# Chapter 9
# Weakly-Nonlinear Oscillators

Many systems function through repeated cycles of operation—from the spinning of gears in a machine, the physiology of heartbeats, biological behaviour on the 24 h circadian cycle, to seasonal climate changes during the motion of the earth around the sun each year. The most basic models for such systems take the form of oscillator equations, having regular predictable periodic solutions; the simplest such model is the *linear oscillator* equation,

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0, \tag{9.1}$$

with *natural frequency* $\omega_0$. It has the general solution $x(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t)$, with the constants $A$ and $B$ being determined by the initial conditions imposed on the system.

More detailed models of oscillatory systems include additional terms that describe other influences that modify the solutions. If these effects are weak, as indicated by a dimensionless system parameter being small, then the model can be written as

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \varepsilon f\left(x, \tfrac{dx}{dt}, t\right), \quad \varepsilon \to 0. \tag{9.2}$$

Such models are called *weakly-nonlinear oscillators* since they reduce to the linear oscillator (9.1) when the perturbation terms (potentially including nonlinearities) are suppressed. In the context of mechanical systems, Eq. (9.2) can describe the small-amplitude motion of a pendulum, or a mass attached to a nonlinear spring.

In the framework of the earlier chapters on perturbation methods, (9.2) may appear to be a straightforward regular perturbation problem, but we will see that regular perturbation expansions will not be able to address the question of greatest interest for oscillating systems, which is,

> *If we understand the behaviour of the system for a single cycle of the oscillation, can we determine how the perturbation forcing terms cumulatively affect the problem over long times and many oscillation periods?*



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Two perturbation methods will be described to illustrate how weak influences can be incorporated into the leading order solution to obtain more accurate long-time predictions of oscillatory behaviour.

## 9.1 Review of Solutions of the Linear Problem

We begin by briefly reviewing the essential results for the linear oscillator equation that form the basis for the perturbation methods for (9.2). As discussed above, the unforced linear oscillator equation is characterised by a natural frequency $\omega_0 > 0$ and has the general homogeneous periodic solution $x_h(t)$,

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0 \qquad \Longrightarrow \qquad x_h(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t). \tag{9.3}$$

Now consider the inhomogeneous version of this equation, driven by a *periodic* forcing function $f(t)$,

$$x'' + \omega_0^2 x = f(t).$$

The function $f$ can be directly replaced by its Fourier series, $f(t) = \sum_k C_k \sin(\gamma_k t) + D_k \cos(\gamma_k t)$, and then by the linearity of the equation, the overall solution will be the sum of the contributions from each of the terms in the series for $f$, yielding $x(t) = x_h(t) + \sum_k x_k(t)$. Each Fourier term yields a problem of the form

$$x_k'' + \omega_0^2 x_k = C_k \sin(\gamma_k t) + D_k \cos(\gamma_k t). \tag{9.4}$$

If $\gamma_k \neq \omega_0$ then the general solution is given as a combination of homogeneous and particular solutions as

$$x_k(t) = \underbrace{A\sin(\omega_0 t) + B\cos(\omega_0 t)}_{\text{Homogeneous solution}} + \underbrace{\frac{C_k}{\omega_0^2 - \gamma_k^2}\sin(\gamma_k t) + \frac{D_k}{\omega_0^2 - \gamma_k^2}\cos(\gamma_k t)}_{\text{Particular solution}}.$$

The particular solution can be obtained from the method of undetermined coefficients or other elementary approaches. This solution is not valid for $\gamma_k = \omega_0$, which is called the case of *resonant forcing* (forcing at a natural frequency of the system). For resonant forcing, the method of undetermined coefficients suggests a different form for the particular solution, $x_p(t) = c_1 t \sin(\omega_0 t) + c_2 t \cos(\omega_0 t)$. Substituting this into (9.4) and matching coefficients of corresponding terms yields the general solution as

$$x_{\omega_0}(t) = \underbrace{A\sin(\omega_0 t) + B\cos(\omega_0 t)}_{\text{Homogeneous solution}} + \underbrace{\frac{C}{2\omega_0} t\cos(\omega_0 t) + \frac{D}{2\omega_0} t\sin(\omega_0 t)}_{\text{Resonant forced response}}.$$

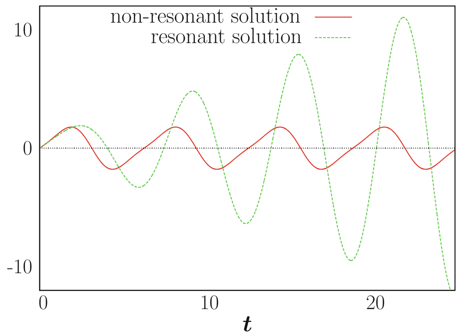


**Fig. 9.1** Solutions of the forced linear oscillator equation (9.4) with $\omega_0 = 1$: the non-resonant solution for $\gamma_k = 2$ and resonant solution for $\gamma_k = 1$

The resonant response, while being oscillatory with period $2\pi/\omega_0$, is notable for having an amplitude that exhibits unbounded growth with increasing time. The terms $t \cos \omega_0 t$ and $t \sin \omega_0 t$ are commonly called *secular growth terms* (see Fig. 9.1). We will see that the occurrence of such terms is the central issue that must be addressed in constructing accurate long-time asymptotic solutions for perturbed oscillators.

