$ through the point $g_0(0) = 0$ normal to $F(0, 0)$. Consider x_0 on this hyperplane, $x_0 = (x_0^1, \ldots, x_0^{d-1}, 0)$. Then for small $|\mu|$, the equation (2.8) has a unique solution for p, x_0, in terms of μ if the Jacobian matrix of $\xi(t, x_0, \mu) - x_0$ with respect to $x_0^1, \ldots, x_0^{d-1}$ and t is nonsingular at $(t, x_0, \mu) = (p_0, 0, 0)$.

The matrix $Y(t)$, in which the columns are the vectors $\partial\xi/\partial x_0^1, \ldots, \partial\xi/\partial x_0^{d-1}$ and ξ' at $(x_0, \mu) = (0, 0)$, is a fundamental matrix for (1.2) and its last column is $F(g_0(t), 0)$. At $t = 0$,

$$Y(0) = \operatorname{diag}[I_{d-1}, \alpha] = Z(0), \tag{2.10}$$

by (1.5). Since (1.2) has, up to constant factors, only the last column $g_0'(t)$ of $Y(t)$ at $(x_0, \mu) = (0, 0)$ as a periodic solution of period p_0, the matrix $Y(p_0) - Y(0)$ annihilates vectors c of the form $c = (0, \ldots, 0, c^d)$ and no others.

The Jacobian matrix J of $\xi(t, x_0, \mu) - x_0$ with respect to $x_0^1, \ldots, x_0^{d-1}$, and t at $(t, x, \mu) = (p_0, 0, 0)$ is

$$J = Y(p_0) - \operatorname{diag}[I_{d-1}, 0],$$

and the last column of $Y(p_0)$ is $F(g_0(0), 0) = (0, \ldots, 0, \alpha)$, so that $J = [Y(p_0) - Y(0)] + \operatorname{diag}[0, \ldots, 0, \alpha]$. If J is singular, then there exists a vector $c = (c^1, \ldots, c^d) \neq 0$ such that $Jc = 0$; i.e.,

$$[Y(p_0) - Y(0)]c + (0, \ldots, 0, \alpha c^d) = 0.$$

In view of (2.10) and $Z(0) = Z(p_0)$, this is the same as

$$Z(0)\{(e^{Rp_0} - I)c + (0, \ldots, 0, c^d)\} = 0 \qquad \text{or}$$
$$(e^{Rp_0} - I)c + (0, \ldots, 0, c^d) = 0.$$

If $c^d = 0$, then $c = 0$ for $e^{Rp_0} - I$ only annihilates vectors of the form $(0, \ldots, 0, c^d)$. If $c^d \neq 0$, then $(e^{Rp_0} - I)^2c = 0$. But this implies that $\lambda = 1$ is at least a double eigenvalue of e^{Rp_0}. This contradiction shows that J is nonsingular.

Hence the implicit function theorem is applicable to (2.8) and gives the desired functions $x_0^1(\mu), \ldots, x_0^{d-1}(\mu)$, and $p(\mu)$. Correspondingly, if $x_0(\mu) = (x_0^1(\mu), \ldots, x_0^{d-1}(\mu), 0)$, then $x(t, \mu) = \xi(t, x_0(\mu), \mu)$ is a periodic solution of (2.9) and is the only periodic solution having an initial point x_0, with $x_0^d = 0$, near to $g_0(0)$ and a period near to p_0. This proves Theorem 2.4.

Exercise 2.1. Let $\dim x = 2$; $F(t, x)$ continuous for all t and x, periodic of period p in t for fixed x. Let the solution $x = x(t, t_0, x_0)$ of

$$x' = F(t, x) \tag{2.11}$$

satisfying $x(t_0) = x_0$ be unique for all t_0, x_0 and exist for $t \geqq t_0$. Finally, for some (t_0, x_0), let $x(t, t_0, x_0)$ be bounded for $t \geqq t_0$. Then (2.11) has at least one periodic solution of period p. See Massera [1].

Exercise 2.2. Let $\alpha(t) = (\alpha^1(t), \ldots, \alpha^d(t))$, $\beta(t) = (\beta^1(t), \ldots, \beta^d(t))$ be piecewise continuously differentiable for $0 \leqq t \leqq p$; $\alpha^j(t) \leqq \beta^j(t)$ for $j = 1, \ldots, d$; and $\alpha(0) = \alpha(p)$, $\beta(0) = \beta(p)$. Let

$$f(t, y) = (f^1(t, y), \ldots, f^d(t, y))$$

be continuous on an open set containing $\Omega^0 = \{(t, y): \alpha^j(t) \leqq y^j \leqq \beta^j(t)$ for $0 \leqq t \leqq p\}$ and let $f(t, y)$ be uniformly Lipschitz continuous with respect to y. Suppose finally that the functions $u^j(t, y) = \alpha^{j\prime}(t) - f^j(t, y^1, \ldots, y^{j-1}, \alpha^j(t), y^{j+1}, \ldots, y^d)$ and $v^j(t, y) = \beta^{j\prime}(t) - f^j(t, y^1, \ldots, y^{j-1}, \beta^j(t),$

$y^{j+1}, \ldots, y^d$) do not change signs (e.g., $u^j \geqq 0$ or $u^j \leqq 0$) and that $u^j v^j \leqq 0$ for all $(t, y) \in \Omega^0$. Then $y' = f(t, y)$ has at least one solution $y = y(t)$, $0 \leqq t \leqq p$, such that $(t, y(t)) \in \Omega^0$ and $y(0) = y(p)$. See Knobloch [1].

PART II. SECOND ORDER BOUNDARY VALUE PROBLEMS

3. Linear Problems

This part of the chapter concerns boundary value problems involving a system of second order equations. Consider first a linear inhomogeneous system of the form

$$x'' = B(t)x + F(t)x' + h(t) \tag{3.1}$$

and the corresponding homogeneous system

$$x'' = B(t)x + F(t)x' \tag{3.2}$$

for a d-dimensional vector x (with real- or complex-valued components). The problem involves solutions satisfying boundary conditions

$$x(0) = x_0, \qquad x(p) = x_p, \tag{3.3}$$

when $p > 0$, x_0, x_p are given. For the inhomogeneous equation (3.1), the conditions (3.3) are not more general than

$$x(0) = 0, \qquad x(p) = 0, \tag{3.4}$$

for if $x - [(x_p - x_0)t/p + x_0]$ is introduced as a new dependent variable, the equation (3.1) goes over into another equation of the same form with $h(t)$ replaced by $h(t) + B(t)(x_p - x_0)t/p + B(t)x_0 + F(t)(x_p - x_0)/p$.

Actually, the theory of the boundary value problem (3.1), (3.4) is contained in § 1. In order to see this, write (3.1) as a first order system

$$y' = A(t)y + g(t), \tag{3.5}$$

where $y = (x, x')$ is a $2d$-dimensional vector, $g(t) = (0, h(t))$, and $A(t)$ is a $2d \times 2d$ matrix

$$A(t) = \begin{pmatrix} 0 & I \\ B(t) & F(t) \end{pmatrix}. \tag{3.6}$$

The boundary conditions (3.4) can be written as

$$My(0) - Ny(p) = 0, \tag{3.7}$$

where M, N are the constant $2d \times 2d$ matrices

$$M = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}. \tag{3.8}$$

Note that

$$\operatorname{rank}(M, N) = \operatorname{rank}\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{pmatrix} = 2d.$$

Instead of restricting M, N to be of the type (3.8), it is possible to choose more general matrices; in this case, (3.4) is replaced by conditions of the form

$$M_{j1}x(0) + M_{j2}x'(0) - N_{j1}x(p) - N_{j2}x'(p) = 0 \qquad \text{for} \quad j = 1, 2,$$

where M_{jk}, N_{jk} are constant $d \times d$ matrices such that

$$(M, N) = \begin{pmatrix} M_{11} & M_{12} & N_{11} & N_{12} \\ M_{21} & M_{22} & N_{21} & N_{22} \end{pmatrix}$$

is of rank $2d$. For the sake of simplicity, only the choice (3.8), i.e., only the boundary conditions (3.4), will be considered.

Lemma 1.1 implies the following:

Lemma 3.1. *Let $B(t)$, $F(t)$ be continuous $d \times d$ matrices for $0 \leqq t \leqq p$; $U(t)$ the $d \times d$ matrix solution of*

$$U'' = B(t)U + F(t)U', \qquad U(0) = 0, \qquad U'(0) = I. \tag{3.9}$$

Then (3.2) *has a nontrivial solution* ($\not\equiv 0$) *solution satisfying* (3.4) *if and only if $U(p)$ is singular. In fact, the number k, $0 \leqq k \leqq d$, of linearly independent solutions of* (3.2), (3.4) *is the number of linearly independent vectors c satisfying $U(p)c = 0$.*

The corresponding corollary of Theorem 1.1 is

Theorem 3.1. *Let $B(t)$, $F(t)$ be continuous for $0 \leqq t \leqq p$. Then* (3.1) *has a solution $x(t)$ satisfying* (3.4) *for every $h(t)$ continuous on $[0, p]$ if and only if* (3.2), (3.4) *has no nontrivial* ($\not\equiv 0$) *solution. In this case, $x(t)$ is unique and there exists a constant K such that*

$$\|x(t)\|, \|x'(t)\| \leqq K\int_0^p \|h(s)\|\, ds. \tag{3.10}$$

Exercise 3.1. Verify Theorem 3.1.

The homogeneous adjoint system for (3.5) is $y' = -A^*(t)y$ which is not equivalent to a second order system without additional assumptions on B or F. The simplest assumption of this type is that $F(t)$ is continuously differentiable. In this case, the homogeneous adjoint system $y' = -A^*(t)y$ is equivalent to

$$z'' = [B^*(t) - F^{*\prime}(t)]z - F^*(t)z' \tag{3.11}$$

and the corresponding inhomogeneous system is

$$z'' = [B^*(t) - F^{*\prime}(t)]z - F^*(t)z' + f(t). \tag{3.12}$$

[Actually, the differentiability condition can be avoided by writing the terms involving F^* as $(F^*z)'$, and interpreting (3.11), (3.12) as first order systems for the $2d$-dimensional vector $(-z' - F^*z, z)$.]

In order to obtain the corresponding Green's relation, multiply (3.1) scalarly by z, (3.12) by x, subtract and integrate over $[0, p]$ to obtain

$$\int_0^p [h(t) \cdot z(t) - x(t) \cdot f(t)]\, dt = [x' \cdot z - x \cdot z' - Fx \cdot z]_0^p. \tag{3.13}$$

Thus, if x satisfies (3.4) and z satisfies

$$z(0) = 0, \qquad z(p) = 0 \tag{3.14}$$

then

$$\int_0^p [h(t) \cdot z(t) - x(t) \cdot f(t)]\, dt = 0, \tag{3.15}$$

so that (3.4) and (3.14) are adjoint boundary conditions.

Exercise 3.2. Verify that (3.2), (3.4) and (3.11), (3.14) are adjoint boundary problems in the sense of § 1.

Lemma 3.2. *Let $B(t)$ be continuous and $F(t)$ continuously differentiable for $0 \leqq t \leqq p$. Then* (3.2), (3.4) *have the same number of linearly independent solutions as the adjoint problem* (3.11), (3.14).

Finally, a corollary of Theorem 1.2 is

Theorem 3.2. *Let $B(t)$ be continuous and $F(t)$ continuously differentiable on $[0, p]$ and such that* (3.2), (3.4) *has k, $1 \leqq k \leqq d$, linearly independent solutions. Let $z_1(t), \ldots, z_k(t)$ be k linearly independent solutions of* (3.11), (3.14). *Let $h(t)$ be continuous on $[0, p]$. Then* (3.1), (3.4) *has a solution if and only if*

$$\int_0^p h(t) \cdot z_j(t)\, dt = 0 \qquad \text{for} \quad j = 1, \ldots, k. \tag{3.16}$$

The next uniqueness theorem has no analogue in § 1.

Theorem 3.3. *Let $B(t)$, $F(t)$ be continuous $d \times d$ matrices on $0 \leqq t \leqq p$ such that*

$$\text{Re}\, [(B(t) - \tfrac{1}{4}F(t)F^*(t))x \cdot x] \geqq 0 \tag{3.17}$$

for all vectors x (i.e., let the Hermitian part of the matrix $B - \frac{1}{4}FF^*$ be non-negative definite). Let $g(t)$ be continuous for $0 \leqq t \leqq p$. Then

$$x'' = B(t)x + F(t)x' + h(t) \tag{3.18}$$

has at most one solution satisfying given boundary conditions $x(0) = x_0$, $x(p) = x_p$.

Remark 1. Actually, Theorem 3.3 remains valid if (3.17) is relaxed to

$$(3.19) \qquad 2\,\mathrm{Re}\,[(B(t) - \tfrac{1}{4}F(t)F^*(t))x \cdot x] > -(\pi/p)^2\,\|x\|^2$$

for all vectors $x \neq 0$; cf. Exercise 3.3.

Proof. Since the difference of two solutions of the given boundary value problem is a solution of

$$(3.20) \qquad x'' = B(t)x + F(t)x', \qquad x(0) = x(p) = 0,$$

it suffices to show that the only solution of (3.20) is $x \equiv 0$.

Let $x(t)$ be a solution of (3.20). Put $r(t) = \|x(t)\|^2$. Then $r' = 2\,\mathrm{Re}\,x \cdot x'$ and $r'' = 2\,\mathrm{Re}\,(x \cdot x'' + \|x'\|^2)$, so that $r'' = 2\,\mathrm{Re}\,[(B(t)x + F(t)x') \cdot x + \|x'\|^2]$. It is readily verified that

$$\mathrm{Re}\,(B(t)x + F(t)x') \cdot x + \|x'\|^2 = \|x' + \tfrac{1}{2}F^*x\|^2 + \mathrm{Re}\,(Bx - \tfrac{1}{4}FF^*x) \cdot x.$$

Thus

$$(3.21) \qquad r'' = 2\,\|x' + \tfrac{1}{2}F^*x\|^2 + 2\,\mathrm{Re}\,[(B - \tfrac{1}{4}FF^*)x \cdot x].$$

Hence (3.17) implies that $r'' \geqq 0$. Since the last part of (3.20) means that $r(0) = r(p) = 0$, it follows that $r(t) \equiv 0$ for $0 \leqq t \leqq p$. This proves Theorem 3.3.

