MA 543 Spring 2025 (Aaron N. K. Yip)
Homework 5, due on Thursday, Apr. 24th, in class

In the following, [M] refers to our official textbook by Meiss, *revised edition, 2017*, which is available online through the Purdue Library page.

As mentioned in the course policy, you can submit as a group consisting of up to three people. You are also allowed to consult and utilize online resouces and information, such as Wikipedia, plotting routines, and so forth. But submitting your solution as a complete "duplication" of online output is not acceptable. Your solution should explain your *solution and thought process in a clear and comprehensive way.*

1. [M Section 6.9]: #10, 12

2. Consider the following perturbed pendulum system:

$$\ddot{x} + \sin x = \mu f(t, x, \dot{x}), \quad |\mu| \ll 1 \text{ and } f \text{ is } T\text{-periodic in } t.$$

   (a) At $\mu = 0$. Write the system as a 2×2 system by introducing $y = \dot{x}$. Show that the equilibium points are given by $(x = n\pi, y = 0)$ where $n = 0, \pm 1, \pm 2, \pm 3 \ldots$. Use your favorite plotting software to plot the trajectory/phase-plot of the system.

   (b) At $\mu = 0$ again. Find the linearized system and fundamental matrix at the equilibrium points corresponding to $n = 0$ and $n = 1$. (Note that due to the periodicity of $\sin x$, these are essentially the only two equilibium points. All the other equilibrium points can be treated in the exactly the same way.)

   (c) Now for $\mu \neq 0$ but $|\mu| \ll 1$. Find (sufficient) condition(s) for T such that there is periodic solutions near the equilibrium point $n = 0$ and $n = 1$.

3. This problem is motivated by the phenomena of *Turing Instability*. Consider the following two matrices ($\epsilon, \delta > 0$):

$$A = \begin{pmatrix} -\delta & -\epsilon \\ 1 & -\delta \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} -\delta & 1 \\ -\epsilon & -\delta \end{pmatrix}$$

   (a) Describe the stability property corresponding to the linear system $\dot{X} = AX$ and $\dot{X} = BX$.

   (b) Now for $0 < \delta, \epsilon \ll 1$. Describe the stability property corresponding to the linear system $\dot{X} = (A + B)X$. Any interesting conclusion?

   (c) Use your favorite plotting software to plot the trajectory/phase-plot of the linear system corresponding to A, B and $A + B$.