## 9.2 The Failure of Direct Regular Expansions

In order to illustrate the shortcomings of the standard regular perturbation expansion approach, we consider two simple perturbed linear oscillator problems in the limit $\varepsilon \to 0$, namely

$$x'' + x = -\varepsilon x, \qquad x(0) = 1, \qquad x'(0) = 0, \tag{9.5}$$

and

$$x'' + x = -\varepsilon x', \qquad x(0) = 1, \qquad x'(0) = -\tfrac{1}{2}\varepsilon. \tag{9.6}$$

Assuming the solutions to be perturbation expansions of the form

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots, \tag{9.7}$$

yields an ordered sequence of initial value problems for each $x_n(t)$ with the leading order for both (9.5) and (9.6) reducing to (9.3) with $\omega_0 = 1$. Solving these sequences of sub-problems up to $O(\varepsilon^2)$, yields the solution of (9.5) as

$$x(t) \sim \cos t - \tfrac{1}{2}\varepsilon t \sin t + \tfrac{1}{8}\varepsilon^2 \left(t \sin t - t^2 \cos t\right), \tag{9.8}$$

while the solution of (9.6) is given by

$$x(t) \sim \cos t - \tfrac{1}{2}\varepsilon t \cos t + \tfrac{1}{8}\varepsilon^2 \left(t \sin t + t^2 \cos t\right). \tag{9.9}$$

While these two expansions appear quite similar in form, this masks the fact that they come from problems predicting very different behaviours. Problem (9.5) can be re-written in the form (9.3) as $x'' + (1 + \varepsilon)x = 0$ and it is straight forward to show that its exact solution is

$$x_{\text{exact}}(t) = \cos(\sqrt{1 + \varepsilon}\, t). \tag{9.10}$$

In other words, the exact solution has a slightly perturbed natural frequency, $\omega_0 = \sqrt{1 + \varepsilon}$, compared to $\omega_0 = 1$ for the leading order problem, $x_0'' + x_0 = 0$. In contrast, (9.6) is a weakly damped linear oscillator of the form $x'' + \varepsilon x' + x = 0$, and its exact solution is given by

$$x_{\text{exact}}(t) = e^{-\varepsilon t/2} \cos\left(\sqrt{1 - \tfrac{1}{4}\varepsilon^2}\; t\right). \tag{9.11}$$

The respective periodic and decaying oscillatory behaviours of these two solutions are shown in Fig. 9.2 along with plots of the expansions (9.8) and (9.9). While the expansions match their corresponding solutions for early times, both dramatically diverge when their later time behaviours become dominated by the secular growth terms.

While these results might suggest that the perturbation expansions have produced incorrect descriptions, we should not dismiss them too soon. In fact, taking the Taylor series expansions of (9.10) and (9.11) for $\varepsilon \to 0$ at fixed finite times ($t = O(1)$) directly reproduces (9.8) and (9.9). Equations (9.8) and (9.9) are indeed correct, but they must be used with caution.

Applying the fundamental assumption of asymptotic ordering to the terms in (9.7), the expansions are valid only when

$$|x_0(t)| \gg |\varepsilon x_1(t)| \gg |\varepsilon^2 x_2(t)| \gg \cdots, \qquad \varepsilon \to 0. \tag{9.12}$$

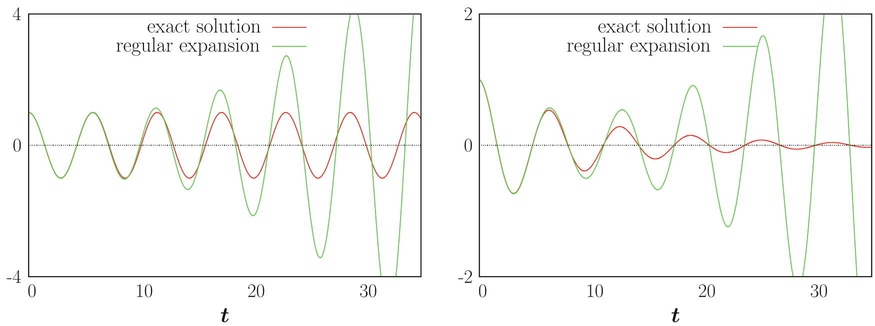


**Fig. 9.2** Exact solutions and regular expansions for problem (9.5): (9.10) and (9.8) (*Left*) and for problem (9.6): (9.11) and (9.9) (*Right*), both with $\varepsilon = 1/5$

Considering the first two terms from (9.8), $|\cos t| \gg |\frac{1}{2}\varepsilon t \sin t|$ holds only if $1 \gg \varepsilon t$. In other words, the expansions can only be expected to hold while this product is small ("for a limited time only" is an appropriate phrase, here applying for $0 \le t \ll 1/\varepsilon$).

These examples are be a bit troubling—we have worked out higher order terms in the asymptotic expansions since the leading order solution by itself, $x_0(t) = \cos t$, does not capture the differences that distinguish these two problems from each other. Yet the additional terms yield contributions that restrict the validity of the expansion to relatively short times, when effects like slow growth or decay or changes in the oscillation frequency have not yet fully developed. This suggests the need for other forms of perturbation expansions that can overcome these limitations.