Exercise 3.3. (*a*) Show that if there exists a continuous real-valued function $q(t)$, $0 \leqq t \leqq p$, such that the equation

$$r'' + q(t)r = 0$$

has no solution $r(t) \not\equiv 0$ with two zeros on $0 \leqq t \leqq p$ [e.g., $q(t) < (\pi/p)^2$] and (3.17) is relaxed to

$$(3.22) \qquad 2\,\mathrm{Re}\,[(B(t) - \tfrac{1}{4}F(t)F^*(t))x \cdot x] \geqq -q(t)\,\|x\|^2$$

for all vectors x, then the conclusion of Theorem 3.3 remains valid. (*b*) Let there exist a continuously differentiable $d \times d$ matrix $K(t)$ on $[0, p]$ such that

$$(3.23) \qquad \mathrm{Re}\,[B - K' + (\tfrac{1}{2}F - K^H)(\tfrac{1}{2}F^* - K^H)]x \cdot x \geqq 0$$

for all vectors x and $0 \leqq t \leqq p$, where $K^H = \tfrac{1}{2}(K + K^*)$. Then the conclusion of Theorem 3.3 is valid. [Note that (3.23) reduces to (3.17) if $K(t) \equiv 0$, so that (*b*) generalizes Theorem 3.3, but not part (*a*) of this exercise.] The 2 in (3.22), hence in (3.19), is not needed if $F \equiv 0$.

Remark 2. If $F(t)$ has a continuous derivative, then (3.20) implies that $x \equiv 0$ if and only if $z \equiv 0$ is the only solution of

$$(3.24) \qquad z'' = [B^*(t) - F^{*\prime}(t)]z - F^*(t)z', \qquad z(0) = z(p) = 0;$$

cf. Lemma 3.2. Hence, the conclusion of Theorem 3.3 is valid if B, F in the criteria (3.17), (3.22), (3.23) are replaced by $B^* - F^{*\prime}$, $-F^*$, respectively.

4. Nonlinear Problems

Let x and f denote vectors with real-valued components. This section deals with second order equations of the form

$$x'' = f(t, x, x') \tag{4.1}$$

and the question of the existence of solutions satisfying the boundary conditions

$$x(0) = 0, \qquad x(p) = 0 \tag{4.2}$$

or, for given x_0 and x_p,

$$x(0) = x_0, \qquad x(p) = x_p. \tag{4.3}$$

The equation (4.1) will be viewed as an "inhomogeneous form" of

$$x'' = 0. \tag{4.4}$$

The problem (4.2), (4.4) has no nontrivial solution. Thus, by Theorem 3.1, an equation

$$x'' = h(t) \tag{4.5}$$

has a unique solution satisfying (4.2). In fact, this solution is given by

$$x(t) = -\frac{1}{p}\left[(p - t)\int_0^t sh(s)\,ds + t\int_t^p (p - s)h(s)\,ds\right]. \tag{4.6}$$

This can be verified by differentiating (4.6) twice; cf. (XI 2.18). The relation (4.6) can be abbreviated to

$$x(t) = -\int_0^p G(t, s)h(s)\,ds, \tag{4.7}$$

where

$$G(t, s) = \frac{1}{p}(p - t)s \qquad \text{or} \qquad G(t, s) = \frac{1}{p}t(p - s) \tag{4.8}$$

according as $0 \leqq s \leqq t \leqq p$ or $0 \leqq t \leqq s \leqq p$. Thus

$$\begin{gathered} 0 \leqq G(t, s) \leqq \frac{p}{4}, \qquad \int_0^p G(t, s)\,ds = \frac{1}{2}t(p - t) \leqq \frac{p^2}{8}, \\ \int_0^p |G_t(t, s)|\,ds = \frac{1}{2p}[t^2 + (p - t)^2] \leqq \frac{p}{2}, \end{gathered} \tag{4.9}$$

where $G_t = \partial G/\partial t$. Thus (4.6) or (4.7) and its differentiated form imply

$$\|x(t)\| \leqq \frac{p^2}{8}\max\|h(s)\|, \qquad \|x'(t)\| \leqq \frac{p}{2}\max\|h(s)\|, \tag{4.10}$$

where the max refers to $0 \leqq s \leqq p$.

Theorem 4.1. *Let $f(t, x, x')$ be continuous for $0 \leqq t \leqq p$ and all (x, x') and satisfy a Lipschitz condition with respect to x, x' of the form*

$$\|f(t, x_1, x_1') - f(t, x_2, x_2')\| \leqq \theta_0 \|x_1 - x_2\| + \theta_1 \|x_1' - x_2'\| \tag{4.11}$$

with Lipschitz constants θ_0, θ_1 so small that

$$\frac{\theta_0 p^2}{8} + \frac{\theta_1 p}{2} < 1. \tag{4.12}$$

Then (4.1) *has a unique solution satisfying* (4.2).

Remark 1. Instead of requiring f to be defined for $0 \leqq t \leqq p$ and all (x, x'), it is sufficient to have f defined for $0 \leqq t \leqq p$, $\|x\| \leqq R$, $\|x'\| \leqq 4R/p$, where R satisfies either

$$\frac{mp^2}{8} \leqq R\left[1 - \left(\frac{\theta_0 p^2}{8} + \frac{\theta_1 p}{2}\right)\right] \tag{4.13}$$

if $m = \max \|f(t, 0, 0)\|$ for $0 \leqq t \leqq p$, or merely

$$\frac{Mp^2}{8} \leqq R \tag{4.14}$$

if $M = \max \|f(t, x, x')\|$ for $\|x\| \leqq R$, $\|x'\| \leqq 4R/p$.

Proof. Let $\mathfrak{D}$ be the Banach space of functions $h(t)$, $0 \leqq t \leqq p$, having continuous first derivatives and the norm

$$|h| = \max\left(\max_{0\leqq t\leqq p} \|h(t)\|, \frac{p}{4} \max_{0\leqq t\leqq p} \|h'(t)\|\right). \tag{4.15}$$

Consider an $h(t)$ in the sphere $|h| \leqq R$ of $\mathfrak{D}$. Let $x(t)$ be the unique solution of

$$x'' = f(t, h(t), h'(t)) \tag{4.16}$$

satisfying $x(0) = x(p) = 0$. Define an operator T_0 on the sphere $|h| \leqq r$ of $\mathfrak{D}$ by putting $T_0[h(t)] = x(t)$.

If $x_0 = T_0[0]$ and $\|f(t, 0, 0)\| \leqq m$, then

$$\|x_0(t)\| \leqq \frac{mp^2}{8}, \qquad \frac{p}{4}\|x_0'(t)\| \leqq \frac{mp^2}{8} \tag{4.17}$$

by the case $h = f(t, 0, 0)$ of (4.10). Thus the norm $x_0(t) = T_0[0] \in \mathfrak{D}$ satisfies

$$|T_0[0]| \leqq \frac{mp^2}{8}. \tag{4.18}$$

Also, if $x_1 = T_0[h_1]$, $x_2 = T_0[h_2]$, then, by (4.10) and (4.11),

$$\|x_1(t) - x_2(t)\| \leqq \frac{p^2}{8}(\theta_0 \max \|h_1 - h_2\| + \theta_1 \max \|h_1' - h_2'\|),$$

$$\|x_1'(t) - x_2'(t)\| \leqq \frac{p}{2}(\theta_0 \max \|h_1 - h_2\| + \theta_1 \max \|h_1' - h_2'\|).$$

If the last inequality is multiplied by $p/4$ and $\theta_1(p^2/8) \max \|h_1' - h_2'\|$ is written as $(\theta_1 p/2)[(p/4) \max \|h_1' - h_2'\|]$, it follows that

$$(4.19) \qquad |T_0[h_1] - T_0[h_2]| \leqq \left(\frac{\theta_0 p^2}{8} + \frac{\theta_1 p}{2}\right) |h_1 - h_2|\,.$$

Thus the inequalities (4.12), (4.13) and (4.18) show that Theorem 0.1 is applicable and give Theorem 4.1.

Similarly, if $\|f(t, x, x')\| \leqq M$ for $\|x\| \leqq R$, $\|x'\| \leqq 4R/p$, then the derivation of (4.17) shows that if $|h| \leqq R$, then $x = T_0[h]$ satisfies $|x| \leqq Mp^2/8$. Thus if (4.14) holds, T_0 maps the sphere $|h| \leqq R$ into itself and the Remark following Theorem 0.1 is applicable in view of (4.12). Hence the proof of Theorem 4.1 and Remark 1 following it is complete.

Corollary 4.1. *Let $f(t, x, x')$ be continuous for $0 \leqq t \leqq p$, $\|x\| \leqq R_0$, $\|x'\| \leqq R_1$ and satisfy* (4.11), (4.12) *and $\|f(t, x, x')\| \leqq M$. Let*

$$(4.20) \qquad \frac{Mp^2}{8} + \|x_0\| \leqq R_0, \qquad \frac{Mp}{2} + \frac{\|x_0\|}{p} \leqq R_1.$$

Then (4.1) *has a unique solution satisfying*

$$(4.21) \qquad x(0) = 0 \quad \text{and} \quad x(p) = x_0.$$

Exercise 4.1. (*a*) Prove Corollary 4.1. (*b*) In Corollary 4.1, let $\|f(t, x, x')\| \leqq M$ be relaxed to $\|f(t, tx_0/p, x_0/p)\| \leqq m$ for $0 \leqq t \leqq p$ and R be defined by replacing "$\leqq$" by "$=$" in (4.13). Show that the conclusion of Corollary 4.1 remains valid if $R + \|x_0\| \leqq R_0$, $4R/p + \|x_0\|/p \leqq R_1$ replaces (4.20).

Theorem 4.2. *Let $f(t, x, x')$ be continuous and bounded, say,*

$$\|f(t, x, x')\| \leqq m,$$

for $0 \leqq t \leqq p$ and all (x, x'). Then (4.1) *has at least one solution $x(t)$ satisfying $x(0) = x(p) = 0$ and*

$$(4.22) \qquad \|x(t)\| \leqq \frac{mp^2}{8}, \qquad \|x'(t)\| \leqq \frac{mp}{2}\,.$$

It is sufficient to require that $f(t, x, x')$ be defined only for $\|x\| \leqq mp^2/8$, $\|x'\| \leqq mp/2$.

Proof. Let $\mathfrak{D}$ be the Banach space of continuously differentiable functions $h(t)$, $0 \leqq t \leqq p$, with norm $|h|$ defined (4.15). Consider $h(t)$ in the sphere $|h| \leq mp^2/8$ of $\mathfrak{D}$. For such an h, put $x = T_0[h]$, where $x(t)$ is the unique solution of (4.16) satisfying $x(0) = x(p) = 0$. Then $\|x(t)\| \leq mp^2/8$ and $\|x'(t)\| \leq mp/2$, so that T_0 maps the sphere $|h| \leq mp^2/8$ into itself.

If $|h_1|$, $|h_2| \leq mp^2/8$ and $x_1 = T_0[h_1]$, $x_2 = T_0[h_2]$, then (4.7) and (4.9) imply that

$$|x_1 - x_2| \leq \frac{p}{4}\int_0^p \|f(t, h_1(t), h_1'(t)) - f(t, h_2(t), h_2'(t))\| \, dt.$$

Since f is a continuous function, it follows that if $|h_1 - h_2| \to 0$, then $|x_1 - x_2| \to 0$. Thus T_0 is continuous.

For any $x(t)$ in the range of T_0, i.e., $x = T_0[h]$ for some h, (4.16) implies $\|x''(t)\| \leq m$. It follows that the set of functions $x(t)$ in the range of $T_0[h]$, $|h| \leq mp^2/8$, are such that $x(t)$, $x'(t)$ are bounded and equicontinuous since

$$\|x(t_1) - x(t_2)\| \leq \tfrac{1}{2}mp\,|t_1 - t_2|, \quad \|x'(t_1) - x'(t_2)\| \leq m\,|t_1 - t_2|.$$

Hence Arzela's theorem implies that the range of $T_0[h]$ has a compact closure. Consequently, Tychonov's theorem is applicable and gives Theorem 4.2.

Corollary 4.2. *Let $f(t, x, x')$ be continuous and satisfy $\|f\| \leq M$ for $0 \leq t \leq T$, $\|x\| \leq R_0$, $\|x'\| \leq R_1$. Let p and x_0 satisfy $0 < p \leq T$ and* (4.20). *Then* (4.1) *has a solution satisfying* (4.21). (*In particular, if* $0 < T < \min((8R_0/M)^{1/2}, 2R_1/M)$, *then there exists a $\delta > 0$ such that if $\|x_0\| \leq \delta$, then* (4.1) *has a solution satisfying* (4.21) *for $p = T$.*)

Exercise 4.2. Prove Corollary 4.2.

Exercise 4.3. Let $f(t, x, x')$ be continuous for $0 \leq t \leq p$, $\|x\| \leq R_0$, and arbitrary x'. Let there exist positive constants a, b such that $\|f(t, x, x')\| \leq a\,\|x'\|^2 + b$ for $0 \leq t \leq p$, $\|x\| \leq R_0$. Assume that $a, b, \|x_0\|$ are such that $a(bp^2 + 2\,\|x_0\|) < 1$ and $r^* = (ap)^{-1}\{1 - [1 - a(bp^2 + 2\,\|x_0\|)]^{1/2}\}$ satisfies $r^*p + 3\,\|x_0\| \leq 4R_0$. Then the boundary value problem (4.1),(4.21) has a solution.

