Homework 3

Due on Sep 15th before 10am on gradescope.

1. (10 pts) Prove that

   (a) the limit of $\left(\frac{n+1}{n}\right)^{\frac{3}{2}}$ is 1,

   (b) the limit of $n^{-\frac{3}{2}}$ is 0.

   Use only the Squeeze Theorem and Theorem 5.1 (Page 61). You cannot use any other theorems which have not been proven in class. For instance, we can easily verify $1/n \to 0$ by definition. Then by Linear and Product Theorems, we have

$$1/n \to 0 \Longrightarrow 1 + 1/n \to 1 \Longrightarrow (1 + 1/n)^2 \to 1.$$

2. (20 pts) Page 74, 5.2/3. Let $a_n = \sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}$. Prove that $a_n \sim \frac{2}{3}n^{\frac{3}{2}}$, i.e.,

$$\lim_{n\to\infty} \frac{a_n}{\frac{2}{3}n^{\frac{3}{2}}} = 1.$$

   Hint:$\sqrt{k}$ is the area of rectangle with width 1 and height $\sqrt{k}$. And the point $(k, \sqrt{k})$ lies on the curve $y = \sqrt{x}$. There are two ways to generate such a rectangle: use the interval $[k-1, k]$ as the shorter side or use the interval $[k, k+1]$ as the shorter side. If using $[k-1, k]$, then the rectangle's area is larger than area below the increasing curve $y = \sqrt{x}$ on the same interval $[k-1, k]$. If using $[k, k+1]$, then the rectangle's area is smaller than area below the curve on the same interval $[k, k+1]$. We obtain a lower and an upper estimate of a_n by using these two different ways. You can use the limits proven in Problem 1 directly.

3. (20 pts) Page 74, 5.3/6. Prove that $a_{n+1}/a_n \longrightarrow 0 \Rightarrow a_n \longrightarrow 0$.
   Hint: First show $|a_{n+1}| < \frac{1}{2}|a_n|, n \gg 1$. Then show $a_n \to 0$ by using $|a_{n+1}| < \frac{1}{2}|a_n|, n \gg 1$.

4. (10 pts) Page 75, Problem 5-1.
   Hint: The given "proof" is problematic because the proof for convergence of $\sqrt{a_n}$ is not provided. There are many ways to establish a valid proof. For instance, if we want to try proof by contradiction, then by definition of the limit, negation of $\sqrt{a_n} \to \sqrt{L}$ means that there exists a

particular number $\epsilon > 0$ s.t. for any integer N, there exists $n \geq N$ s.t. $|\sqrt{a_n} - \sqrt{L}| \geq \epsilon$.
To avoid any confusion with conflict of sympbols, we can replace ϵ by a different sympbol c and replace n by m. So if we assume $\sqrt{a_n} \to \sqrt{L}$ is not true, then there exists a particular number $c > 0$ s.t. for any integer N, there exists $m \geq N$ s.t. $|\sqrt{a_m} - \sqrt{L}| \geq c$.

5. (20 pts) Page 75, Problem 5-2.
Assume $a_{n+1}/a_n \longrightarrow L < 1$ and $a_n > 0$. Prove that (a) $\{a_n\}$ is decreasing for n large; (b) $a_n \longrightarrow 0$.
For (b), just show one proof (either one you prefer). This problem is similar to Problem 3 (replace 1/2 by a positive constant $c < 1$, the same proof should still work).

6. (20 pts) Page 75, Problem 5-7.
Define a sequence a_n, which satisfies $a_{n+1} = \sqrt{2a_n}$, $a_0 > 0$. Prove that
   (a) a_n is monotone and bounded.
   (b) Determine the limit of a_n, and show it is independent of the choice of $a_0 > 0$.

Hint:

- To show it's monotone: notice that $a_{n+1} \geq a_n \Leftrightarrow \sqrt{2a_n} \geq a_n \Leftrightarrow a_n \leq 2$. So we can show $a_0 \geq 2 \Rightarrow 2 \leq a_{n+1} \leq a_n$ by Mathematical Induction (read A.4): first if assuming $a_0 \geq 2$, we have $a_1 \geq a_0$ and $a_1 = \sqrt{2a_0} \geq \sqrt{4} = 2$; second, assume $2 \leq a_n \leq a_{n-1}$, we have $a_{n+1} \leq a_n$ and $a_{n+1} = \sqrt{2a_n} \geq \sqrt{4} = 2$. Similarly, discuss the case $a_0 \leq 2$.
- The limit should be 2 (but why?).
- To prove the limit, introduce error term $e_n = a_n - 2$. For the case $a_0 \geq 2$, we have $e_n \geq 0$ and

$$(e_{n+1} + 2) = \sqrt{2(e_n + 2)} \Rightarrow (e_{n+1} + 2)^2 = 2(e_n + 2).$$

  Try to derive a useful recursive inequality for e_{n+1} and e_n so that you can establish lower ($e_n \geq 0$) and upper bounds for e_n to show $e_n \to 0$ by Squeeze Theorem.