## 9.3 Poincare–Lindstedt Expansions

The failure of the regular expansion to describe periodic motion on long timescales in the context of astronomy (for the motion of the planets) motivated the development of an improved approach known as the Poincare–Lindstedt method. A key idea behind the method is that the regular perturbation expansion is too restrictive in its form and does not allow for the possibility that the solution may have a frequency that is shifted from the leading order natural frequency $\omega_0$ (as in (9.10)).

The Poincare–Lindstedt approach begins with a change of variables in terms of a frequency that can depend on the perturbation parameter,

$$x(t) = \tilde{x}(\theta) \qquad \text{with} \qquad \theta = \Omega(\varepsilon)t, \tag{9.13}$$

such that $\Omega(0) = \omega_0$. We then seek regular perturbation expansions in terms of the new unknowns,

$$\tilde{x}(\theta) = \tilde{x}_0(\theta) + \varepsilon\tilde{x}_1(\theta) + \cdots, \qquad \Omega(\theta) = \omega_0 + \varepsilon\omega_1 + \cdots. \tag{9.14}$$

We will now give an example to illustrate how this seemingly minor change allows the Poincare–Lindstedt method to eliminate secular growth terms and obtain perturbation expansions that are valid over longer times for some problems.

Consider a problem similar to (9.5),

$$x'' + 4x = -\varepsilon x, \qquad x(0) = 3, \qquad x'(0) = -8. \tag{9.15}$$

After the change of variables (9.13), we arrive at the modified problem for $\tilde{x}(\theta)$,

$$\Omega^2 \frac{d^2\tilde{x}}{d\theta^2} + 4\tilde{x} = -\varepsilon\tilde{x}, \qquad \tilde{x}(0) = 3, \qquad \Omega \left.\frac{d\tilde{x}}{d\theta}\right|_{\theta=0} = -8. \tag{9.16}$$

Substituting expansions (9.14) into (9.16) and separating by orders of $\varepsilon$ yields, at $O(\varepsilon^0)$,

$$\omega_0^2 \frac{d^2\tilde{x}_0}{d\theta^2} + 4\tilde{x}_0 = 0, \qquad \tilde{x}_0(0) = 3, \qquad \omega_0 \frac{d\tilde{x}_0}{d\theta}\bigg|_{\theta=0} = -8, \tag{9.17a}$$

and at $O(\varepsilon^1)$,

$$\omega_0^2 \frac{d^2\tilde{x}_1}{d\theta^2} + 4\tilde{x}_1 = -\tilde{x}_0 - 2\omega_0\omega_1 \frac{d^2\tilde{x}_0}{d\theta^2}, \tag{9.17b}$$

$$\tilde{x}_1(0) = 0, \qquad \omega_0\tilde{x}_1'(0) = -\omega_1\tilde{x}_0'(0),$$

and so on at higher orders. Note that lower-order terms from the expansion of the solution ($\tilde{x}_k$ for $k = 0, 1, \ldots, n-1$) should be shifted to the right-hand side of the equation for $\tilde{x}_n$ and be treated as known parts of the inhomogeneous forcing.

Identifying the natural frequency as $\omega_0 = 2$ from (9.15) with $\varepsilon = 0$, the leading order solution can then be obtained from (9.17a) as

$$\tilde{x}_0(\theta) = -4\sin\theta + 3\cos\theta. \tag{9.18}$$

Substituting these results into the $O(\varepsilon)$ problem (9.17b) yields

$$4\frac{d^2\tilde{x}_1}{d\theta^2} + 4\tilde{x}_1 = [16\omega_1 - 4]\sin\theta + [-12\omega_1 + 3]\cos\theta, \tag{9.19}$$

$$\tilde{x}_1(0) = 0, \qquad 2\tilde{x}_1'(0) = 4\omega_1.$$

At this point $\omega_1$ is an undetermined constant. Problem (9.19) can be solved for any value $\omega_1$ to obtain $\tilde{x}_1(\theta)$. However, noting the presence of the resonant forcing terms, $\sin\theta$ and $\cos\theta$, on the right-hand side of the equation, the solution would include secular growth terms. But, if those resonant terms in (9.19) were eliminated by an appropriate choice of $\omega_1$, then $\tilde{x}_1(\theta)$ would be bounded and $\tilde{x}_0 + \varepsilon\tilde{x}_1$ would remain asymptotically well-ordered for all times. For this problem, this criterion selects $\omega_1 = 1/4$. This choice yields $\tilde{x}_1(\theta) = \frac{1}{2}\sin\theta$, and then using (9.13) and (9.14) the solution of the original problem can be written as

$$x(t) \sim (-2 + \tfrac{1}{2}\varepsilon)\sin([2 + \tfrac{1}{4}\varepsilon]t) + 3\cos([2 + \tfrac{1}{4}\varepsilon]t). \tag{9.20}$$

This solution holds over $0 \le t \ll 1/\varepsilon^2$; compare this with (9.8), which was valid only over $0 \le t \ll 1/\varepsilon$, see Fig. 9.3. In fact, even without going through the work of solving for $\tilde{x}_1(\theta)$, the Poincare–Lindstedt approach has yielded an improved solution by determining $\omega_1$ through suppressing the resonant terms in the $O(\varepsilon)$ equation; the condition on $\omega_1$ is another example of a *solvability condition*.