Note that Corollaries 4.1 and 4.2 are similar except that in Corollary 4.1 there is the extra assumption that (4.11) and (4.12) hold; correspondingly, there is the extra assertion that the solution of (4.1), (4.21) is unique. We can prove another type of uniqueness theorem.

Theorem 4.3. *Let $f(t, x, x')$ be continuous for $0 \leq t \leq p$ and for (x, x') on some 2d-dimensional convex set. Let $f(t, x, x')$ have continuous partial derivatives with respect to the components of x and x'. Let the Jacobian matrices of f with respect to x, x'*

$$(4.23) \qquad B(t, x, x') = \partial_x f(t, x, x'), \qquad F(t, x, x') = \partial_{x'} f(t, x, x')$$

satisfy

(4.24) $$2(B - \tfrac{1}{4}FF^*)z \cdot z > -\frac{\pi^2}{p^2}\|z\|^2$$

for all (constant) vectors $z \neq 0$. *Then* (4.1) *has at most one solution satisfying given boundary conditions* $x(0) = x_0$, $x(p) = x_p$.

By the use of Exercise 3.3(*a*), condition (4.24) can be relaxed to

$$2(B - \tfrac{1}{4}F^*F)z \cdot z \geqq -q(t)\|z\|^2,$$

where $q(t)$ satisfies the conditions of Exercise 3.3(*a*). Here and in (4.24), "2" is not needed if f is independent of x'.

Proof. Suppose that there exist two solutions $x_1(t)$, $x_2(t)$. Put $x(t) = x_2(t) - x_1(t)$, so that

$$x'' = f(t, x_2(t), x_2'(t)) - f(t, x_1(t), x_1'(t)), \qquad x(0) = x(p) = 0.$$

This can be written as

$$x'' = B_1(t)x + F_1(t)x', \qquad x(0) = x(p) = 0,$$

where

(4.25) $$B_1(t) = \int_0^1 B\,ds, \qquad F_1(t) = \int_0^1 F\,ds,$$

and the argument of B, F in (4.25) is

(4.26) $$(t, (1 - s)x_1(t) + sx_2(t), (1 - s)x_1'(t) + sx_2'(t)).$$

This is a consequence of Lemma V 3.1.

For any constant vector z, an application of Schwarz's inequality to the formula in (4.25) for each component of $F_1^*(t)z$ gives

$$\|F_1^*(t)z\|^2 \leqq \int_0^1 \|F^*z\|^2\,ds,$$

where the argument of F^* is (4.26). Hence,

$$[B_1(t) - \tfrac{1}{4}F_1(t)F_1^*(t)]z \cdot z \geqq \int_0^1 [B - \tfrac{1}{4}FF^*]z \cdot z\,ds.$$

Thus by (4.24)

$$2[B_1(t) - \tfrac{1}{4}F_1(t)F_1^*(t)]z \cdot z > -\frac{\pi^2}{p^2}\|z\|^2$$

for all vectors $z \neq 0$. Consequently, Theorem 3.3 and Remark 1 following it imply that $x(t) \equiv 0$. This proves the theorem.

Exercise 4.4. Let $f(t, x, x')$ be continuous for $0 \leqq t \leqq p$ and (x, x') on some $2d$-dimensional domain and satisfy a Lipschitz condition of the form (4.11), where

(4.27) $$2\theta_0 + \tfrac{1}{4}\theta_1^2 < \frac{\pi^2}{p^2}.$$

Then (4.1) has at most one solution satisfying given boundary conditions $x(0) = x_0$, $x(p) = x_p$.

Exercise 4.5. Let $f(t, x, x')$ be continuous for $0 \leqq t \leqq p$ and (x, x') on a $2d$-dimensional domain. Let $\Delta x = x_2 - x_1$, $\Delta x' = x_2' - x_1'$, $\Delta f = f(t, x_2, x_2') - f(t, x_1, x_1')$, where x_1, x_2, x_1', x_2' are independent variables and assume that

$$\Delta x \cdot \Delta f + |\Delta x'|^2 > 0 \quad \text{if} \quad \Delta x \neq 0, \quad \Delta x \cdot \Delta x' = 0.$$

Then the boundary value problem $x'' = f(t, x, x')$, $x(0) = x_0$, $x(p) = x_p$ has at most one solution.

Exercise 4.6. (*a*) Let x be a real variable. Let $f(t, x, x')$ be continuous and strictly increasing in x for fixed (t, x'). Then (4.1) can have at most one solution satisfying given boundary conditions $x(0) = x_0$, $x(p) = x_p$. (*b*) Show that (*a*) is false if "strictly increasing" is replaced by "nondecreasing." (*c*) Show that if, in part (*a*), "strictly increasing" is replaced by "nondecreasing" and, in addition, f satisfies a uniform Lipschitz condition with respect to x', then the conclusion in (*a*) is valid. [For an existence theorem under the conditions of part (*c*), see Exercise 5.4.]

Exercise 4.7 (*Continuity Method*). Let x be a real variable. Let $\alpha(t, x')$, $\beta(t, x')$ be real-valued, continuous functions for $-\infty < t$, $x' < \infty$ with the properties that (i) α, β are periodic of period $p > 0$ in t for fixed x'; (ii) $\alpha > 0$; (iii) $|\beta(t, x')| \to \infty$ and $|\alpha(t, x')/\beta(t, x')| \to 0$ as $|x'| \to \infty$ uniformly in t. (*a*) Show that

$$x'' = x\alpha(t, x') + \beta(t, x') \tag{4.28}$$

has at most one solution of period p,

$$x(0) - x(p) = 0, \qquad x'(0) - x'(p) = 0. \tag{4.29}$$

(*b*) Show that if $C = \max |\beta(t, 0)|/\alpha(t, 0)$ and K is so large that $C\alpha(t, x') \leqq \frac{1}{2}|\beta(t, x')|$ and $|\beta(t, 0)| \leqq |\beta(t, x')|/4$ when $|x'| \geqq K$, then any periodic solution $x(t)$ of (4.28) satisfies $|x(t)| \leqq C$, $|x'(t)| \leqq K$. (*c*) Assume that α, β are of class C^1. By showing that the set of λ-values on $0 \leqq \lambda \leqq 1$ for which

$$x'' = x\alpha(t, x') + \beta(t, x') - \beta(t, 0) + \lambda\beta(t, 0)$$

has a periodic solution is open and closed on $0 \leqq \lambda \leqq 1$, prove that (4.28) has a unique periodic solution. (*d*) Show that the assumption in (*c*) that α, β are of class C^1 can be omitted.

Exercise 4.8 (*Continuation*). Let $\alpha(t, x, x')$, $\beta(t, x, x')$ be continuous for $-\infty < t, x, x' < \infty$ with the properties that (i) α, β are periodic of period $p > 0$ in t for fixed (x, x'); (ii) $\alpha > 0$; (iii) there is a constant C such that $|\beta(t, x, 0)| \leqq C\alpha(t, x, 0)$ for $-\infty < t, x < \infty$; (iv) $|\beta(t, x, x')| \to \infty$ and $|\alpha(t, x, x')/\beta(t, x, x')| \to 0$ as $|x'| \to \infty$ uniformly on bounded

(t, x)-sets. Show that

$$x'' = x\alpha(t, x, x') + \beta(t, x, x')$$

has at least one periodic solution.

5. A Priori Bounds

The proofs for the existence theorems for solutions of boundary value problems in the last section depended on finding bounds for the solution and its derivative. This section deals with more a priori bounds and their applications. The main problem to be considered is of the following type: Given a d-dimensional vector function $x(t)$ of class C^2 on some interval $0 \leqq t \leqq p$, a bound for $\|x(t)\|$, and some majorants for $\|x''\|$, find a bound for $\|x'\|$. The following result holds for the case when x is a real-valued function:

Lemma 5.1. *Let $\varphi(s)$, where $0 \leqq s < \infty$, be a positive continuous function satisfying*

$$\int^{\infty} \frac{s\,ds}{\varphi(s)} = \infty. \tag{5.1}$$

Let $R \geqq 0$ and $\tau > 0$. Then there exists a number M [depending only on $\varphi(s)$, R, τ] with the following property: If $x(t)$ is a real-valued function of class C^2 for $0 \leqq t \leqq p$, where $p \geqq \tau$, satisfying

$$|x| \leqq R, \qquad |x''| \leqq \varphi(|x'|), \tag{5.2}$$

then $|x'| \leqq M$ for $0 \leqq t \leqq p$.

Proof. In view of (5.1), there exists a number M such that

$$\int_{2R/\tau}^{M} \frac{s\,ds}{\varphi(s)} = 2R. \tag{5.3}$$

It will be shown that M has the desired property. [Instead of assumption (5.1), it would be sufficient to assume the existence of an M satisfying (5.3).]

Let $|x'(t)|$ assume its maximum value at a point $t = a$, $0 \leqq a \leqq p$. We can suppose that $x'(a) > 0$, otherwise x is replaced by $-x$. If $x'(a) > 2R/\tau$, then there exists a point t on $0 \leqq t \leqq p$ where $x'(t) \leqq 2R/p \leqq 2R/\tau$. Otherwise $x(p) - x(0) > 2R$ which contradicts $|x| \leqq R$. Assume $x'(a) > 2R/\tau$ and let $t = b$ be a point nearest $t = a$ where $x'(t) = 2R/\tau$. For sake of definiteness, let $b > a$. Thus $0 \leqq 2R/\tau = x'(b) \leqq x'(t) \leqq x'(a)$ for $a \leqq t \leqq b$.

If the second inequality in (5.2) is multiplied by $x'(t) > 0$, a quadrature over $a \leqq t \leqq b$ gives

$$\left| \int_a^b \frac{x'(t)x''(t)\,dt}{\varphi(x'(t))} \right| \leqq \int_a^b x'(t)\,dt \leqq 2R.$$

Even though it is not assumed that $x'' \neq 0$, the formal change of variables $s = x'(t)$ is permitted on the left and gives

$$\int_{2R/\tau}^{x'(a)} \frac{s\,ds}{\varphi(s)} \leqq 2R;$$

cf. Lemma I 4.1. From (5.3), it is seen that $x'(a) \leqq M$. Thus it follows that either $x'(a) \leqq 2R/\tau$ or $x'(a) \leqq M$. In either case $x'(a) \leqq M$. Since $x'(a) = \max |x'(t)|$ for $0 \leqq t \leqq p$, the lemma follows.

Lemma 5.1 is false if x is a d-dimensional vector, $d \geqq 2$, and absolute values are replaced by norms in (5.2). In order to see this, note that $\varphi(s) = \gamma s^2 + C > 0$, where γ and C are constants, satisfies the condition of Lemma 5.1. Let $x(t)$ denote the binary vector $x(t) = (\cos nt, \sin nt)$. Thus $\|x\| = 1$, $\|x'(t)\| = |n|$, $\|x''(t)\| = n^2 = \|x'\|^2$. Thus the inequalities analogous to (5.2),

$$(5.4) \qquad \|x\| \leqq R, \qquad \|x''\| \leqq \varphi(\|x'\|),$$

hold for $R = 1$, $\varphi(s) = s^2 + 1$. But there does not exist a number M such that $\|x'(t)\| \leqq M$ for all choices of n. The main result for vector-valued functions will be the next lemma.

Lemma 5.2. *Let $\varphi(s)$, where $0 \leqq s < \infty$, be a positive continuous function satisfying* (5.1). *Let α, K, R, τ be non-negative constants. Then there exists a constant M [depending only on $\varphi(s)$, α, R, τ, K] with the following property: If $x(t)$ is a vector-valued function of class C^2 on $0 \leqq t \leqq p$, where $p \geqq \tau$, satisfying* (5.4) *and*

$$(5.5) \qquad \|x\| \leqq R, \qquad \|x''\| \leqq \alpha r'' + K, \qquad \text{where} \quad r = \|x\|^2,$$

then $\|x'\| \leqq M$ on $0 \leqq t \leqq p$.

