$$\min_{x} F(x) := f(x) + g(x)$$

$$x_{k+1} = (I + \eta \partial f)^{-1} (I - \eta \nabla g)(x_k)$$

$$= \operatorname{Prox}^{\eta}_{f} [x_k - \eta \nabla g(x_k)]$$

Sufficient Decrease Lemma

Assume
1. $f(x)$ is convex
2. $g(x)$ is convex
3. $\nabla g(x)$ is L-continuous with L

$$\bar{x} = \operatorname{Prox}^{\eta}_{f} (x - \eta \nabla g(x))$$

$$F(x) - F(\bar{x}) \geq \left(\frac{1}{\eta} - \frac{L}{2}\right) \|\bar{x} - x\|^2$$

Stable step size $\eta \leq \frac{2}{L}$

Theorem [Prox-Grad Inequality]

Assume
1. $f(x)$ is convex
2. $\nabla g(x)$ is L-continuous with L

Let $\bar{y} = \operatorname{Prox}^{\eta}_{f} (y - \eta \nabla g(y))$, and $\eta \leq \frac{1}{L}$

$$F(x) - F(\bar{y}) \geq \frac{L}{2} \|x - \bar{y}\|^2 - \frac{L}{2} \|x - y\|^2$$
$$+ g(x) - g(y) - \langle \nabla g(y), x - y \rangle$$

Theorem Assume
① $f(x)$ is convex
② $g(x)$ is convex
③ $\nabla g(x)$ is L-continuous with L

$F(x) = f(x) + g(x)$ is convex

$\{x_k\}$ generated by Proximal Gradient Method ($\eta \le \frac{1}{L}$) satisfies:

$$\|x_{k+1} - x_*\| \le \|x_k - x_*\| \quad \text{for any minimizer } x_*$$

Proof:

$$\boxed{F(x) - F(\bar{y}) \ge \frac{1}{2\eta}\|x - \bar{y}\|^2 - \frac{1}{2\eta}\|x - y\|^2 + g(x) - g(y) - \langle \nabla g(y), x - y \rangle}$$

$$0 \ge 2\eta\left[F(x_*) - F(x_{k+1})\right] \ge \|x_* - x_{k+1}\|^2 - \|x_* - x_k\|^2$$

$y = x_k \Rightarrow \bar{y} = x_{k+1}$ $\quad x = x_*$

because $\quad g(x) - g(y) - \langle \nabla g(y), x - y \rangle \ge 0$

Theorem [$O(\frac{1}{k})$ rate]

Assume
① $f(x)$ is convex
② $g(x)$ is convex
③ $\nabla g(x)$ is L-continuous with L
④ $F(x)$ has one minimizer

Proximal Gradient Method with $\eta = \frac{1}{L}$ satisfies:

$$F(x_k) - F(x_*) \le \frac{L}{2}\|x_0 - x_*\|^2 \cdot \frac{1}{k}$$

Proof: $\quad 2\eta\left[F(x_*) - F(x_{k+1})\right] \ge \|x_* - x_{k+1}\|^2 - \|x_* - x_k\|^2$

$$\Rightarrow \frac{2}{L}\sum_{k=0}^{n-1}\left[F(x_*) - F(x_{k+1})\right] \ge \|x_* - x_n\|^2 - \|x_* - x_0\|^2$$

$$\geq -\|x_* - x_0\|^2$$

$$\Rightarrow \quad \frac{2}{L}\sum_{k=0}^{n-1}\left[F(x_{k+1}) - F(x_*)\right] \leq \|x_0 - x_*\|^2$$

$$\boxed{\begin{array}{l}\text{Sufficient Decrease Lemma}\\ F(x) - F(\bar{x}) \geq \left(\frac{1}{\eta} - \frac{L}{2}\right)\|\bar{x} - x\|^2\end{array}} \Rightarrow F(x_{k+1}) \leq F(x_k)$$

$$\Rightarrow \quad \text{LHS} \geq \frac{2}{L}\cdot n\cdot\left[F(x_n) - F(x_*)\right]$$

$$\Rightarrow \quad \frac{2}{L}\cdot n\cdot\left[F(x_n) - F(x_*)\right] \leq \|x_0 - x_*\|^2$$

Strongly Convex Case

Theorem [Linear Rate] $F(x)$ is strongly convex

Assume
① $f(x)$ is convex
② $g(x)$ is strongly convex with μ
③ $\nabla g(x)$ is L-continuous with L