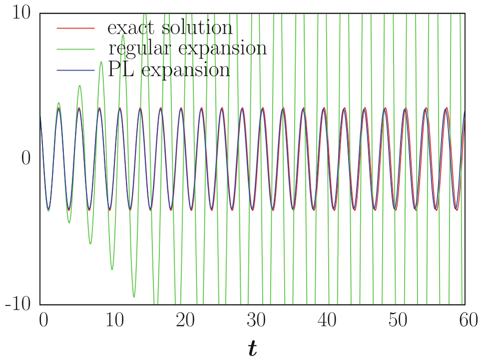


**Fig. 9.3** Comparison of the regular expansion and the Poincare–Lindstedt leading order solution against the exact solution of (9.15). Slight shifts between the Poincare–Lindstedt and exact solution are visible for larger times

While the example above was a linear problem, the Poincare–Lindstedt method extends directly to nonlinear equations with perturbation terms of the form $\varepsilon f(x)$. However, readers are right to suspect that the approach has some limitations. Noting that the general solution of the leading order problem has two independent solution terms, $\tilde{x}_0 = A\sin\theta + B\cos\theta$, and that each leads to resonant forcing in (9.19), it should be a little surprising that the coefficients of both resonant terms can be zeroed using only one degree of freedom, $\omega_1$. In fact, this is only possible when $x(t)$ is a *periodic solution*, leading to a degenerate linear system for the coefficients of the resonant forcing terms. Hence, the Poincare–Lindstedt method can only be used to obtain periodic solutions, and cannot, for example, generate the slowly decaying solution (9.11). To overcome this limitation, we consider another related perturbation method.

## 9.4 The Method of Multiple Time-Scales

Like the Poincare–Lindstedt method, the *method of multiple time-scales* (MMTS) determines solutions to perturbed oscillators by suppressing resonant forcing terms that would yield spurious secular terms in the asymptotic expansions. The method of multiple time-scales makes a less restrictive assumption on the form of the solution than employed by the Poincare–Lindstedt method; it assumes that the solution can be expressed as a function of multiple (for our purposes, just two) time variables,

$$x(t) = X(t, \tau), \tag{9.21}$$

where $t$ is the regular (or "fast") time variable and $\tau$ is a new variable describing the "slow-time" dependence of the solution. In a physical context, $t$ could represent a circadian rhythm of a daily cycle, while $\tau$ might describe changes to this cycle that are only noticeable over the timespan of years.

The simplest approach to determining the choice of the slow time variable, is to identify the combinations of $\varepsilon$ and $t$ present in secular terms in the regular expansion. In the examples from Sect. 9.2, $\tau = \varepsilon t$, so we focus on this case.[1]

The first step is to perform the change of variables (9.21), where by using the chain rule, we arrive at

$$\frac{dx}{dt} = \frac{\partial X}{\partial t} + \frac{\partial X}{\partial \tau}\frac{d\tau}{dt} = \frac{\partial X}{\partial t} + \varepsilon\frac{\partial X}{\partial \tau}, \tag{9.22}$$

and similarly,

$$\frac{d^2x}{dt^2} = \frac{\partial^2 X}{\partial t^2} + 2\varepsilon\frac{\partial^2 X}{\partial t\partial \tau} + \varepsilon^2\frac{\partial^2 X}{\partial \tau^2}. \tag{9.23}$$

The autonomous weakly-nonlinear oscillator equation

$$\frac{d^2x}{dt^2} + x = \varepsilon f\left(x, \tfrac{dx}{dt}\right) \tag{9.24}$$

then becomes

$$\frac{\partial^2 X}{\partial t^2} + 2\varepsilon\frac{\partial^2 X}{\partial t\partial \tau} + \varepsilon^2\frac{\partial^2 X}{\partial \tau^2} + X = \varepsilon f(X, X_t + \varepsilon X_\tau)$$

which can be rearranged into a more useful form as

$$\frac{\partial^2 X}{\partial t^2} + X = \varepsilon\big(f(X, X_t + \varepsilon X_\tau) - 2X_{t\tau}\big) - \varepsilon^2 X_{\tau\tau}. \tag{9.25}$$

We still have a perturbed linear oscillator, but now, we must specify that the oscillation is with respect to the fast time variable $t$, and we note that the right-hand side perturbation involves additional terms and derivatives with respect to $t$ and $\tau$. It may seem peculiar to replace the simpler ODE (9.24) by a more complicated partial differential equation, but (9.25) can still be solved through application of ODE methods and it provides the degrees of freedom necessary to properly describe the system over longer times.

The next step is to expand the MMTS solution as a regular perturbation expansion,

$$X(t, \tau) = X_0(t, \tau) + \varepsilon X_1(t, \tau) + O(\varepsilon^2), \tag{9.26}$$

and substitute into (9.25). Analogous to the Poincare–Lindstedt method, we want to suppress any resonant terms that occur in order to determine the currently unspecified parts of the solution. The final step is then to reconstruct the relationship between fast and slow times, say $\tau = \varepsilon t$, to obtain the final solution as $x(t) \sim X_0(t, \varepsilon t)$.