Proof. The first step of the proof is to show that (5.5) alone implies the existence of a bound for $\|x'(t)\|$ on any interval $[\mu, p - \mu]$, $0 < \mu \leqq \frac{1}{2}p$. Let $0 < \mu < p$ and $0 \leqq t \leqq p - \mu$, then

$$(5.6) \qquad x(t + \mu) - x(t) - \mu x'(t) = \int_t^{t+\mu} (t + \mu - s)x''(s)\,ds,$$

$t + \mu - s \geqq 0$, and (5.5) imply that

$$\mu \|x'(t)\| \leqq 2R + \int_t^{t+\mu} (t + \mu - s)(\alpha r''(s) + K)\,ds.$$

This inequality and the analogue of (5.6) in which x is replaced by r give

$$\mu \|x'(t)\| \leqq 2R + \alpha[r(t + \mu) - r(t) - \mu r'(t)] + \tfrac{1}{2}K\mu^2;$$

hence

$$(5.7) \quad \mu \|x'(t)\| \leqq 2R(1 + \alpha R) + \tfrac{1}{2}K\mu^2 - \alpha\mu r'(t) \qquad \text{for} \quad 0 \leqq t \leqq p - \mu$$

Similarly, for $\mu \leqq t \leqq p$, the relation

$$x(t) - x(t-\mu) - \mu x'(t) = -\int_{t-\mu}^{t} (t-\mu-s)x''(s)\,ds$$

implies that

(5.8) $\mu \|x'(t)\| \leqq 2R(1+\alpha R) + \frac{1}{2}K\mu^2 + \alpha\mu r'(t) \quad \text{for} \quad \mu \leqq t \leqq p.$

Let

(5.9) $$M_1(\tfrac{1}{2}p) = \frac{4R(1+\alpha R)}{p} + 4Kp.$$

The choice $\mu = \frac{1}{2}p$ in (5.7) and (5.8) gives

(5.10) $$\|x'(t)\| \leqq M_1(\tfrac{1}{2}p) - \alpha r'(t) \quad \text{for} \quad 0 \leqq t \leqq \tfrac{1}{2}p,$$

(5.11) $$\|x'(t)\| \leqq M_1(\tfrac{1}{2}p) + \alpha r'(t) \quad \text{for} \quad \tfrac{1}{2}p \leqq t \leqq p.$$

Adding (5.10), (5.11) for $t = p/2$ shows that

(5.12) $$\|x'(\tfrac{1}{2}p)\| \leqq M_1(\tfrac{1}{2}p).$$

The assumption (5.4) and (5.10)–(5.11) imply that

(5.13) $$\frac{|x' \cdot x''|}{\varphi(\|x'\|)} \leqq \|x'\| \leqq M_1(\tfrac{1}{2}p) \pm \alpha r',$$

where $\pm$ is required according as $t \geqq \frac{1}{2}p$ or $t \leqq \frac{1}{2}p$. Let $\Phi(s)$ be defined by

(5.14) $$\Phi(s) = \int_0^s \frac{u\,du}{\varphi(u)}.$$

Then, by Lemma I 4.1,

(5.15) $$|\Phi(\|x'(t)\|) - \Phi(\|x'(\tfrac{1}{2}p)\|)| = \left| \int x' \cdot x'' \frac{dt}{\varphi(\|x'\|)} \right|,$$

where the integral is taken over the t-interval with endpoints t and $p/2$. In view of (5.13), the integral is majorized by

$$\tfrac{1}{2}pM_1(\tfrac{1}{2}p) + \alpha\,|r(t) - r(\tfrac{1}{2}p)| \leqq \tfrac{1}{2}pM_1(\tfrac{1}{2}p) + 2\alpha R^2.$$

Hence

$$\Phi(\|x'(t)\|) \leqq \Phi(\|x'(\tfrac{1}{2}p)\|) + \tfrac{1}{2}pM_1(\tfrac{1}{2}p) + 2\alpha R^2.$$

In view of (5.12) and the fact that Φ is an increasing function, $\|x'(t)\| \leqq M(p)$, where

$$M(p) = \Phi^{-1}[\Phi(M_1(\tfrac{1}{2}p)) + \tfrac{1}{2}pM_1(\tfrac{1}{2}p) + 2\alpha R^2]$$

and Φ^{-1} is the function inverse to Φ. If $p \geqq \tau$, then $t \in [0, p]$ is contained in an interval of length τ in $[0, p]$. Thus the considerations just completed show that p can be replaced by τ, and the lemma is proved with $M(\tau)$ as an admissible choice of M.

Exercise 5.1. Show that an analogue of Lemma 5.2 remains valid if (5.5) is replaced by

$$\|x\| \leqq R, \qquad \|x''\| \leqq \rho'',$$

where $\rho(t)$ is real-valued function of class C^2 on $0 \leqq t \leqq p$ such that $|\rho(t)| \leqq K_1$. In this case, M depends only on $\varphi(s)$, α, R, τ, and K_1.

The choice $\varphi(s) = \gamma s^2 + C$ in Lemma 5.1 gives the following:

Corollary 5.1. *Let γ, C, α, K, R, τ be non-negative constants. Then there exists a constant M [depending only on γ, C, α, R, τ, K] such that if $x(t)$ is of class C^2 on $0 \leqq t \leqq p$, where $p \geqq \tau$, satisfying* (5.5) *and*

$$\|x\| \leqq R, \qquad \|x''\| \leqq \gamma \|x'\|^2 + C, \tag{5.16}$$

then $\|x'\| \leqq M$ for $0 \leqq t \leqq p$.

Remark 1. If γ in (5.16) satisfies $\gamma R < 1$, then (5.5) holds with

$$\alpha = \frac{\gamma}{2(1 - \gamma R)}, \qquad K = \frac{C}{1 - \gamma R}. \tag{5.17}$$

Thus assumption (5.5) is redundant in Corollary 5.1 when $\gamma R < 1$ (but the example preceding Lemma 5.2 shows that (5.5) cannot be omitted if $\gamma R = 1$). Also if α in (5.5) satisfies $2\alpha R < 1$, then (5.16) holds with

$$\gamma = \frac{2\alpha}{1 - 2\alpha R} \quad \text{and} \quad C = \frac{K}{1 - 2\alpha R}, \tag{5.18}$$

so that (5.16) is redundant in this case. Even if $d = 1$ (so that $x(t)$ is real-valued), condition (5.16) cannot be omitted if $2\alpha R > 1$).

In order to verify the first part of Remark 1, note that

$$r'' = 2(x \cdot x'' + \|x'\|^2). \tag{5.19}$$

Hence (5.16) shows that $r'' \geqq 2[(1 - \gamma R)\|x'\|^2 - CR]$. Another application of (5.16) gives $\gamma r'' \geqq 2[(1 - \gamma R)(\|x''\| - C) - CR\gamma] = 2[(1 - \gamma R)\|x''\| - C]$. This is the same as (5.5) with the choices (5.17). The proof of the remark concerning (5.18) is similar.

Exercise 5.2. Show that if $2\alpha R > 1$, then assumption (5.5) cannot be dropped in Corollary 5.1.

The following simple fact will be needed subsequently.

Lemma 5.3. *Let $f(t, x, x')$ be a continuous function on a set*

$$E(p, R) = \{(t, x, x'):\ 0 \leqq t \leqq p,\ \|x\| \leqq R,\ x' \text{ arbitrary}\}, \tag{5.20}$$

and let f have one or more of the following properties:

$$x \cdot f + \|x'\|^2 > 0 \quad \text{when} \quad x \cdot x' = 0 \quad \text{and} \quad \|x\| > 0, \tag{5.21}$$

$$x \cdot f + \|x'\|^2 > 0 \quad \text{when} \quad x \cdot x' = 0 \quad \text{and} \quad \|x\| = R, \tag{5.22}$$

$$\|f\| \leqq \varphi(\|x'\|), \tag{5.23}$$

$$\|f\| \leqq 2\alpha(x \cdot f + \|x'\|^2) + K. \tag{5.24}$$

Let $M > 0$. Then there exists a continuous bounded function $g(t, x, x')$ defined for $0 \leqq t \leqq p$ and arbitrary (x, x') satisfying

$$g(t, x, x') \equiv f(t, x, x') \quad \text{for} \quad 0 \leqq t \leqq p, \quad \|x\| \leqq R, \quad \|x'\| \leqq M \tag{5.25}$$

and having the corresponding set of properties among the following:

$$x \cdot g + \|x'\|^2 > 0 \quad \text{when} \quad x \cdot x' = 0 \quad \text{and} \quad \|x\| \geqq 0, \tag{5.21'}$$

$$x \cdot g + \|x'\|^2 > 0 \quad \text{when} \quad x \cdot x' = 0 \quad \text{and} \quad \|x\| \geqq R, \tag{5.22'}$$

$$\|g\| \leqq \varphi(\|x'\|), \tag{5.23'}$$

$$\|g\| \leqq 2\alpha(x \cdot g + \|x'\|^2) + K. \tag{5.24'}$$

Proof. We can obtain such a function g as follows: Let $\delta(s)$, where $0 \leqq s < \infty$, be a real-valued continuous function satisfying $\delta = 1$, $0 < \delta < 1$, $\delta = 0$ according as $\delta \leqq M$, $M < s \leqq 2M$, $s > 2M$. Put

$$g(t, x, x') = \delta(\|x'\|)f(t, x, x') \qquad \text{on} \quad E(p, R),$$

$$g(t, x, x') = \frac{R}{\|x\|}\, g\left(t, \frac{Rx}{\|x\|}, x'\right) \qquad \text{for} \quad \|x\| > R.$$

On $E(p, R)$, the identity

$$x \cdot g + \|x'\|^2 = \delta(\|x'\|)(x \cdot f + \|x'\|^2) + [1 - \delta(\|x'\|)]\, \|x'\|^2$$

makes it clear that g has the desired properties on $E(p, R)$. Furthermore the validity of any of the relations (5.21')–(5.24') for $\|x\| = R$ implies its validity for $\|x\| > R$. This proves the lemma.

Note that inequalities of the type (5.23), (5.24) imply that solutions of

$$x'' = f(t, x, x') \tag{5.26}$$

satisfy (5.4), (5.5), respectively; cf. (5.19).

Theorem 5.1. *Let $f(t, x, x')$ be a continuous function on the set $E(p, R)$ in* (5.20) *satisfying*

$$x \cdot f + \|x'\|^2 \geqq 0 \quad \text{if} \quad x \cdot x' = 0 \quad \text{and} \quad \|x\| = R, \tag{5.27}$$

(5.24) *and* (5.23), *where $\varphi(s)$, $0 \leqq s < \infty$, is a positive continuous function satisfying* (5.1). *Let $\|x_0\|$, $\|x_p\| \leqq R$. Then* (5.26) *has at least one solution satisfying $x(0) = x_0$, $x(p) = x_p$.*

It will be clear from the proof that assumption (5.23) can be omitted if $2\alpha R < 1$. Furthermore, if f satisfies

$$\|f\| \leqq \gamma \|x'\|^2 + C, \tag{5.28}$$

where γ, C are non-negative constants and $\gamma R < 1$, then both assumptions (5.23) and (5.24) can be omitted.

If the vector x is 1-dimensional, Lemma 5.1 can be used in the proof instead of Lemma 5.2. This gives the following:

Corollary 5.2. *Let x be a real variable and $f(t, x, x')$ be a real-valued function in Theorem* 5.1. *Then the conclusion of Theorem* 5.1 *remains valid if condition* (5.24) *is omitted.*

Note that, in this case, condition (5.27) becomes simply $f(t, +R, 0) \geqq 0$ and $f(t, -R, 0) \leqq 0$ for $0 \leqq t \leqq p$.

Proof of Theorem 5.1. The proof will be given first for the case that f satisfies (5.22) instead of (5.27). Let $M > 0$ be a constant (with $p = \tau$) supplied by Lemma 5.2. Let $g(t, x, x')$ be a continuous bounded function for $0 \leqq t \leqq p$ and arbitrary (x, x') satisfying (5.25), (5.22′), (5.23′), and (5.24′). By Theorem 4.2, the boundary value problem

$$x'' = g(t, x, x'), \qquad x(0) = x_0, \qquad \text{and} \quad x(p) = x_p$$

has a solution $x(t)$. Condition (5.22′) means that $r = \|x(t)\|^2$ satisfies $r'' > 0$ if $r' = 0$ and $r \geqq R^2$; cf. (5.19). Hence $r(t)$ does not have a maximum at any point t, $0 < t < p$, where $r(t) \geqq R^2$. Since $r(0) = \|x_0\|^2$, $r(p) = \|x_p\|^2$ satisfy $r(0)$, $r(p) \leqq R^2$, it follows that $r(t) \leqq R^2$ (i.e., $\|x(t)\| \leqq R$) for $0 \leqq t \leqq p$. By virtue of $x'' = g$ and (5.23′), (5.24′), Lemma 5.2 is applicable to $x(t)$ and implies that $\|x'(t)\| \leqq M$ for $0 \leqq t \leqq p$.

Consequently, (5.25) shows that $x(t)$ is a solution of (5.26). This proves Theorem 5.1 provided that (5.27) is strengthened to (5.22). In order to remove this proviso, note that if $\epsilon > 0$, the function $f(t, x, x') + \epsilon x$ satisfies the conditions of Theorem 5.1 as well as (5.22) if φ, K in (5.23), (5.24) are replaced by $\varphi + \epsilon R$, $K + \epsilon R$, respectively. Hence

$$x'' = f(t, x, x') + \epsilon x$$

has a solution $x = x_\epsilon(t)$ satisfying the boundary conditions. It is clear that $\|x_\epsilon(t)\| \leqq R$ and that there exists a constant M (independent of ϵ, $0 < \epsilon \leqq 1$) such that $\|x_\epsilon'(t)\| \leqq M$. Consequently, if $N = \max \|f(t, x, x')\| + 1$

for $0 \leqq t \leqq p$, $\|x\| \leqq R$, $\|x'\| \leqq M$, then $\|x_\epsilon''(t)\| \leqq N$. Thus the family of functions $x_\epsilon(t)$, $x_\epsilon'(t)$ for $0 \leqq t \leqq p$ are uniformly bounded and equicontinuous. By Arzela's theorem, there is a sequence $1 > \epsilon_1 > \epsilon_2 > \cdots$ such that $\epsilon_n \to 0$ as $n \to \infty$, and $x(t) = \lim x_\epsilon(t)$ exists as $\epsilon = \epsilon_n \to 0$ and is a solution of (5.26) satisfying $x(0) = x_0$, $x(p) = x_p$. This completes the proof of Theorem 5.1.

Exercise 5.3. Show that if (5.27) in Theorem 5.1 is strengthened to

$$x \cdot f + \|x'\|^2 \geqq 0 \quad \text{when} \quad x \cdot x' = 0, \tag{5.29}$$

then (5.26) has a solution $x(t)$ satisfying $x(0) = x_0$, $x(p) = 0$, and

$$r \geqq 0, \quad r' \leqq 0 \quad \text{if} \quad r = \|x\|^2. \tag{5.30}$$

Exercise 5.4. Let u be a real variable. Let $h(t, u, u')$ be real-valued and continuous for $0 \leqq t \leqq p$ and all (u, u'), and satisfy the following conditions: (i) h is a nondecreasing function of u for fixed (t, u'); (ii) $|h| \leqq \varphi(|u'|)$ where $\varphi(s)$ is a positive, continuous, nondecreasing function for $s \geqq 0$ satisfying (5.1); (iii) $u'' = h(t, u, u')$ has at least one solution $u_0(t)$ which exists on $0 \leqq t \leqq p$ [e.g., (ii) and (iii) hold if $|h| \leqq \alpha |u'| + K$ for constants α, K]. Let u_0, u_p be arbitrary numbers. Then $u'' = h(t, u, u')$ has at least one solution $u(t)$ satisfying $u(0) = u_0$, $u(p) = u_p$. [For a related uniqueness assertion, see Exercise 4.6(*c*).]

