I Previously on nonsmooth case:

$f(x)$ and $g(x)$ are convex

$g(x)$ is differentiable

$$\min_x f(x)$$

Example: $f(x) = \|x\|_1$

① Subgradient Method

$$x_{k+1} = x_k - \eta \partial f(x_k)$$

② Proximal Point Method

$$x_{k+1} = (I + \eta \partial f)^{-1}(x_k)$$

$$\min_x f(x) + g(x)$$

Example: $\|x\|_1 + \frac{\lambda}{2}\|Ax-b\|^2$

③ Proximal Gradient Method

$$x_{k+1} = (I + \eta \partial f)^{-1}(x_k - \eta \nabla g(x_k))$$

④ Fast Proximal Gradient

II

$$\min_x f(x) + g(x)$$

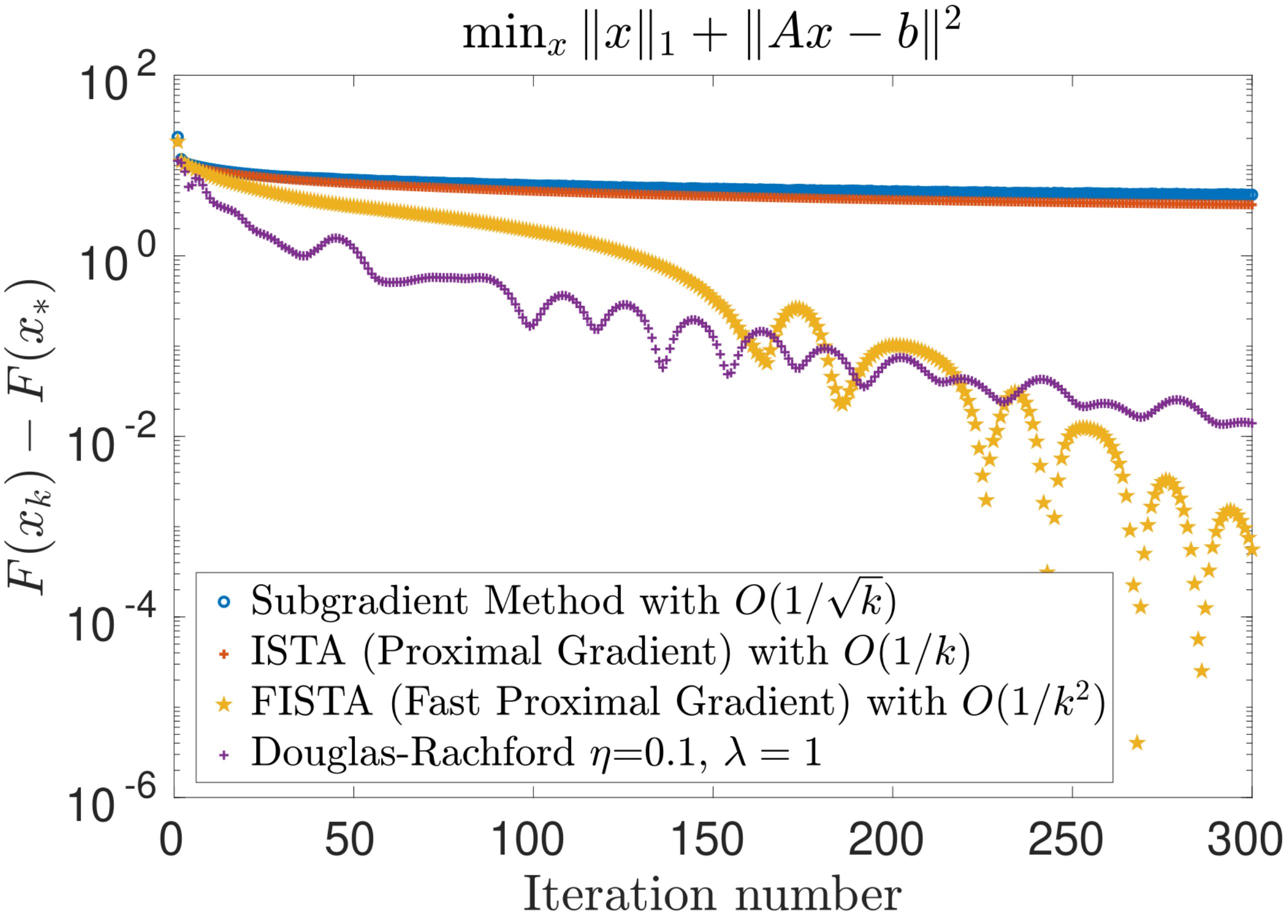
f & g are both nonsmooth

Example: $\min_x \|x\|_1 + i_{\{x: Ax=b\}}$

⑤ Douglas-Rachford splitting (1979)

A simple version

$$\begin{cases} y_{k+1} = \dfrac{I + R_f R_g}{2}(y_k) \\ R_f = 2 \cdot \mathrm{Prox}_f^{\eta} - I \\ R_g = 2 \cdot \mathrm{Prox}_g^{\eta} - I \end{cases}$$



(a) Douglas-Rachford splitting converges for any step size $\eta > 0$ but proximal operator for $g(\mathbf{x}) = \|A\mathbf{x} - b\|^2$ requires $(I + \eta 2A^T A)^{-1}$.