[1] Other problems, and additional timescales needed for higher order expansions could involve higher-order (slower) timescales such as $\tau_k = \varepsilon^k t$ for $k = 2, 3, \ldots$.

As an example, we consider a damped nonlinear oscillator,

$$\frac{d^2x}{dt^2} + x = -\varepsilon\kappa\frac{dx}{dt} + \varepsilon x^3, \qquad x(0) = 1, \qquad x'(0) = -2, \tag{9.27}$$

for $\varepsilon \to 0$. Setting $\tau = \varepsilon t$, and following the method of multiple time-scales decomposes the problem into the leading order problem

$$X_{0tt} + X_0 = 0, \qquad X_0(0,0) = 1, \qquad X_{0t}(0,0) = -2, \tag{9.28a}$$

and at $O(\varepsilon)$,

$$X_{0tt} + X_0 = -\kappa X_{0t} + X_0^3 - 2X_{0t\tau}, \tag{9.28b}$$

$$X_1(0,0) = 0, \qquad X_{1t}(0,0) = -X_{0\tau}(0,0).$$

With respect to the fast-time variable $t$, (9.28a) has a solution that is a linear combination of $\sin t$ and $\cos t$. However, unlike (9.3), since other (slow time) variables appear, the coefficients in the linear combination are not constants, but are functions dependent on the slow-time variable $\tau$,

$$X_0(t,\tau) = A(\tau)\sin t + B(\tau)\cos t. \tag{9.29}$$

From the initial conditions on $X_0$ at $t = \tau = 0$, we find that

$$A(0) = -2, \qquad B(0) = 1, \tag{9.30}$$

but otherwise, $A(\tau)$ and $B(\tau)$ are as-yet undetermined functions.

Moving on to the $O(\varepsilon)$ problem (9.28b), after substituting in (9.29), we obtain

$$\begin{aligned} X_{1tt} + X_1 = &\left(-\kappa A + \tfrac{3}{4}A^2B + \tfrac{3}{4}B^3 - 2\frac{dA}{d\tau}\right)\cos t \\ &+ \left(\kappa B + \tfrac{3}{4}A^3 + \tfrac{3}{4}AB^2 + 2\frac{dB}{d\tau}\right)\sin t \\ &+ \left(\tfrac{1}{4}B^3 - \tfrac{3}{4}A^2B\right)\cos(3t) - \left(\tfrac{1}{4}A^3 - \tfrac{3}{4}AB^2\right)\sin(3t). \end{aligned} \tag{9.31}$$

Note that many of the terms on the right-hand side of (9.31) are a consequence of the nonlinear term $X_0^3$. The forcing terms **must** be expanded out as the sum of sines and cosines of the fast time scale with coefficients that can only depend on the slow time scale. For simple nonlinear products, like $X_0^3$, this can done be through the use of trigonometric identities (see Appendix A). For more complicated types of nonlinear forcing terms, the right side of (9.25) should be replaced by its Fourier series expansion (also see Appendix A). These Fourier series decompositions are essential in separating out the resonant and non-resonant forcing terms. For (9.31), $\cos t$ and $\sin t$ are resonant terms, while $\cos(3t)$, $\sin(3t)$ are non-resonant.

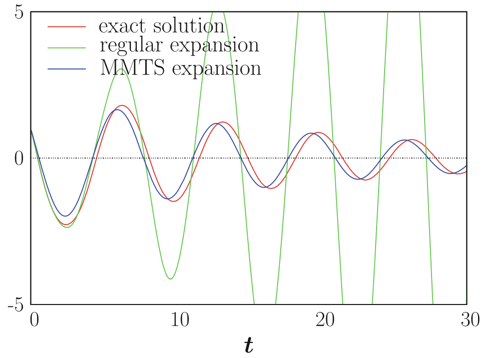


**Fig. 9.4** Comparison of the regular expansion and the method of multiple time-scales leading order solution against the exact solution of (9.27)

To eliminate the resonant terms that would cause expansion (9.26) to break-down, we set the respective coefficients to zero,

$$-\kappa A + \tfrac{3}{4}A^2 B + \tfrac{3}{4}B^3 - 2\frac{dA}{d\tau} = 0, \qquad \kappa B + \tfrac{3}{4}A^3 + \tfrac{3}{4}AB^2 + 2\frac{dB}{d\tau} = 0. \tag{9.32}$$

These two equations are the solvability conditions for this problem. They are coupled ordinary differential equations in terms of the slow-time variable that describe the evolution of the amplitude coefficients $A(\tau)$, $B(\tau)$ in solution (9.29); consequently, they are also often called *amplitude equations*.

Solving the amplitude equations to determine $A(\tau)$, $B(\tau)$ subject to their initial conditions (9.30) yields the MMTS leading order approximation of the solution (9.29), as illustrated in Fig. 9.4.