Theorem 5.2. *Let $f(t, x, x')$ be continuous in*

$$E(R) = \{(t, x, x'):\ 0 \leqq t < \infty,\ \|x\| \leqq R,\ x' \text{ arbitrary}\}. \tag{5.31}$$

For every $p > 0$, let f satisfy the conditions of Theorem 5.1 *on $E(p, R)$ in* (5.20), *where $\varphi(s)$ and the constants α, K in* (5.23), (5.24) *can depend on p. Let $\|x_0\| \leqq R$. Then* (5.26) *has a solution $x(t)$ which satisfies $x(0) = x_0$ and exists for $t \geqq 0$.*

Exercise 5.5. (*a*) Prove Theorem 5.2. (*b*) Show that if, in addition, (5.27) is strengthened to (5.29) in Theorem 5.2, then the solution $x(t)$ can be chosen so that (5.30) holds. (*c*) Furthermore, if (5.29) is strengthened to $x \cdot f + \|x'\|^2 \geqq 0$, then $r \geqq 0$, $r' \leqq 0$, $r'' \geqq 0$ for $t \geqq 0$. (*d*) If x is 1-dimensional, show that condition (5.24) can be omitted from Theorem 5.2 and parts (*b*) and (*c*) of this exercise.

Exercise 5.6. Let $f(t, x, x')$ be continuous on the set $E(R)$ in (5.31). For every m, $0 < m < R$, let there exist a continuous function $h(t) = h(t, m)$ for large t such that $\int^\infty th(t)\,dt = \infty$ and $x \cdot f(t, x, x') \geqq h(t) \geqq 0$ for large t, $0 < m \leqq \|x\| \leqq R$, x' arbitrary. Let $x(t)$ be a solution of (5.26) for large t. Then $x(t) \to 0$ as $t \to \infty$.

Exercise 5.7. Let $f(t, x, x')$ be continuous on $E(R)$ in (5.31) and have continuous partial derivatives with respect to the components of x, x'; let the Jacobian matrices (4.23) satisfy $\frac{1}{2}(B + B^*) - \frac{1}{4}FF^* \geqq 0$; cf. (3.17).

Let $\|x_0\| \leqq R$. Then (5.26) has at most one solution satisfying $x(0) = x_0$ and $\|x(t)\| \leqq R$ for $t \geqq 0$.

Remark 2. The main role of the assumptions involving (5.23) and/or (5.24) in Theorems 5.1, 5.2 is to assure that the following holds:

Assumption (A_p). There exists a constant $M = M(p)$ with the property that if $x(t)$ is a solution of $x'' = f(t, x, x')$ for $0 \leqq t \leqq p$ satisfying $\|x(t)\| \leqq R$, then $\|x'(t)\| \leqq M$ for $0 \leqq t \leqq p$.

Exercise 5.8. Let $f(t, x, x')$ be continuous on $E(R)$ in (5.31) and satisfy assumption (A_p) for all $p \geqq p_0 > 0$. Suppose that, for each x_0 in $\|x_0\| \leqq R$, (5.26) has exactly one solution $x(t) = x(t, x_0)$ satisfying $x(0) = x_0$ and existing for $t \geqq 0$ (cf., e.g., Theorem 5.2 and Exercise 5.7.) (*a*) Show that $x(t, x_0)$ is a continuous function of (t, x_0) for $t \geqq 0$, $\|x_0\| \leqq R$. (*b*) Suppose, in addition, that $f(t, x, x')$ is periodic of period p_0 in t for fixed (x, x'). Then (5.26) has at least one solution $x(t)$ of period p_0.

PART III. GENERAL THEORY

6. Basic Facts

The main objects of study in this part of the chapter will be a linear inhomogeneous system of differential equations

$$y' = A(t)y + g(t), \tag{6.1}$$

the corresponding homogeneous system

$$y' = A(t)y, \tag{6.2}$$

and a related nonlinear system

$$y' = A(t)y + f(t, y). \tag{6.3}$$

Let J denote a fixed t-interval $J: 0 \leqq t < \omega \, (\leqq \infty)$. The symbols $x, y, f, g, \ldots$ denote elements of a d-dimensional Banach space Y over the real or complex number field with norms $\|x\|$, $\|y\|$, $\|f\|$, $\|g\|, \ldots$. (Here $\|x\|$ is not necessarily the Euclidean norm.) In (6.1), $g = g(t)$ is a locally integrable function on J (i.e., integrable on every closed, bounded subinterval of J). $A(t)$ is an endomorphism of Y for (almost all) fixed t and is locally integrable on J. Thus if a fixed coordinate system is chosen on Y, $A(t)$ is a locally integrable $d \times d$ matrix function on J.

When $y(t)$ is a solution of (6.1) on the interval $[0, a] \subset J$, the fundamental inequality

$$(6.4) \qquad \|y(t)\| \leqq \left\{\|y(t')\| + \int_0^a \|g(s)\|\, ds\right\} \exp \int_0^a \|A(s)\|\, ds \quad \text{for } 0 \leqq t, t' \leqq a$$

follows from Lemma IV 4.1. If this relation is integrated with respect to t' over $[0, a]$, we obtain

$$(6.5) \qquad \|y(t)\| \leqq \left\{\frac{1}{a}\int_0^a \|y(s)\|\, ds + \int_0^a \|g(s)\|\, ds\right\} \exp \int_0^a \|A(s)\|\, ds$$

for $0 \leqq t \leqq a$.

Let $L = L_J$ denote the space of real-valued functions $\varphi(t)$ on J with the topology of convergence in the mean L^1 on compact intervals of J. Thus L is a Fréchet (= complete, linear metric) space. For example, the following metric, which will not be used below, can be introduced on L: let $0 = t_0 < t_1 < t_2 < \ldots, t_n \to \omega$ as $n \to \infty$, and let the distance between $\varphi, \psi \in L$ be

$$d(\varphi, \psi) = \sum_{n=1}^{\infty} \frac{2^n}{2^n[1 + I(n)]}, \qquad \text{where} \quad I(n) = \int_0^{t_n} |\varphi - \psi|\, dt.$$

Correspondingly, let $C = C_J$ denote the space of continuous, real-valued functions $\varphi(t)$ on J with the topology of uniform convergence on compact interval of J. Thus C is also a Fréchet space. A metric on C, e.g., is

$$d(\varphi, \psi) = \sum_{n=1}^{\infty} \frac{m(n)}{2^n[1 + m(n)]}, \qquad \text{where} \quad m(n) = \max_{[0,t_n]} |\varphi(t) - \psi(t)|.$$

The symbols $L^p = L_J^p$, $1 \leqq p \leqq \infty$, denote the usual Banach spaces of real-valued functions $\varphi(t)$ on $J: 0 \leqq t < \omega\ (\leqq \infty)$ with the norm

$$|\varphi|_p = \left(\int_J |\varphi(t)|^p\, dt\right)^{1/p} \quad \text{if} \quad 1 \leqq p < \infty,$$

$$|\varphi|_\infty = \operatorname*{ess\,sup}_J |\varphi(t)| \quad \text{if} \quad p = \infty.$$

L_0^∞ is the subspace of L^∞ consisting of functions $\varphi(t)$ satisfying $\varphi(t) \to 0$ as $t \to \omega$. For other Banach spaces B of real-valued, measurable functions $\varphi(t)$ in J, the notation $|\varphi|_B$ will be used for the norm of $\varphi(t)$ in B.

Remark. Strictly speaking, the spaces $L, L^\infty, L_0^\infty, \ldots$ are not spaces of "real-valued functions" but rather spaces of "equivalence classes of real-valued functions," where two functions are in the same equivalence class if they are equal except on a set of Lebesgue measure zero. Since no confusion will arise, however, over this "abuse of language," the abbreviated terminology will be used. In this terminology, the meaning of a "continuous function in L" or the "intersection $L \cap C$" is clear.

$L(Y), L^p(Y), B(Y), \ldots$ will represent the space of measurable vector-valued functions $y(t)$ on $J: 0 \leqq t < \omega$ ($\leqq \infty$) with values in Y such that $\varphi(t) = \|y(t)\|$ is in $L, L^p, B, \ldots$. With L^p or B, the norm $|\varphi|_p$ or $|\varphi|_B$ will be abbreviated to $|y|_p$ or $|y|_B$.

A Banach space $\mathfrak{D}$ will be said to be *stronger than* $L(Y)$ when (i) $\mathfrak{D}$ is contained in $L(Y)$ algebraically and (ii) for every a, $0 < a < \omega$, there is a number $\alpha = \alpha_{\mathfrak{D}}(a)$ such that $y(t) \in \mathfrak{D}$ implies

$$\int_0^a \|y(t)\|\, dt \leqq \alpha\, |y|_{\mathfrak{D}}, \qquad \text{where} \quad \alpha = \alpha_{\mathfrak{D}}(a). \tag{6.6}$$

[It is easily seen from the Open Mapping Theorem 0.3 that condition (ii) is equivalent to: "convergence in $\mathfrak{D}$ implies convergence in $L(Y)$."]

If $\mathfrak{D}$ is a Banach space stronger than $L(Y)$, a $\mathfrak{D}$-solution $y(t)$ of (6.1) or (6.2) means a solution $y(t) \in \mathfrak{D}$. Let $Y_{\mathfrak{D}}$ denote the set of initial points $y(0) \in Y$ of $\mathfrak{D}$-solutions $y(t)$ of (6.2). Then $Y_{\mathfrak{D}}$ is a subspace of Y. Let Y_1 be a subspace of Y complementary to $Y_{\mathfrak{D}}$; i.e., Y_1 is a subspace of Y such that $Y = Y_{\mathfrak{D}} \oplus Y_1$ is the direct sum of $Y_{\mathfrak{D}}$ and Y_1, so that every element $y \in Y$ has a unique representation $y = y_0 + y_1$ with $y_0 \in Y_{\mathfrak{D}}, y_1 \in Y_1$ (e.g., if Y is a Euclidean space, Y_1 can be, but need not be, the subspace of Y orthogonal to $Y_{\mathfrak{D}}$). Let P_0 be the projection of Y onto $Y_{\mathfrak{D}}$ annihilating Y_1; thus if $y = y_0 + y_1$ with $y_0 \in Y_{\mathfrak{D}}, y_1 \in Y_1$, then $P_0 y = y_0$.

Lemma 6.1. *Let $A(t)$ be locally integrable on J and let $\mathfrak{D}$ be a Banach space stronger than $L(Y)$. Then there exist constants C_0, C_1 such that if $y(t)$ is a $\mathfrak{D}$-solution of* (6.2), *then*

$$|y|_{\mathfrak{D}} \leqq C_0 \|y(0)\| \quad \text{and} \quad \|y(0)\| \leqq C_1 |y|_{\mathfrak{D}}. \tag{6.7}$$

Proof. $Y_{\mathfrak{D}}$ is a subspace of the finite dimensional space Y. In addition, there is a one-to-one, linear correspondence between solutions $y(t)$ of (6.2) and their initial points $y(0)$. Thus the set of $\mathfrak{D}$-solutions of (6.2) is a finite dimensional subspace of $\mathfrak{D}$ which is in one-to-one, linear correspondence with $Y_{\mathfrak{D}}$. It is a well known and easily verified fact that if two finite-dimensional, normed linear spaces can be put into one-to-one correspondence, then the norm of an element of one space is majorized by a constant times the norm of the corresponding element of the other space. [For example, an admissible choice of C_1 is a

$$a^{-1}\alpha_{\mathfrak{D}}(a) \exp \int_0^a \|A(s)\|\, ds$$

for any a, $0 < a < \omega$. This follows from (6.6) and the choices $t = 0$, $g(s) = 0$ in (6.5).]

Let $\mathfrak{B}$, $\mathfrak{D}$ be Banach spaces stronger than $L(X)$. Define an operator $T = T_{\mathfrak{B}\mathfrak{D}}$ from $\mathfrak{D}$ to $\mathfrak{B}$ as follows: The domain $\mathscr{D}(T) \subset \mathfrak{D}$ of T is the set of functions $y(t)$, $t \in J$, which are absolutely continuous (on compact subintervals of J), $y(t) \in \mathfrak{D}$, and $y'(t) - A(t)\,y(t) \in \mathfrak{B}$. For such a function $y(t)$, Ty is defined to be $y'(t) - A(t)\,y(t)$. In other words, $Ty = g$, where $g(t) \in \mathfrak{B}$ is given by (6.1).

Lemma 6.2. *Let $A(t)$ be locally integrable on J and let $\mathfrak{B}$, $\mathfrak{D}$ be Banach spaces stronger than $L(Y)$. Then $T = T_{\mathfrak{B}\mathfrak{D}}$ is a closed operator; that is the graph of T, $\mathscr{G}(T) = \{(y(t), g(t)) : y(t) \in \mathscr{D}(T), g = Ty\}$, is a closed set of the Banach space $\mathfrak{D} \times \mathfrak{B}$.*

Proof. In order to prove this, it must be shown that if $y_1(t), y_2(t), \ldots$ are elements of $\mathscr{D}(T)$, $g_n = Ty_n$, $y(t) = \lim y_n(t)$ exists in $\mathfrak{D}$ and $g(t) = \lim g_n(t)$ exists in $\mathfrak{B}$, then $y(t) \in \mathscr{D}(T)$ and $g(t) = Ty$.