Proximal Gradient Method with $\eta = \frac{1}{L}$ satisfies:

① $\|x_{k+1} - x_*\|^2 \le (1 - \frac{\mu}{L}) \|x_k - x_*\|^2$

② $\|x_k - x_*\|^2 \le (1 - \frac{\mu}{L})^k \|x_0 - x_*\|^2$

③ $F(x_k) - F(x_*) \le \frac{L}{2} (1 - \frac{\mu}{L})^k \|x_0 - x_*\|^2$

Proof:

$$\boxed{F(x) - F(\bar{y}) \ge \frac{1}{2\eta}\|x - \bar{y}\|^2 - \frac{1}{2\eta}\|x - y\|^2 + g(x) - g(y) - \langle \nabla g(y), x - y \rangle}$$

$$\Rightarrow F(x_*) - F(x_{k+1}) \ge \frac{L}{2}\|x_* - x_{k+1}\|^2 - \frac{L}{2}\|x_* - x_k\|^2 + \underbrace{g(x_*) - g(x_k) - \langle \nabla g(x_k), x_* - x_k \rangle}_{\ge \frac{\mu}{2}\|x_* - x_k\|^2}$$

$$\Rightarrow F(x_*) - F(x_{k+1}) \ge \frac{L}{2}\|x_* - x_{k+1}\|^2 - \frac{L - \mu}{2}\|x_* - x_k\|^2$$

$$\Rightarrow \|x_{k+1} - x_*\|^2 \le (1 - \frac{\mu}{L})\|x_k - x_*\|^2$$

$$\Rightarrow \|x_k - x_*\|^2 \leq (1-\frac{\mu}{L})^k \|x_0 - x_*\|^2$$

$$\begin{aligned} F(x_{k+1}) - F(x_*) &\leq \frac{L-\mu}{2}\|x_k - x_*\|^2 - \frac{L}{2}\|x_{k+1} - x_*\|^2 \\ &\leq \frac{L-\mu}{2}\|x_k - x_*\|^2 \\ &= \frac{L}{2}(1-\frac{\mu}{L})\|x_k - x_*\|^2 \\ &\leq \frac{L}{2}(1-\frac{\mu}{L})^{k+1}\|x_0 - x_*\|^2. \end{aligned}$$

<u>Def</u> $\{x_k\}$ is Fejer Monotone if there is y s.t.

$$\|x_{k+1} - y\| \leq \|x_k - y\|, \quad \forall k$$

<u>Theorem</u> [Convergence]

Assume
① $f(x)$ is convex
② $g(x)$ is convex
③ $\nabla g(x)$ is L-continuous with L
④ $F(x)$ has a minimizer

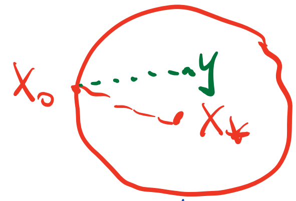
Proximal Gradient Method with $\eta = \frac{1}{L}$ satisfies:

$\{x_k\}$ converges to one minimizer of $F(x)$.

Proof:

$\|x_{k+1} - x_*\| \leq \|x_k - x_*\|$ for any minimizer x_*

$\Rightarrow \; \|x_k - x_*\| \le \|x_0 - x_*\|, \; \forall k$



$\Rightarrow \; \{x_k\}$ is in the ball centered at x_* with radius $\|x_0 - x_*\|$

Real Analysis

$\boxed{\text{Bounded Sequence in } \mathbb{R}^n \text{ has a convergent subsequence}}$

$\Rightarrow \quad x_{k_j} \to y, \; j \to \infty$

$F(x_{k_j}) \to F(x_*)$

$F(x)$ on $\mathbb{R}^n$ is convex $\Rightarrow F(x)$ is continuous

$\Rightarrow \; F(x_{k_j}) \to F(y)$

$\Rightarrow \; F(y) = F(x_*)$

$\Rightarrow \; y$ is a minimizer

$\Rightarrow \; \|x_{k+1} - y\| \le \|x_k - y\|$

$$\Rightarrow \left.\begin{array}{l} \|x_k - y\| \text{ is decreasing} \\ \|x_{k_j} - y\| \to 0 \end{array}\right\} \Rightarrow \|x_k - y\| \to 0$$

$\Rightarrow \; \{x_k\}$ converges to y.