Homework 1

Due before 10am on September 1st on gradescope.

1. (20 pts) Proposition: If the sequences $\{a_n\}$ and $\{b_n\}$ are bounded above, then $\{a_n b_n\}$ is bounded above.

   a) Prove this is false by giving a counterexample.

   b) Strengthen the hypotheses and prove your amended proposition.

   (Read top P. 405 in textbook for "stronger statement": here the "statements" are the hypotheses on the two sequences in the Proposition. In other words, besides requiring $\{a_n\}$ and $\{b_n\}$ being bounded above, find what additional assumptions can ensure $\{a_n b_n\}$ being bounded above)

2. (20 pts) Let $c_1, c_2, \cdots, c_N$ and a be real numbers. Prove the following:

$$\left|\sum_{n=1}^{N} c_n \sin(na)\right| \geq 1 \Rightarrow |c_n| > \frac{1}{2^n} \quad \text{for some} \quad n \leq N.$$

   Prove it by contraposition (read A.2 in textbook): not B $\Rightarrow$ not A, but write the contrapositive statement avoiding all negative words like "not","no" and symbols for them. The phrase *for some* n means *for at least one value of* n.

3. (20 pts) Page 46: 3.1/1(c). Do it directly from Definition 3.1 of limit; don't use any limit theorems you know from calculus (in Chapter 5 here).

4. (20 pts)

   a) Prove $\{x_n\}$ defined by $x_{n+1} = \frac{n^2+10}{(n+1)(n+3)} x_n, x_0 > 0$ is monotone for $n \gg 1$. (Two ways to show a positive sequence a_n is increasing are to show the ratio $a_{n+1}/a_n \geq 1$ or show the difference $a_{n+1} - a_n \geq 0$.) Analogously for decreasing: use $\leq 1, \leq 0$.

   b) For what n will $\frac{3n}{n+2} \underset{\epsilon}{\approx} 3$ if (i) $\epsilon = 0.1$ (ii) $\epsilon = 0.01$?

5. (20 pts)

   a) Prove that if $\{x_n\}$ converges, it is bounded for $n \gg 1$.

b) Then prove that it is bounded (i.e., for all n).