The basic inequality (6.5) combined with (6.6) and the analogue of (6.6) for the space $\mathfrak{B}$ give

$$\|y_n(t) - y_m(t)\| \leqq \left\{\frac{1}{a}\alpha_{\mathfrak{D}}\,|y_n - y_m|_{\mathfrak{D}} + \alpha_{\mathfrak{B}}\,|g_n - g_m|_{\mathfrak{B}}\right\} \exp \int_0^a \|A(s)\|\,ds.$$

Hence $y(t)$ is the uniform limit of $y_1(t), y_2(t), \ldots$ on any interval $[0, a] \subset J$.

The differential equation (6.1) is equivalent to the integral equation

$$y(t) = y(a) + \int_a^t A(s)y(s)\,ds + \int_a^t g(s)\,ds.$$

Since the convergence of $g_1, g_2, \ldots$ in $\mathfrak{B}$ implies its convergence in $L(Y)$, it follows that (6.1) holds where $y = \lim y_n(t)$ in $\mathfrak{D}$, $g = \lim g_n(t)$ in $\mathfrak{B}$. Finally, $y \in \mathfrak{D}$, $g \in \mathfrak{B}$ show that $y \in \mathscr{D}(T)$. This proves Lemma 6.2.

The pair of Banach spaces $(\mathfrak{B}, \mathfrak{D})$ is said to be *admissible* for (6.1) or for $A(t)$ if each is stronger than $L(Y)$, and, for every $g(t) \in \mathfrak{B}$, the differential equation (6.1) has a $\mathfrak{D}$-solution. In other words, the map $T = T_{\mathfrak{B}\mathfrak{D}} : \mathscr{D}(T) \to \mathfrak{B}$ is onto, i.e., the range of T is $\mathfrak{B}$. (For example, if $J: 0 \leqq t < \infty$, $A(t)$ is continuous of period p, and $\mathfrak{B} = \mathfrak{D}$ is the Banach space of continuous functions $y(t)$ of period p with norm $|y|_{\mathfrak{D}} = \sup \|y(t)\|$, then $(\mathfrak{B}, \mathfrak{D})$ is admissible for (6.1) if and only if (6.2) has no nontrivial solution of period p; see Theorem 1.1.)

Lemma 6.3. *Let $A(t)$ be locally integrable on J, let $(\mathfrak{B}, \mathfrak{D})$ be admissible for* (6.1), *and let $y_0 \in Y_{\mathfrak{D}}$. Then, if $g(t) \in \mathfrak{B}$,* (6.1) *has a unique $\mathfrak{D}$-solution $y(t)$ such that $P_0 y(0) = y_0$. Furthermore, there exist positive constants C_0 and K, independent of $g(t)$, satisfying*

$$|y|_{\mathfrak{D}} \leqq C_0\|y_0\| + K\,|g|_{\mathfrak{B}}. \tag{6.8}$$

Proof. Consider first the case that $y_0 = 0$, so that we seek $\mathfrak{D}$-solutions $y(t)$ with $y(0) \in Y_1$. For any $g \in \mathfrak{B}$, (6.1) has a solution $y(t) \in \mathfrak{D}$, by assumption. Let $y(0) = y_0 + y_1$, where $y_0 = P_0 y(0) \in Y_{\mathfrak{D}}$, $y_1 \in Y_1$. Let $y_0(t)$ be the solution of the homogeneous equation (6.2) such that $y_0(0) = y_0$, so that $y_0(t) \in \mathfrak{D}$. Then $y_1(t) = y(t) - y_0(t) \in \mathfrak{D}$ is a solution of (6.1) and $y_1(0) = y_1 \in Y_1$.

It is clear that $y_1(t)$ is a unique $\mathfrak{D}$-solution of (6.2) with initial point in Y_1. Thus there is a one-to-one linear correspondence between $g \in \mathfrak{B}$ and $\mathfrak{D}$-solutions $y_1(t)$ of (6.2) with $y_1(0) \in Y_1$. The proof of Lemma 6.2 shows that if T_1 is the restriction of $T = T_{\mathfrak{B}\mathfrak{D}}$ with domain consisting of elements $y(t) \in \mathscr{D}(T)$ satisfying $y(0) \in Y_1$, then T_1 is closed. Thus T_1 is a closed, linear, one-to-one operator which maps its domain in $\mathfrak{D}$ onto $\mathfrak{B}$. By the Open Mapping Theorem 0.3, there is a constant K such that if $T_1 y = g$, then $|y|_{\mathfrak{D}} \leqq K\,|g|_{\mathfrak{B}}$. This proves the theorem for $y_0 = 0$.

If $y_0 \neq 0$, let $y_1(t)$ be the unique $\mathfrak{D}$-solution of (6.2) satisfying $y_1(0) \in Y_1$. Let $y_0(t)$ be the unique $\mathfrak{D}$-solution of the homogeneous equation (6.2) satisfying $y_0(0) = y_0$. Then $y(t) = y_0(t) + y_1(t)$ is a $\mathfrak{D}$-solution of (6.1), $P_0 y(0) = y_0$, and $|y|_{\mathfrak{D}} \leqq |y_0(t)|_{\mathfrak{D}} + |y_1(t)|_{\mathfrak{D}}$. By the part of the lemma already proved, $|y_1(t)|_{\mathfrak{D}} \leqq K\,|g|_{\mathfrak{B}}$ and, by Lemma 6.1, $|y_0(t)|_{\mathfrak{D}} \leqq C_0\,\|y_0\|$. This completes the proof of Lemma 6.3.

7. Green's Functions

Let $h_{0a}(t)$ be the characteristic function of the interval $0 \leqq t \leqq a$, so that $h_{0a}(t) = 1$ or 0 according as $0 \leqq t \leqq a$ does or does not hold. Similarly, let $h_a(t)$ be the characteristic function of the half-line $t \geqq a$, so that $h_a(t) = 1$ or 0 according as $t \geqq a$ or $t < a$.

A Banach space $\mathfrak{B}$ of functions on J: $0 \leqq t < \omega\ (\leqq \infty)$ will be called *lean at* $t = \omega$ if $\psi(t) \in \mathfrak{B}$ and $0 < a < \omega$ imply that $h_{0a}(t)\psi(t),\ h_a(t)\psi(t) \in \mathfrak{B}$; $|h_{0a}\psi|_{\mathfrak{B}},\ |h_a\psi|_{\mathfrak{B}} \leqq |\psi|_{\mathfrak{B}}$; and $|h_a\psi|_{\mathfrak{B}} \to 0$ as $a \to \omega$. Since $h_a(t)\psi(t) = \psi(t) - h_{0a}(t)\psi(t)$ on J, the property "lean at $t = \omega$" implies that the set of functions $h_{0a}(t)\psi(t)$ of $\mathfrak{B}$ vanishing outside of compact intervals $[0, a] \subset J$ is dense in $\mathfrak{B}$.

Let $\mathfrak{D}$ be a Banach space stronger than $L(Y)$. As above, let $Y_1 = Y_{1\mathfrak{D}}$ be a subspace of Y complementary to $Y_{\mathfrak{D}}$. Let $P_0 = P_{0\mathfrak{D}}$ be the projection of Y onto $Y_{\mathfrak{D}}$ annihilating Y_1, and $P_1 = I - P_0$ the projection of Y onto Y_1 annihilating $Y_{\mathfrak{D}}$. In terms of a fixed basis on Y, P_0 and P_1 are representable as matrices.

Let $U(t)$ be the fundamental matrix for (6.1) on $0 \leqq t < \omega$ satisfying $U(0) = I$. For $0 \leqq s, t < \omega$, define a (matrix) function $G(t, s)$ by

$$(7.1)\qquad \begin{aligned} G(t, s) &= U(t)P_0U^{-1}(s) \qquad \text{for}\quad 0 \leqq s \leqq t,\\ G(t, s) &= -U(t)P_1U^{-1}(s) \qquad \text{for}\quad 0 \leqq t < s. \end{aligned}$$

For a fixed t, $G(t, s)$ is continuous on $0 \leqq s < \omega$, except at $s = t$, where it has left and right limits, $U(t)P_0U^{-1}(t)$ and $-U(t)P_1U^{-1}(t)$.

Theorem 7.1. *Let $A(t)$ be locally integrable on J. Suppose that $\mathfrak{B}$, $\mathfrak{D}$ are Banach spaces stronger than $L(Y)$; that $\mathfrak{B}$ is lean at ω; and that*

$\mathfrak{D}$ has the property that if $y(t)$, $y_1(t)$ are continuous functions from J to Y and $y(t) - y_1(t) \equiv 0$ near $t = \omega$ (i.e., $y_1 - y_2 = 0$ except on an interval $[0, a] \subset J$), then $y(t) \in \mathfrak{D}$ implies that $y_1(t) \in \mathfrak{D}$. Then $(\mathfrak{B}, \mathfrak{D})$ is admissible for (6.1) *if and only if, for every $g(t) \in \mathfrak{B}$,*

$$y(t) = \int_0^\omega G(t, s)g(s)\,ds = \lim_{a\to\omega} \int_0^a G(t, s)g(s)\,ds \tag{7.2}$$

exists in $\mathfrak{D}$. In this case, the limit is uniform on compact intervals of J and is the unique $\mathfrak{D}$-solution of (6.1) *with $y(0) \in Y_1$.*

Proof. *"Only if".* Let $g(t) \in \mathfrak{B}$, $g_a(t) = h_{0a}(t)g(t)$. Then (7.2) becomes

$$y_a(t) = \int_0^\omega G(t, s)g_a(s)\,ds = \int_0^a G(t, s)g(s)\,ds, \tag{7.3}$$

where the integral exists as a Lebesgue integral for every fixed t, since $G(t, s)$ is bounded for $0 \leqq s \leqq a$ and $g_a(s)$ is integrable over J. In view of the first part of (7.1), the contribution of $0 \leqq s \leqq t$ to (7.3) is

$$U(t)P_0\int_0^t U^{-1}(s)g_a(s)\,ds = U(t)\int_0^t U^{-1}(s)g_a(s)\,ds - U(t)P_1\int_0^t U^{-1}(s)g_a(s)\,ds.$$

Hence, by the second part of (7.1), (7.3) is

$$y_a(t) = U(t)\int_0^t U^{-1}(s)g_a(s)\,ds + U(t)y_a(0), \tag{7.4}$$

where

$$y_a(0) = -P_1\int_0^\omega U^{-1}(s)g_a(s)\,ds. \tag{7.5}$$

It follows from (7.4) and Corollary IV 2.1 that $y_a(t)$ is a solution of (6.1) when $g(t)$ is replaced by $g_a(t)$.

An analogue of the derivation of (7.4) gives

$$y_a(t) = -U(t)\int_t^a U^{-1}(s)g_a(s)\,ds + U(t)P_0\int_0^a U^{-1}(s)g_a(s)\,ds.$$

Hence

$$U^{-1}(a)y_a(a) = P_0\int_0^a U^{-1}(s)g_a(s)\,ds \in Y_{\mathfrak{D}}.$$

Thus for $a \leqq t < \omega$, $y_a(t)$ is identical with the solution $U(t)\,U^{-1}(a)y_a(a)$ of the homogeneous equation (6.2). Since the initial point of the latter solution is in $Y_{\mathfrak{D}}$, the property assumed for $\mathfrak{D}$ implies that $y_a(t) \in \mathfrak{D}$.

Since $y_a(0) \in Y_1$ by (7.5), it follows that $y_a(t)$ is the unique solution of (6.1), where $g = g_a(t)$, satisfying $y_a(0) \in Y_1$. Hence, by Lemma 6.3, $|y_a|_{\mathfrak{D}} \leqq K|g_a|_{\mathfrak{B}}$.

Let $0 < a < b < \omega$. Then, since $\mathfrak{B}$ is lean at $t = \omega$,

$$|y_a - y_b|_{\mathfrak{D}} \leqq K\,|g_a - g_b|_{\mathfrak{B}} \leqq 2K\,|h_a g|_{\mathfrak{B}} \to 0 \qquad \text{as} \quad a \to \omega$$

Thus $y = \lim y_a(t)$ exists in $\mathfrak{D}$ as $a \to \omega$. Also $g = \lim g_a(t)$ in $\mathfrak{B}$. Since $T = T_{\mathfrak{B}\mathfrak{D}}$ in Lemma 6.2 is closed, $y(t)$ is a $\mathfrak{D}$-solution of (6.1). The proof of Lemma 6.3 shows that $y = \lim y_a(t)$ uniformly on compact intervals of J. Hence, $y(0) = \lim y_a(0) \in Y_1$. This proves "only if" in Theorem 7.1. The "if" part is easy.

Corollary 7.1. *Let $\omega = \infty$; B and D be Banach spaces of class $\mathscr{T}^{\#}$; B' be the space associate to B; cf.* § XIII 9. *For the admissibility of* $(B(Y), D(Y))$,(i) *it is necessary that $\|G(t, \cdot)\| \in B'$ for fixed t—thus the integrals in* (7.2) *are Lebesgue integrals;* (ii) *when B is lean at ∞, it is necessary and sufficient that* (7.2) *define a bounded operator $g \to y$ from $B(Y)$ to $D(Y)$;* (iii) *it is sufficient that $r(t) \in D$ where $r(t) = |\,\|G(t, \cdot)\|\,|_{B'}$;* (iv) *when $D = L^\infty$, it is necessary and sufficient that $r(t) \in L^\infty$.*

Exercise 7.1. Verify this corollary.

