MA 543 Spring 2025 (Aaron N. K. Yip)
Homework 2, due on Thursday, Feb. 13th, in class

In the following, [M] refers to our official textbook by Meiss, *revised edition, 2017*, which is available online through the Purdue Library page.

As mentioned in the course policy, you can submit as a group consisting of up to three people. You are also allowed to consult and utilize online resouces and information, such as Wikipedia, plotting routines, and so forth. But submitting your solution as a complete "duplication" of online output is not acceptable. Your solution should explain your *solution and thought process in a clear and comprehensive way.*

1. [M Section 3.6]: #7, 13, 14.

2. This problem shows that $AB = BA$ *if and only if* $e^{At}e^{Bt} = e^{(A+B)t}$ *for all* t. It is based on [M Section 2.9, #6] but you can just follow the instructions below.

   (Beware: unless $AB = BA$, it is *just not* true that $(A+B)^2 = A^2 + 2AB + B^2$, but rather $(A+B)^2 = A^2 + AB + BA + B^2$.)

   (a) [M Section 2.9, #6(a)] *If* $AB = BA$, show that $e^{At}e^{Bt} = e^{(A+B)t}$ for all t.
   (Hint: find the series expansions of e^{At} and e^{Bt}, multiply them together and then compare it with the series expanson of $e^{(A+B)t}$. You can assume/accept/should know the fact that within radius of convergence of power series, you can treat them just like normal polynomials, with finite degrees. So you can add, subtract, multiply and long-divide power series. In addition, the radius of convergence for the power series for the exponential is *infinity*.)

   (b) [M Section 2.9, #6(d)] *If* $e^{At}e^{Bt} = e^{(A+B)t}$ for all t, then $AB = BA$.
   (Hint: differentiate both sides twice and set $t = 0$.)

   (c) A repetition of the above. *If* $e^{At}e^{Bt} = e^{(A+B)t}$ for all t, then $AB = BA$.
   (Hint: find the power series of $e^{At}e^{Bt}$ and $e^{(A+B)t}$ and compare the coefficients of the t^2 terms.)

3. Find e^{At} where A is the following matrix:

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 \end{pmatrix}$$

(Hint: look up/search for the concept of Jordan form. *(i) If $A = D+N$ and $DN = ND$, then $e^{At} = e^{Dt}e^{Nt}$; (ii) If D is diagonal, then e^{Dt} is easy to compute; (iii) If N is nilpotent, i.e $N^k = 0$, then e^{Nt} is easy to compute by using power series.* Find the D and N in this problem.)

4. Though the theory and concept of Jordan form is powerful, it is not easy to compute and in fact, it is not stable under small perturbation of the matrix. There is a result, called *Schur's Triangulation* of matrices – feel free to look it up, e.g. [B, p.21, Chapter 1, Section 10, Thm 6], that can be much more easily proved and in fact produces almost all the results which can be obtained by using the Jordan form.

   Consider the following matrix

$$A = \begin{pmatrix} -3 & a_{12} & a_{13} \\ 0 & -2 & a_{23} \\ 0 & 0 & -2 \end{pmatrix}$$

   where a_{12}, a_{12} and a_{23} are some constants. Consider the differential equation $\dot{X} = AX$, $X(0) = X_0$.

   (a) Solve for $X(t)$ explicitly (component by component).

   (Hint: solve for the last component of $X(t)$ first.

   (b) Show that given any X_0, the solution $X(t)$ will go to zero exponentially fast as $t \longrightarrow +\infty$, i.e. $\|X(t)\| \leq Ce^{-Kt}\|X_0\|$ for $t > 0$. What is the value of K you would use?