Theorem If $f(x)$ & $g(x)$ are convex, the simple Douglas-Rachford splitting iteration converges.

Proof:

$$\boxed{\begin{array}{l} \text{Theorem The following are equivalent:} \\ \text{① } T \text{ is firmly nonexpansive} \\ \text{② } I-T \text{ is firmly nonexpansive} \\ \text{③ } 2T-I \text{ is nonexpansive} \end{array}}$$

$f(x)$ is convex $\Rightarrow \mathrm{Prox}^{\eta}_{f}$ is firmly nonexpansive

$\Leftrightarrow R_f = 2\,\mathrm{Prox}^{\eta}_{f} - I$ is nonexpansive

$\Rightarrow T = R_f R_g$ is nonexpansive

$\Rightarrow Y_{k+1} = [(1-\theta)I + \theta T](Y_k)$ converges

$\theta \in (0,1)$

$\theta = \frac{1}{2}$ gives simplest DR splitting

$S = \frac{1}{2}I + \frac{1}{2}R_f R_g$ is firmly nonexpansive

$\Updownarrow$

$2S - I = R_f R_g$ is nonexpansive

The general Douglas-Rachford splitting is written as

$$y_{k+1} = \left[\theta I + (1-\theta) R_f R_g\right](y_k)$$

$\theta \in (0,1)$

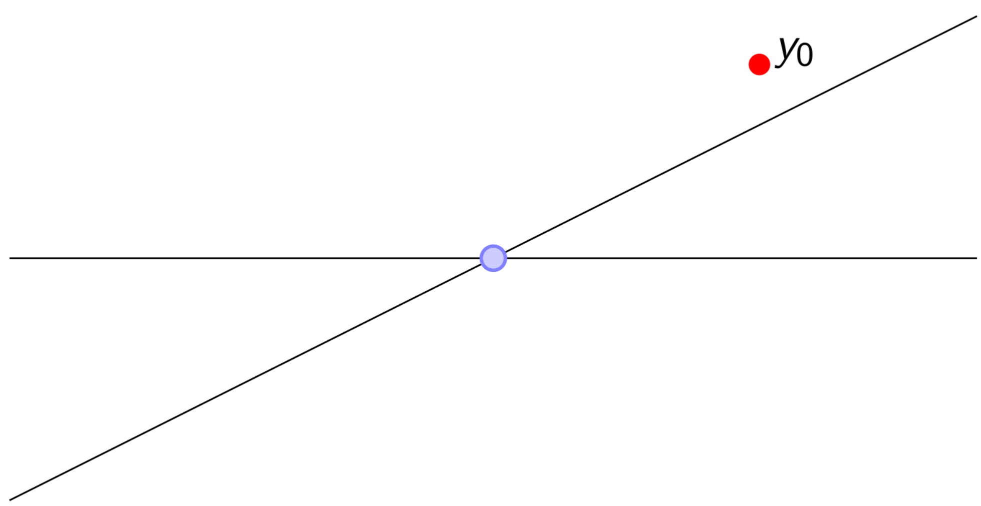
$\boxed{\text{Theorem}}$ Assume $T: \mathbb{R}^n \to \mathbb{R}^n$ is nonexpansive, then T has at least one fixed point.