8. Nonlinear Equations

Lemmas 6.1–6.3 will be used to study the nonlinear equation

$$y' = A(t)y + f(t, y). \tag{8.1}$$

Let $\mathfrak{B}$, $\mathfrak{D}$ be Banach spaces stronger than $L(Y)$ and Σ_ρ the closed ball

$$\Sigma_\rho = \{y(t)\colon\ y(t) \in \mathfrak{D}, \qquad |y|_{\mathfrak{D}} \leqq \rho\} \qquad \text{in } \mathfrak{D}.$$

Theorem 8.1. *Let J; $0 \leqq t < \omega$ $(\leqq \infty)$; $A(t)$ a locally, integrable $d \times d$ matrix function on J, and $(\mathfrak{B}, \mathfrak{D})$ admissible for* (6.1). *Let $f(t, y(t))$ be an element of $\mathfrak{B}$ for every $y(t) \in \Sigma_\rho$ and satisfy*

$$|f(t, y_1(t)) - f(t, y_2(t))|_{\mathfrak{B}} \leqq \theta |y_1(t) - y_2(t)|_{\mathfrak{D}} \tag{8.2}$$

for all $y_1(t), y_2(t) \in \Sigma_\rho$ and some constant θ; $r = |f(t, 0)|_{\mathfrak{B}}$; $y_0 \in Y_{\mathfrak{D}}$. Suppose that if C_0, K are the constants in Lemma 6.3, *then θ, r, $\|y_0\|$ are so small that*

$$C_0\|y_0\| + Kr \leqq \rho(1 - \theta K) \qquad \text{and} \quad \theta K < 1. \tag{8.3}$$

Then (8.1) *has a unique solution $y(t) \in \Sigma_\rho$ satisfying*

$$P_0 y(0) = y_0. \tag{8.4}$$

It will be clear from the proof that the first part of (8.3) can be replaced by the assumption

$$C_0\,\|y_0\| + K\,|\,f(t, y(t)|_{\mathfrak{B}} \leqq \rho \qquad \text{for all} \quad y(t) \in \Sigma_\rho. \tag{8.5}$$

In fact, the role of the assumption in (8.3) is to assure (8.5). In (8.4), P_0 is the projection of Y onto $Y_{\mathfrak{D}}$ annihilating a fixed subspace Y_1, where $Y = Y_{\mathfrak{D}} \oplus Y_1$.

Proof. Theorem 8.1 is an immediate consequence of Theorem 0.1 and Lemma 6.3. Since $f(t, x(t)) \in \mathfrak{B}$ for any $x(t) \in \Sigma_\rho$, Lemma 6.3 and the assumption that $(\mathfrak{B}, \mathfrak{D})$ is admissible imply that

$$(8.6) \qquad y' = A(t)y + f(t, x(t))$$

has a unique $\mathfrak{D}$-solution $y(t)$ satisfying (8.4) and (6.8), where $g(t) = f(t, x(t))$. Define the operator T_0 from Σ_ρ into $\mathfrak{D}$ by $y(t) = T_0[x(t)]$. In particular, if $m = |T_0[0]|_{\mathfrak{D}}$, then

$$(8.7) \qquad m \leqq C_0 \|y_0\| + Kr, \qquad \text{where} \quad r = |f(t, 0)|_{\mathfrak{B}}.$$

If $x_1(t), x_2(t) \in \Sigma_\rho$ and $y_1 = T_0[x_1]$, $y_2 = T_0[x_2]$, it follows that $y_1(t) - y_2(t)$ is the unique $\mathfrak{D}$-solution of

$$y' = A(t)y + f(t, x_1(t)) - f(t, x_2(t))$$

satisfying $P_0 y(0) = 0$. Hence, by Lemma 6.3 and by (8.2),

$$(8.8) \qquad |y_1 - y_1|_{\mathfrak{D}} \leqq \theta K |x_1 - x_2|_{\mathfrak{D}}.$$

Consequently, Theorem 0.1 is applicable, and so T_0 has a unique fixed point $y(t) \in \Sigma_\rho$. This proves Theorem 8.1.

The statement of the next theorem involves the space $C(Y)$ of continuous functions $y(t)$ from J to Y with the topology of uniform convergence on compact intervals in J. The theorem will also involve an assumption concerning the continuity of the map $T_1[y(t)] = f(t, y(t))$ from the closure of the subset $\Sigma_\rho \cap C(Y)$ of $C(Y)$ into $\mathfrak{B}$. This condition is rather natural in dealing with Banach spaces $\mathfrak{B}$, $\mathfrak{D}$ of continuous functions on J with norms which imply uniform convergence on J. This is the case in Parts I and II, where J is replaced by a closed bounded interval $0 \leqq t \leqq p$. This continuity condition will also be satisfied under different circumstances in Corollary 8.1.

Theorem 8.2. *Let $A(t)$ be locally integrable on J; $\mathfrak{B}$, $\mathfrak{D}$ Banach spaces stronger that $L(Y)$; Σ_ρ the closed ball of radius ρ in $\mathfrak{D}$; and S the closure of $\Sigma_\rho \cap C(Y)$ in $C(Y)$. Let $A(t)$ and $f(t, y)$ satisfy* (i) $(\mathfrak{B}, \mathfrak{D})$ *is admissible for* (6.1); (ii) $y(t) \to f(t, y(t))$ *is a continuous map of the subset S of the space $C(Y)$ into $\mathfrak{B}$*; (iii) *there exists an $r > 0$ such that*

$$(8.9) \qquad |f(t, y(t))|_{\mathfrak{B}} \leqq r \qquad \text{for} \quad y(t) \in S;$$

and (iv) *there exists a function $\lambda(t) \in L$ such that*

$$(8.10) \qquad \|f(t, y(t))\| \leqq \lambda(t) \qquad \text{for} \quad t \in J, \quad y(t) \in S.$$

Let C_0, K be the constants of Lemma 6.3 *and let $y_0 \in Y_{\mathfrak{D}}$. Let r, $\|y_0\|$ be so small that*

$$(8.11) \qquad C_0\|y_0\| + Kr \leqq \rho.$$

Then (8.1) *has at least one solution $y(t) \in \Sigma_\rho$ satisfying $P_0y(0) = y_0$.*

Proof. As in the last proof, define an operator T_0 of S into $\mathfrak{D}$ by putting $y = T_0[x]$, where $x(t) \in S$ and $y(t)$ is the unique $\mathfrak{D}$-solution of (8.6) satisfying (8.4). Thus, by Lemma 6.3,

$$|y|_{\mathfrak{D}} \leqq C_0 \|y_0\| + K\,|f(t, x(t))|_{\mathfrak{B}} \leqq C_0 \|y_0\| + Kr.$$

Hence assumption (8.11) implies that T_0 maps S into itself, in fact, into $\Sigma_\rho \cap C(Y) \subset S$.

Note that the basic inequality (6.5) implies that

$$\|y(t)\| \leqq \left\{\frac{1}{a}\int_0^a \|y(s)\|\,ds + \int_0^a \|g(s)\|\,ds\right\} \exp \int_0^a \|A(s)\|\,ds$$

for $0 \leqq t \leqq a$ if $g(t) = f(t, x(t))$. Since $\mathfrak{D}$ is stronger than $L(Y)$, (6.6) holds. Also there is a similar inequality for elements $g \in \mathfrak{B}$ with a suitable constant $\alpha_{\mathfrak{B}}(a)$. Hence, for $0 \leqq t \leqq a$,

$$(8.12) \qquad \|y(t)\| \leqq \left\{\frac{1}{a}\,\alpha_{\mathfrak{D}}(a)\,|y|_{\mathfrak{D}} + \alpha_{\mathfrak{B}}(a)\,|g|_{\mathfrak{B}}\right\} \exp \int_0^a \|A(s)\|\,ds.$$

It will first be verified that T_0: $S \to S$ is continuous where S is considered to be a subset of $C(Y)$. Let $x_j(t) \in S$, $g_j(t) = f(t, x_j(t))$, $y_j(t) = T_0[x_j(t)]$ for $j = 1, 2$, then $y_1(t) - y_2(t)$ is the unique $\mathfrak{D}$-solution of (6.1), where $g = g_1 - g_2$, satisfying $P_0[y_1(0) - y_2(0)] = 0$. Hence Lemma 6.3 implies that

$$|y_1 - y_2|_{\mathfrak{D}} \leqq K\,|g_1 - g_2|_{\mathfrak{B}}.$$

Also, (8.12) holds if $y = y_1 - y_2$ and $g = g_1 - g_2$. Thus, for $0 \leqq t \leqq a$

$$\|y_1(t) - y_2(t)\| \leqq \left\{\frac{1}{a}\,\alpha_{\mathfrak{D}}(a)K + \alpha_{\mathfrak{B}}(a)\right\} |g_1 - g_2|_{\mathfrak{B}} \exp \int_0^a \|A(s)\|\,ds.$$

Since, by assumption (ii), $x_1(t) \to x_2(t)$ in $C(Y)$ implies $g_1 \to g_2$ in $\mathfrak{B}$, it follows that $y_1(t) \to y_2(t)$ uniformly on intervals $[0, a]$ of J; i.e., $y_1(t) \to y_2(t)$ in $C(Y)$. This proves the continuity of T_0: $S \to S$.

It will now be shown that the image T_0S of S has a compact closure in $C(Y)$. It follows from (8.12), where $g(t) = f(t, x(t))$ and $y(t) = T_0[x(t)]$ that, for $0 \leqq t \leqq a$,

$$\|y(t)\| \leqq \left\{\frac{1}{a}\,\alpha_{\mathfrak{D}}(a)\rho + \alpha_{\mathfrak{B}}(a)r\right\} \exp \int_0^a \|A(s)\|\,ds.$$

Thus the set of functions $y(t) \in T_0S$ are uniformly bounded on every interval $[0, a]$ of J. If $c(a)$ is the number on the right of the last inequality, then (8.6) and (8.10) show that

$$\|y(t) - y(s)\| \leqq c(a)\int_s^t \|A(u)\|\, du + \int_s^t \lambda(u)\, du \qquad \text{for} \quad 0 \leqq s \leqq t \leqq a.$$

Therefore, the functions $y(t)$ in the image T_0S of S are equicontinuous on every interval $[0, a] \subset J$. Consequently, Arzela's theorem shows that T_0S has a compact closure in $C(Y)$. Since S is convex and closed in $C(Y)$, it follows from Corollary 0.1 that T_0 has a fixed point $y(t) \in S$. Thus Theorem 8.2 is a consequence of the fact that $y(t) = T_0[y(t)] \in \Sigma_\rho \cap C(Y)$.

It is convenient to have conditions on $\mathfrak{B}$, $\mathfrak{D}$, $f(t, y)$, $\lambda(t)$ which imply (ii), (iii), (iv) in Theorem 8.2.

Assumption (H_0) on $\mathfrak{B} = B(X)$: *Let $\mathfrak{B} = B(X)$ (cf. § 6), where X is a subspace of Y and B is a Banach space of real-valued functions on J such that (i) B is stronger than L; (ii) B is lean at $t = \omega$ (cf. § 7); (iii) B contains the characteristic function $h_{0a}(t)$ of the intervals $[0, a] \subset J$; and (iv) if $\varphi_1(t) \in B$ and $\varphi_2(t)$ is a measurable function on J such that $|\varphi_2(t)| \leqq |\varphi_1(t)|$, then $\varphi_2(t) \in B$ and $|\varphi_2|_{\mathfrak{B}} \leqq |\varphi_1|_{\mathfrak{B}}$.*

It is important to have $\mathfrak{B} = B(X)$ rather than $\mathfrak{B} = B(Y)$ for applications to higher order equations. If such equations are written as systems of differential equations of the first order, the "inhomogeneous term $f(t, y)$" will generally belong to a subspace X of Y; e.g., $f(t, y)$ might be of the form $(h, 0, \ldots, 0)$.

Examples of spaces B satisfying the conditions in (H_0) are $B = L^p$, $1 \leqq p < \infty$, and $B = L_0^\infty$ (but not $B = L^\infty$). Other such spaces B can be obtained as follows: Let $\psi(t) > 0$ be a measurable function such that $\psi(t)$ and $1/\psi(t)$ are bounded on every interval $0 \leqq t \leqq a$ $(<\omega)$. Denote by $B = L_{\psi 0}^\infty$ the space of functions $\varphi(t)$ on J such that $\varphi(t)/\psi(t) \in L_0^\infty$ with the norm $|\varphi|_{\mathfrak{B}} = |\varphi/\psi|_\infty$. The space $B = L_{\psi 0}^\infty$ satisfies conditions (i)–(iv). For this space, $\lambda(t) \in B$ holds if

$$(8.13) \qquad 0 \leqq \lambda(t) \leqq \psi(t) \quad \text{and} \quad \frac{\lambda(t)}{\psi(t)} \to 0 \quad \text{as} \quad t \to \omega.$$

Assumption (H_1) on $f(t, y)$: *Let $f(t, y)$ be continuous on the product set of J and the ball $\|y\| \leqq \rho$ in Y, let f have values in X, and let there exist a function $\lambda(t) \in L$ such that*

$$(8.14) \qquad \|f(t, y)\| \leqq \lambda(t) \quad \text{for} \quad t \in J,\ \|y\| \leqq \rho.$$

Corollary 8.1. *Let $A(t)$ be locally integrable on J, $(\mathfrak{B}, \mathfrak{D})$ admissible for* (6.1), $\mathfrak{B}$ *satisfies* (H_0), $\mathfrak{D} = L^\infty(Y)$ *[or* $\mathfrak{D} = L_0^\infty(Y)$*]*, $f(t, y)$ *satisfies* (H_1) *and $\lambda(t) \in B$ with $r = |\lambda|_B$. Let $y_0 \in Y_{\mathfrak{D}}$. Then, if* (8.11) *holds,* (8.1) *has* *at least one solution* $y(t)$ *on* $0 \leqq t < \omega$ *satisfying* $P_0 y(0) = y_0$, $\|y(t)\| \leqq \rho$ [*and* $y(t) \to 0$ *as* $t \to \omega$].

Exercise 8.1. Verify Corollary 8.1.

