MA 543 Spring 2025 (Aaron N. K. Yip)
Homework 5, due on Thursday, Mar. 27th, in class

In the following, [M] refers to our official textbook by Meiss, *revised edition, 2017*, which is available online through the Purdue Library page.

As mentioned in the course policy, you can submit as a group consisting of up to three people. You are also allowed to consult and utilize online resouces and information, such as Wikipedia, plotting routines, and so forth. But submitting your solution as a complete "duplication" of online output is not acceptable. Your solution should explain your *solution and thought process in a clear and comprehensive way.*

1. [M Section 5.7]: #1, 4, 8, 9, 10

2. Consider the following system:

$$\begin{aligned} \dot{x} &= xy + ax^3 + by^2x; \\ \dot{y} &= -y + cx^2 + dx^2y. \end{aligned}$$

Note that $(0,0)$ is a non-hyperbolic equilibrium point. This question analyzes the stability behavior of the above system *near the origin* $(0,0)$ *on the center manifold* $W^c(0,0)$ for different cases of the parameters a,b,c,d. As a start, write $W^c(0,0)$ as $(x,h(x))$, where $h(x) = \alpha x^2 + \beta x^2 + \gamma x^4 + \delta x^5 + \epsilon x^6 + \zeta x^7 \cdots$.

(a) Classify the stability of the origin on the W^c for $a+c \neq 0$.

(b) What if $a+c=0$?

(c) What if $a+c=0$ and $cd+bc^2=0$?

Remarks:

(a) How many terms you need in the expansion for h depends on the situation.

(b) This question clearly shows that for non-hyperbolic equilibrium point, you need to analyze very carefully the higher order terms. A more general theory lies in the realm of bifucation theory.

(c) This question also shows that the description of the dynamics *does not depend* on the *four parameters* a,b,c,d *separately.* An effective description in fact depends on much fewer parameters. Such a dimensional reduction is extremely useful in practice.

3. Consider the minimization of a function $f(x)$. A typical method is *negative gradient descent*:

$$\dot{x} = -\nabla f(x). \tag{1}$$

However, it is well-known that a practical optimization algorithm simply does not implement the gradient descent in a naive way. A method to accelerate the convergence rate to the minimizer is use Newton's Second Law with friction:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\nabla f(x) - ay, \end{aligned} \quad \text{or equivalently,} \quad \ddot{x} = -a\dot{x} - \nabla f(x) \tag{2}$$

where $a > 0$ is the frictional coefficient, to be chosen appropriately.

Consider the simplest one dimensional case, $f(x) = \frac{1}{2}\lambda x^2$, which is a convex function and $\lambda > 0$ is called the convexity or the curvature constant.

(a) Solve (1). What is the convergence rate to the optimal point ($x = 0$)?

(b) Solve (2). *If* you know the value of λ and *if* you choose $a = 2\sqrt{\lambda}$, what is the convergence rate to the optimal point ($x = 0$)?

(c) When will (2) gives a better rate than (1)?

Remark. Of course, in practice, the value of λ is not known or is difficult to compute/estimate. Feel free to look up further the topic on accelerated method for optimization which is a also good topic for the course final paper/project.