## 9.5 Further Directions

Perturbation methods for weakly nonlinear oscillators have been developed extensively in connection with many applications ranging from mechanical oscillations and electrical systems to population dynamics. Some more detailed introductory presentations are given in [48, 54, 56, 78], and some advanced treatments are given in [11, 47, 58, 73, 77]. Besides Poincare–Lindstedt and the method of multiple time-scales, other related approaches also exist, including the method of averaging [77, 92, 102] and near-identity transformations [58, 73]. The engineering approach of *harmonic balance* is related to these methods.

The mathematical theory underpinning the solvability conditions coming from the Poincare–Lindstedt and method of multiple time-scales approaches is the *Fredholm alternative theorem* [39, 93]. The Fredholm alternative supplies a criterion for the existence or uniqueness of solutions of inhomogeneous linear problems that applies to these oscillator equations and other classes of perturbation problems (see Exercise 9.13).

## 9.6 Exercises

**9.1** Consider the problem for $x(t)$ with $\varepsilon \to 0$,

$$\frac{d^2x}{dt^2} + x = 32\varepsilon x^3, \qquad x(0) = e^{-\varepsilon}, \qquad x'(0) = 2 + \varepsilon.$$

Obtain the first two terms of the regular expansion, $x(t) = x_0(t) + \varepsilon x_1(t) + O(\varepsilon^2)$. Identify the homogeneous solution, resonant response and non-resonant response in $x_1(t)$.

**9.2** For each problem use the Poincare–Lindstedt expansion with the two-term approximation of the solution, $x(t) \sim \tilde{x}_0(\theta) + \varepsilon\tilde{x}_1(\theta)$ with $\theta = (\omega_0 + \varepsilon\omega_1)t$, for $\varepsilon \to 0$ to find $\tilde{x}_0(\theta)$ and $\omega_0, \omega_1$.

(a) Are there periodic solutions for every value of $a > 0$ for

$$\frac{d^2x}{dt^2} + 4x = -\varepsilon x^3, \qquad x(0) = a, \qquad x'(0) = 0?$$

(b) Find the value for $a(>0)$ that yields a periodic solution of

$$\frac{d^2x}{dt^2} + 9x = -\varepsilon(x^2 - 1)\frac{dx}{dt}, \qquad x(0) = a, \qquad x'(0) = 0.$$

**9.3** Apply the method of multiple scales with $\tau = \varepsilon t$ to the *van der Pol oscillator*

$$\frac{d^2x}{dt^2} + 9x = -\varepsilon(x^2 - 1)\frac{dx}{dt} \qquad \varepsilon \to 0 \tag{9.33}$$

to obtain a solution in the form $x(t) \sim X_0(t, \tau) = A(\tau)\sin(3t) + B(\tau)\cos(3t)$.

(a) Determine the amplitude equations for $A(\tau), B(\tau)$. Determine initial conditions for $A, B$ in terms of $x(0), x'(0)$.
(b) Let $R(\tau) = \sqrt{A^2 + B^2}$. Determine the equation for $dR/d\tau = f(R)$ using the amplitude equations from part (a). Determine the equilibrium values for $R$.

**9.4** Show that the leading order MMTS solution for weakly nonlinear oscillators can be written in the polar form

$$X_0(t, \tau) = R(\tau)\sin(\omega_0 t + \Phi(\tau)). \tag{9.34}$$

(a) Relate the amplitude $R$ and phase $\Phi$ to the coefficients $A, B$ introduced in (9.29).

(b) Show that (9.32) can be rewritten as

$$\frac{dR}{d\tau} = -\tfrac{1}{2}\kappa R, \qquad \frac{d\Phi}{d\tau} = -\tfrac{3}{8}R^2$$

and solve this simpler polar system with appropriate initial conditions.

**9.5** Use the method of multiple scales to investigate the near-resonant behaviour of the damped, driven oscillator for $x(t)$ for $\varepsilon \to 0$,

$$\frac{d^2x}{dt^2} + \varepsilon\beta\frac{dx}{dt} + x + \varepsilon\alpha x^3 = \varepsilon\cos(t + \gamma\varepsilon t),$$

with given parameters $\alpha, \beta, \gamma$. Use the slow-timescale $\tau = \varepsilon t$. Note the presence of $\tau$ in the forcing term on the right-hand side. (Hint: write the forcing as $\cos(t + \gamma\tau)$)

(a) Show that the leading order solution can be written in the complex form

$$X_0(t, \tau) = C(\tau)e^{it} + \overline{C(\tau)}e^{-it},$$

where $\overline{z} = x - iy$ denotes the complex conjugate of $z = x + iy$. Relate the complex-valued function $C$ to the real-valued functions $A, B$ used in (9.29).

(b) Using the result of part (a) in the equation for $X_1(t, \tau)$, find the two solvability conditions. Show that these reduce to a single complex equation for $dC/d\tau$.

(c) *Entrainment* describes a solution locking onto the behaviour entirely set by a forcing term, leaving no direct sign of the natural frequency from the unforced problem (i.e. no homogeneous solution). Setting $C(\tau) = Me^{i\theta}e^{i\gamma\tau}$ in your equation from (b), obtain an equation for $M$, the real-valued constant amplitude of the entrained solution, with $\theta$ being a (real-valued) phase constant. Determine the *detuning relation*, $\gamma = \gamma(M)$, relating the amplitude to the frequency-offset from resonance.