Exercise 8.2. Let Y be expressed as a direct sum $Y_{\mathfrak{D}} \oplus Y_1$; let P_0 be the projection of Y onto $Y_{\mathfrak{D}}$ annihilating Y_1, and $P_1 = I - P_0$ the projection of Y onto Y_1 annihilating $Y_{\mathfrak{D}}$. Let $A(t)$ be locally integrable on J: $0 \leqq t < \infty$. Define $G(t, s)$ by (7.1) and suppose that there exist constants N, $\nu > 0$ such that $\|G(t, s)\| \leqq Ne^{-\nu|t-s|}$ for s, $t \geqq 0$. Let $f(t, y)$ be continuous for $0 \leqq t < \infty$, $\|y\| \leqq \rho$, and let $\|f(t, y)\| \leqq r$. Let $y_0 \in Y_{\mathfrak{D}}$. Show that if $\|y_0\|$ and $r > 0$ are sufficiently small, then (8.1) has a solution $y(t)$ for $0 \leqq t < \infty$ satisfying $\|y(t)\| \leqq \rho$ and $P_0 y(0) = y_0$. (For necessary and sufficient conditions assuring these assumptions on G, see Theorems XIII 2.1 and XIII 6.4.)

9. Asymptotic Integration

In this section, let J be the half-line J: $0 \leqq t < \infty$ (so that $\omega = \infty$). As a corollary of Theorem 8.2, we have:

Theorem 9.1. *Let* $A(t)$ *be continuous on* J: $0 \leqq t < \infty$. *Let* $f(t, y)$ *be continuous for* $t \geqq 0$, $\|y\| \leqq \rho$, *satisfy*

$$\text{(9.1)} \qquad \|f(t, y)\| \leqq \lambda(t) \quad \text{for} \quad t \geqq 0, \quad \|y\| \leqq \rho,$$

and have values in a subspace X *of* Y. *Assume either* (i) *that* $\lambda(t) \in L^1$ *and that* $(L^1(X), \mathfrak{D})$, *where* $\mathfrak{D} = L^\infty(Y)$ [*or* $\mathfrak{D} = L_0^\infty(Y)$], *is admissible for*

$$\text{(9.2)} \qquad y' = A(t)y + g(t);$$

or (ii) *that there exists a measurable function* $\psi(t) > 0$ *on* J *such that* $\psi(t)$ *and* $1/\psi(t)$ *are locally bounded, that*

$$\text{(9.3)} \qquad 0 \leqq \lambda(t) \leqq \psi(t) \quad \text{and} \quad \frac{\lambda(t)}{\psi(t)} \to 0 \quad \text{as} \quad t \to \infty,$$

and that for every $g(t) \in L(X)$, *for which*

$$\text{(9.4)} \qquad \frac{g(t)}{\psi(t)} \to 0 \quad \text{as} \quad t \to \infty,$$

(9.2) *has a* $\mathfrak{D}$*-solution. Then if* t_0 *is sufficiently large, the system*

$$\text{(9.5)} \qquad y' = A(t)y + f(t, y)$$

has a solution for $t \geqq t_0$ *such that* $\|y(t)\| \leqq \rho$ [*and* $y(t) \to 0$ *as* $t \to \infty$].

Remark 1. Assumption (ii) merely means that $(L_{\psi 0}^\infty(X), \mathfrak{D})$ is admissible for (9.2). Actually, assumption (i) is a special case of (ii) but is isolated for convenience. For a discussion of conditions necessary and sufficient for $(L^1(X), L^\infty(X))$ or $(L^1(X), L_0^\infty(X))$ to be admissible for (9.2), where $X = Y$, see Theorem XIII 6.3.

Remark 2. Let $U(t)$ be the fundamental solution for

$$y' = A(t)y \tag{9.6}$$

satisfying $U(0) = I$. Let $y_0 \in Y_{\mathfrak{D}}$. Then if $\|y_0\|$ is sufficiently small and $t_0 \geqq 0$ is sufficiently large, the solution $y(t)$ in Theorem 9.1 can be chosen so as to satisfy

$$U^{-1}(t_0)y(t_0) = y_0.$$

Let C_0, K be the constants of Lemma 6.3 associated with the admissibility of the appropriate pairs of spaces $(L^1(X), \mathfrak{D})$ or $(L_{\psi 0}^\infty(X), \mathfrak{D})$. According as (i) or (ii) is assumed, the conditions of smallness on $\|y_0\|$ and largeness of t_0 are

$$C_0 \|y_0\| + K\int_{t_0}^\infty \lambda(t)\,dt \leqq \rho \quad \text{or} \quad C_0 \|y_0\| + \frac{K\lambda(t)}{\psi(t)} \leqq \rho \quad \text{for } t \geqq t_0.$$

Proof. Let $\mathfrak{B} = L^1(X)$ or $\mathfrak{B} = L_{\psi 0}^\infty(X)$ according as (i) or (ii) is assumed. Then Theorem 9.1 is a consequence of Corollary 8.1 obtained by replacing $f(t, y)$, $\lambda(t)$ by the functions $h_a(t)f(t, y)$, $h_a(t)\lambda(t)$, where $a = t_0$ and $h_a(t)$ is 1 or 0 according as $t \geqq a$ or $t < a$.

Exercise 9.1. The following type of question often arises: Let $y_1(t)$ be a solution of the homogeneous linear system (9.6). When does (9.5) have a solution $y(t)$ for large t such that $y - y_1 \to 0$ as $t \to \infty$? Deduce sufficient conditions from Theorem 9.1.

As an application of Theorem 9.1, consider a second order equation

$$u'' = h(t, u, u') \tag{9.7}$$

for a real-valued function u. Assume that $h(t, u, u')$ is continuous for $t \geqq 0$ and arbitrary (u, u'). Let α, β be constants and consider the question whether (9.7) has a solution for large t satisfying

$$u(t) - \alpha t - \beta \to 0 \quad \text{and} \quad u'(t) - \alpha \to 0 \quad \text{as} \quad t \to \infty \tag{9.8}$$

Introduce the change of variables $u \to v$, where

$$u = \alpha t + \beta + v, \tag{9.9}$$

then (9.7) becomes

$$v'' = h(t, \alpha t + \beta + v, \alpha + v') \tag{9.10}$$

and (9.8) is $v, v' \to 0$ as $t \to \infty$. Theorem 9.1 implies the following:

Corollary 9.1 *Let $h(t, u, u')$ be continuous for $t \geqq 0$ and arbitrary (u, u') such that*

$$|h(t, \alpha t + \beta + u, \alpha + u')| \leqq \lambda(t) \qquad \text{for} \quad |u|, |u'| \leqq \rho,$$

where $\lambda(t)$ *is a function satisfying*

$$\int^{\infty} t\lambda(t)\,dt < \infty.$$

Then (9.7) *has a solution* $u(t)$ *for large* t *satisfying* (9.8).

Exercise 9.2. (*a*) Verify Corollary 9.1. (*b*) Apply it to the case that $h = f(t)g(u)$, where $\alpha \neq 0$ or $\alpha = 0$. (*c*) Generalize it by replacing (9.7) by $u^{(d)} = h(t, u, u', \ldots, u^{(d-1)})$.

Actually Corollary 9.1 is a special case of Theorem X 13.1, but Theorem X 13.1 can itself be deduced from Theorem 9.1; cf. Exercise 9.3 below.

Many problems involving asymptotic integrations can be solved by the use of Theorem 9.1. Often these problems can be put into the following form: Let $Q(t)$ be a continuously differentiable matrix for $t \geqq 0$. Does the nonlinear system (9.5) have a solution $y(t)$ such that if

(9.11) $$y = Q(t)x,$$

then $c = \lim x(t)$ exists as $r \to \infty$? The differential equation for $x(t)$ is

(9.12) $$x' = Q^{-1}(t)[A(t)Q(t) - Q'(t)]x + Q^{-1}(t)f(t, Q(t)x).$$

The change of variables

(9.13) $$z = x - c$$

transforms (9.12) into

(9.14) $$z' = Q^{-1}(AQ - Q')z + g(t, z, c),$$

where

(9.15) $$g(t, z, c) = Q^{-1}(AQ - Q')c + Q^{-1}f(t, Qz + Qc).$$

The problem is thus reduced to the question: Does (9.14) have a solution $z(t)$ for large t such that $z(t) \to 0$ as $t \to \infty$? Clearly, Theorem 9.1 is adapted to answer such questions.

We should point out that if the answer is affirmative, then (9.11) and the conclusion $x(t) - c \to 0$ as $t \to \infty$ need not be very informative unless estimates for $\|x(t) - c\|$ are obtained [e.g., if $Q(t)$ is the 2×2 matrix $Q(t) = (q_{jk}(t))$, where $q_{k1} = (-1)^k e^{-t}$, $q_{k2} = e^t$ for $k = 1, 2$ and $c = (1, 0)$, then we can only deduce $y(t) = o(e^t)$, but not an asymptotic formula of the type $y(t) = (-1 + o(1), 1 + o(1))e^{-t}$ as $t \to \infty$.]

Exercise 9.3. Follow the procedure just mentioned and deduce Theorem X 13.1 by using Theorem 9.1 (instead of Lemma X 4.3).

Notes

INTRODUCTION. The use of fixed point theorems in function spaces was initiated by Birkhoff and Kellogg [1]. For Theorem 0.2, see Tychonov [1]. For Schauder's fixed point theorem, see Schauder [1]. For the remark at the end of the Introduction, see Graves [1]. As mentioned in the text, Theorem 0.3 is a result of Banach [1].

SECTION 1. Results analogous to those of this section but dealing with one equation of the second order, e.g., go back to Sturm. Boundary value problems for systems of second order equations were considered by Mason [1]. The results of this section (except for Theorem 1.3) are due to Bounitzky [1]; the treatment in the text follows Bliss [1]. These results are merely the introduction to the subject which is usually concerned with eigenfunction expansions; see Bliss [1] for older references to Hildebrandt, Birkhoff, Langer, and others. For an excellent recent treatment for the singular, self-adjoint problem; see Brauer [2]. Theorem 1.3 is given by Massera [1], who attributes the proof in the text to Bohnenblust.

SECTION 2. Theorems 2.1 and 2.2 are similar to Theorems 4.1 and 4.2, respectively. Exercise 2.1 is a result of Massera [1] and generalizes a theorem of Levinson [2]; its proof depends on a (2-dimensional) fixed point theorem of Brouwer. Exercise 2.2 is a result of Knobloch [1], who uses a variant of Brouwer's fixed point theorem due to Miranda [1]; cf. Conti and Sansone [1, pp. 438–444].

Theorems 2.3 and 2.4 are due to Poincaré [5, I, chap. 3 and 4]; see Picard [2, III, chap. 8]. Problems concerning "degenerate" cases of Theorems 2.3 and 2.4 when the Jacobians in the proofs vanish were also treated by Poincaré and since then by many others, including Lyapunov. For some more recent work and older references, see E. Hölder [1], Friedrichs [1], and J. Hale [1]; for the problem in a very general setting, see D. C. Lewis [4].

SECTION 3. The scalar case of Theorem 3.3 is a result of Picard [4]; the extension to systems is in Hartman and Wintner [22]. In the scalar case, (3.17) can be relaxed to the condition Re $B(t)x \cdot x \geqq 0$, Rosenblatt [2]; see also Exercise 4.5(*c*). The uniqueness criterion in Exercise 3.3(*b*), among others, is given by Hartman and Wintner [22]. Sturm types of comparison theorems for self-adjoint systems have been given by Morse [1].

SECTION 4. Theorem 4.1 and its proof are due to Picard [4, pp. 2–7]. For related results in the scalar case, see Nagumo [2], [4], references in Hartman and Wintner [8] and Lees [1] to Rosenblatt, Cinquini, Zwirner, and others. Theorem 4.2 is a result of Scroza-Dragoni [1]. The uniqueness Theorem 4.3 is due to Hartman [19]. For Exercise 4.6(*b*), see Hartman and Wintner [8]; for part (*c*), with the additional condition that f has a continuous partial derivative $\partial f/\partial x \geqq 0$, see Rosenblatt [2]. For Exercises 4.7 and 4.8, see Nirenberg [1].

SECTION 5. Lemma 5.1 and Corollary 5.2 are results of Nagumo [2]. The example following Lemma 5.1 is due to Heinz [1]. The other theorems of this section are contained in Hartman [19]. Exercise 5.4 is a generalization of a result of Lees [1] who gives a very different proof from that in the *Hints*. For the scalar case in Exercise 5.5(*d*), see Hartman and Wintner [8]; this result was first proved by A. Kneser [2] (see Mambriani [1]) for the case when f does not depend on x'. For related results, see Exercises XIV 2.8 and 2.9. A generalization of Exercise 5.9 involving almost periodic functions is given in Hartman [19] and is based on a paper of Amerio [1].

SECTION 6. Part III is an outgrowth of a paper of Perron [12], whose results were carried farther by Persidskiĭ [1], Malkin [1], Krein [1], Bellman [2], Kučer [1], and Maizel' [1]. Except for Kučer, these authors deal, for the most part, with the case $\mathfrak{B} = L^\infty(Y)$, $\mathfrak{D} = L^\infty(Y)$. (For a statement concerning the results of these earlier papers, see Massera and Schäffer [1, I].) The results of this section are due to Massera and Schäffer [1] who deal with the more general situation when the space Y need not be finite-dimensional.

SECTION 7. For the notion of "lean at ω," see Schäffer [2, VI]. The Green's functions G of this section occur in Massera and Schäffer [1, I and IV]. Theorem 7.1 and Corollary 7.1 may be new.

SECTION 8. Theorem 8.1 is a result of Corduneanu [1]. Theorem 8.2 is a corrected version of a similar result of Corduneanu[1] (see Hartman and Onuchic [1]); also Massera [8]. For Corollary 8.1, see Hartman and Onuchic [9]. For Exercise 8.2, see Massera and Schäffer [1, I or IV].

SECTION 9. This application of the results of § 8 is given by Hartman and Onuchic [1]. For Corollary 9.1, see Hale and Onuchic [1].