$x_{k+1} = \theta x_k + (1-\theta) T(x_k)$, $\theta \in (0,1)$ satisfies

1) $\{x_k\}$ converges to one fixed point y of T

2) $\|x_{k+1} - x_k\|^2 \le \frac{1}{(k+1)}\left(\frac{1}{\theta} - 1\right)\|x_0 - y\|^2$

For $\min_x f(x) + g(x)$, if $f(x)$ and $g(x)$ are indicator functions of two lines respectively, then the minimizers are the intersection point.



Sometimes, general DR is written as

$$y_{k+1} = \left[(1-\lambda)I + \lambda \frac{I + R_f R_g}{2}\right](y_k), \quad \lambda \in (0,2)$$

Theorem: General DR converges for any convex $f(x)$ & $g(x)$, and any $\eta > 0$.

Proof: $T = \dfrac{I + R_f R_g}{2}$ T is firmly nonexpansive

$\Updownarrow$

$2T - I$ is nonexpansive

$$S = (1-\lambda)I + \lambda T = (1-2\theta)I + 2\theta T, \quad \theta \in (0,1)$$

$$= (1-\theta)\cdot I + \theta \cdot (2T - I)$$

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Developments of Douglas-Rachford Splitting

$u_t = u_{xx} + u_{yy}$

1. Peaceman and Rachford, 1955; Douglas and Rachford, 1956: implicit finite difference in solving heat equations. → ADI
2. Lions and Mercier, 1979: extension to maximal monotone operators; DR is firmly nonexpansive thus convergent.
3. Glowinski and Marroco, 1975; Gabay, 1983: the alternating direction method of multipliers (ADMM) is equivalent to DR. ADMM has been widely used in nonlinear mechanics and convex optimization.
4. Goldstein and Osher, 2009: split Bregman method widely used in image processing problems, which is the same as ADMM.
5. Bauschke, Combettes, and Luke, 2002: the widely used Fienup's Hybrid Input-Output (1982) algorithm for phase retrieval problem can be viewed as Douglas-Rachford splitting.

The following three algorithms are exactly the same:

- Douglas-Rachford Splitting on $\min_x f(x) + g(x)$.
- ADMM on Fenchel Dual $\min_y f^*(y) + g^*(-y)$.
- Split Bregman on Fenchel Dual $\min_y f^*(y) + g^*(-y)$.

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III Douglas-Rachford Splitting (Lions & Mercier 1979)