**9.6** Apply the method of multiple scales with $\tau = \varepsilon t$ and $\varepsilon \to 0$ to the problem

$$\frac{d^2x}{dt^2} + \varepsilon\left|\frac{dx}{dt}\right|\frac{dx}{dt} + x = 0, \qquad x(0) = 0, \qquad x'(0) = 1.$$

Using the polar form (9.34), derive and solve the amplitude equations for $R(\tau)$ and $\Phi(\tau)$ to obtain the leading order solution $x(t) \sim X_0(t, \tau)$.
Hint: You will need to calculate some terms of a Fourier series. Write the series in terms of the variable $s = t + \Phi$ on $-\pi < s < \pi$, namely $f(s) = \sum_k a_k \sin(ks) + b_k \cos(ks)$.

**9.7** In Exercise 3.6, the equation for the parametrically-driven pendulum was derived; for a specific choice of parameters, it can be written as

$$\frac{d^2\theta}{dt^2} + 4\sin\theta = -\varepsilon\sin(4t)\sin\theta.$$

Using the scaling for small-amplitude oscillations, $\theta = \delta x$ for $\delta \to 0$, at leading order this equation yields a form of the *Mathieu equation*,

$$\frac{d^2x}{dt^2} + \left[4 + \varepsilon \sin(4t)\right] x = 0. \tag{9.35}$$

Consider solutions of the Mathieu equation for $\varepsilon \to 0$:

(a) Use the method of multiple scales with $\tau = \varepsilon t$ to determine the amplitude equations for the slowly varying coefficients in the leading order solution.
(b) Solve for the leading order solution that satisfies the initial conditions

$$x(0) = 5, \qquad x'(0) = 6.$$

**9.8** Consider the equation for the complex-valued solution $x(t)$ with $\varepsilon \to 0$,

$$\frac{dx}{dt} + i4x = \varepsilon \cos(4t)x^2$$

Apply the method of multiple time scales with $\tau = \varepsilon t$ and $x(t) \sim X_0(t, \tau) + \varepsilon X_1(t, \tau)$.

(a) Write the equations for $X_0$ and $X_1$.
(b) Write the general solution of the $O(1)$ equation.
(c) Determine the amplitude equation and explain the condition that selects this result.
(d) Determine the leading order solution for $x(t)$ that satisfies the initial condition

$$x(0) = 1 + i.$$

**9.9** For the problem with $\varepsilon \to 0$,

$$\frac{d^2x}{dt^2} + x = \varepsilon x^2, \qquad x(0) = 1, \qquad x'(0) = 0,$$

the slow timescale for MMTS is not the usual one, $\tau \neq \varepsilon t$. Attempt a regular expansion $x \sim x_0 + \varepsilon x_1 + \varepsilon^2 x_2$ to determine the slow time variable. Determine the leading order solution using the Poincare–Lindstedt method.

**9.10** For $\varepsilon \to 0$, show that there is a large-amplitude periodic solution of

$$\frac{d^2x}{dt^2} + x = \varepsilon x^2 + \varepsilon \cos t \qquad x'(0) = 0.$$

To do this, let $x(t) \sim \varepsilon^{-\beta} X(t, \tau)$ with $\tau = \varepsilon^{\alpha} t$ and $X \sim X_0 + \varepsilon^{\gamma} X_1 + \varepsilon^{2\gamma} X_2$ and select $\alpha, \beta, \gamma > 0$ to ensure that $X_0, X_1, X_2$ have no secular growth terms.

**9.11** *Delay differential equations* (DDE) are equations that involve current properties of the solution coupled to the solution at earlier times [35]. Consider the DDE

$$\frac{d^2x}{dt^2} + x + \varepsilon x(t-2) = 0.$$

(a) This is a linear constant coefficient equation and can be solved in terms of trial solutions of the form $x(t) = e^{\lambda t}$. Write the characteristic equation determining $\lambda$ and obtain the solutions to $O(\varepsilon)$ for $\varepsilon \to 0$. Are the solutions of this equation stable or unstable? (see Sect. 1.5)
(b) Apply the method of multiple scales to obtain the amplitude equations.
(c) Similarly analyse the *weakly-delayed* linear oscillator equation,

$$\frac{d^2x}{dt^2} + x(t-\varepsilon) = 0.$$

**9.12** Now we complete the derivation of the KdV equation that was started in Exercise 8.5. There, we obtained the equations for $C(x,t) = C_0 + \varepsilon^2 C_1 + \cdots$ and $F(x,t) = F_0 + \varepsilon^2 F_1 + \cdots$, as
$O(1)$ equations:

$$\frac{\partial C_0}{\partial t} + F_0 = 0, \qquad \frac{\partial F_0}{\partial t} + \frac{\partial^2 C_0}{\partial x^2} = 0 \tag{9.36}$$

$O(\varepsilon^2)$ equations:

$$\frac{\partial C_1}{\partial t} + F_1 = \tfrac{1}{2}\frac{\partial^3 C_0}{\partial t \partial x^2} - \tfrac{1}{2}\left(\frac{\partial C_0}{\partial x}\right)^2, \tag{9.37a}$$

$$\frac{\partial F_1}{\partial t} + \frac{\partial^2 C_1}{\partial x^2} = \tfrac{1}{6}\frac{\partial^4 C_0}{\partial x^4} - F_0\frac{\partial^2 C_0}{\partial x^2} - \frac{\partial F_0}{\partial x}\frac{\partial C_0}{\partial x}. \tag{9.37b}$$

(a) Show from the $O(1)$ equations that $C_0$ and $F_0$ each satisfy the classic wave equation (2.52) with unit speed and having independent left/right moving waves.
(b) We will now restrict attention just to right-moving waves. It can be shown that the $O(\varepsilon^2)$ equations would lead to solutions with secular growth, hence consider a multiple-time scale expansion of the form $C = c_0(z,\tau) + \varepsilon^2 c_1(z,\tau)$ and $F = f_0(z,\tau) + \varepsilon^2 f_1(z,\tau)$ where $z = x - t$ and $\tau = \varepsilon^2 t$. Show that the $O(1)$ equations reduce to a single relation between $c_0$, $f_0$ and that the $O(\varepsilon^2)$ equations reduce to a single compatibility condition given by a partial differential equation for $f_0(z,\tau)$, the KdV equation,

$$\frac{\partial f_0}{\partial \tau} + \tfrac{3}{2} f_0 \frac{\partial f_0}{\partial z} + \tfrac{1}{6}\frac{\partial^3 f_0}{\partial z^3} = 0. \tag{9.38}$$

**9.13** The *Fredholm alternative theorem* is the key principle that determines the solvability conditions in the Poincare–Lindstedt and MMTS methods, but here we illustrate how it also applied to determining the *eigenvalues of perturbed matrices* [47].

Consider the matrix

$$\mathbf{A}(\varepsilon) = \begin{pmatrix} -e^{\varepsilon} & 3-3\varepsilon \\ 3+\varepsilon & -1+2\sin\varepsilon \end{pmatrix}.$$

For the limit $\varepsilon \to 0$, use the following steps to solve the eigenvalue problem,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}.$$

(a) Write the asymptotic expansion for the matrix in powers of $\varepsilon$, $\mathbf{A} \sim \mathbf{A}_0 + \varepsilon\mathbf{A}_1 + \varepsilon^2\mathbf{A}_2$ and similar expansions for each eigenvector and corresponding eigenvalue $\mathbf{x} \sim \mathbf{x}_0 + \varepsilon\mathbf{x}_1 + \varepsilon^2\mathbf{x}_2$, and $\lambda \sim \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2$.

(b) The leading order problem, $\mathbf{A}_0\mathbf{x}_0 = \lambda_0\mathbf{x}_0$, can be written in the form of a homogeneous problem,

$$(\mathbf{A}_0 - \lambda_0\mathbf{I})\mathbf{x}_0 = \mathbf{0}$$

Obtain the eigensolutions for this matrix, $\{\lambda_0^{(k)}, \mathbf{x}_0^{(k)}\}$.

(c) For this problem, every vector $\mathbf{v}$ can be written as a linear combination of the leading order eigenvectors, $\mathbf{v} = c_{(1)}\mathbf{x}_0^{(1)} + c_{(2)}\mathbf{x}_0^{(2)}$. Write the formulas for $c_{(k)}$ for $k = 1, 2$ in terms of $\mathbf{v}$ and $\mathbf{x}_0^{(k)}$.

(d) Write the $O(\varepsilon)$ equation in the form of an inhomogeneous problem with the same matrix operator as the $O(1)$ equation,

$$(\mathbf{A}_0 - \lambda_0\mathbf{I})\mathbf{x}_1 = \mathbf{f}_1(\mathbf{x}_0, \mathbf{A}_1, \lambda_1).$$

Note the similarities between these equations for $\mathbf{x}_0$, $\mathbf{x}_1$ and the oscillator problems, (9.17a) and (9.17b) or (9.28a) and (9.28b). The Fredholm alternative essentially states that

> *The solution of the $O(\varepsilon^n)$ inhomogeneous problem will exist and be unique if the forcing term has no contribution from the solution of the $O(\varepsilon^0)$ homogeneous problem.*[2]

For oscillator problems, this addresses the existence of *periodic* solutions in the absence of resonant forcing terms.

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[2]This is the simplified version for symmetric matrices and self-adjoint differential equations. The general version of the Fredholm alternative is similar:

> *The solution of the non-homogeneous problem* $\mathbf{A}_0\mathbf{x} = \mathbf{b}$ *will be unique if and only if the adjoint problem* $\mathbf{A}_0^{\dagger}\mathbf{y} = \mathbf{0}$ *has only the trivial solution* [39, 93].

For matrix equations, non-existence or non-uniqueness will result if the forcing term includes contributions from the $\mathbf{x}_0^{(k)}$ nullvector corresponding to the $\lambda^{(k)}$ in the matrix operator. Select the values of $\lambda_1$ to eliminate those contributions and obtain the first two terms in the expansions of the eigenvalues, $\lambda^{(k)} \sim \lambda_0^{(k)} + \varepsilon\lambda_1^{(k)}$ for $k = 1, 2$.