$f(x)$ is convex $\Rightarrow \operatorname{Prox}_f^\eta$ is firmly nonexpansive

$\Rightarrow R_f = 2\operatorname{Prox}_f^\eta - I$ is nonexpansive

So $T = R_f R_g$ is nonexpansive

$\Rightarrow S = \frac{T+I}{2}$ is firmly nonexpansive

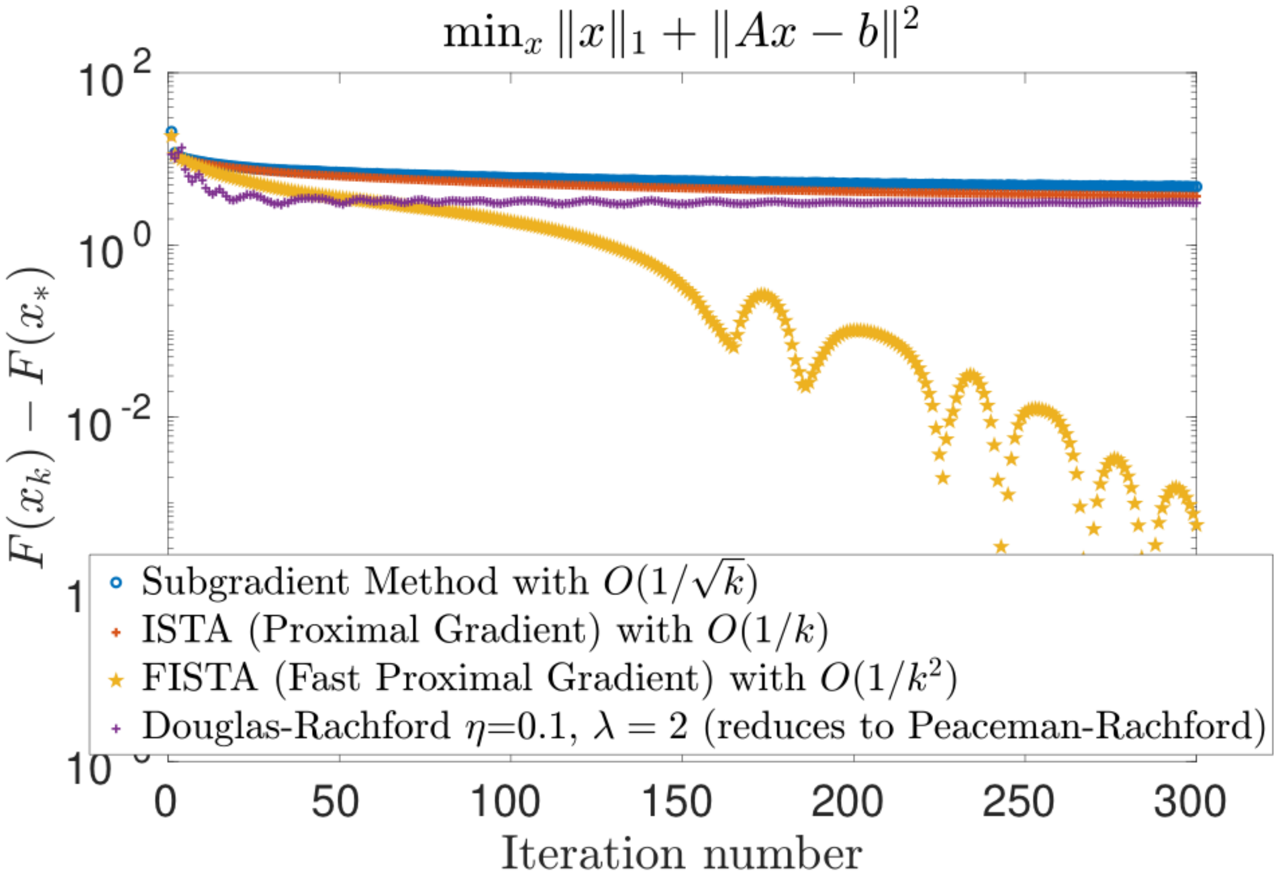
$$\boxed{\begin{aligned} &\text{Simple Douglas-Rachford Splitting} \\ &① \quad y_{k+1} = \frac{I + R_f R_g}{2}(y_k) \qquad \theta = \tfrac{1}{2} \\ &\qquad\quad = \theta y_k + (1-\theta) R_f R_g (y_k) \end{aligned}}$$

$x_{k+1} = \theta x_k + (1-\theta) T(x_k)$ converges if T is nonexpansive
$0 < \theta < 1$

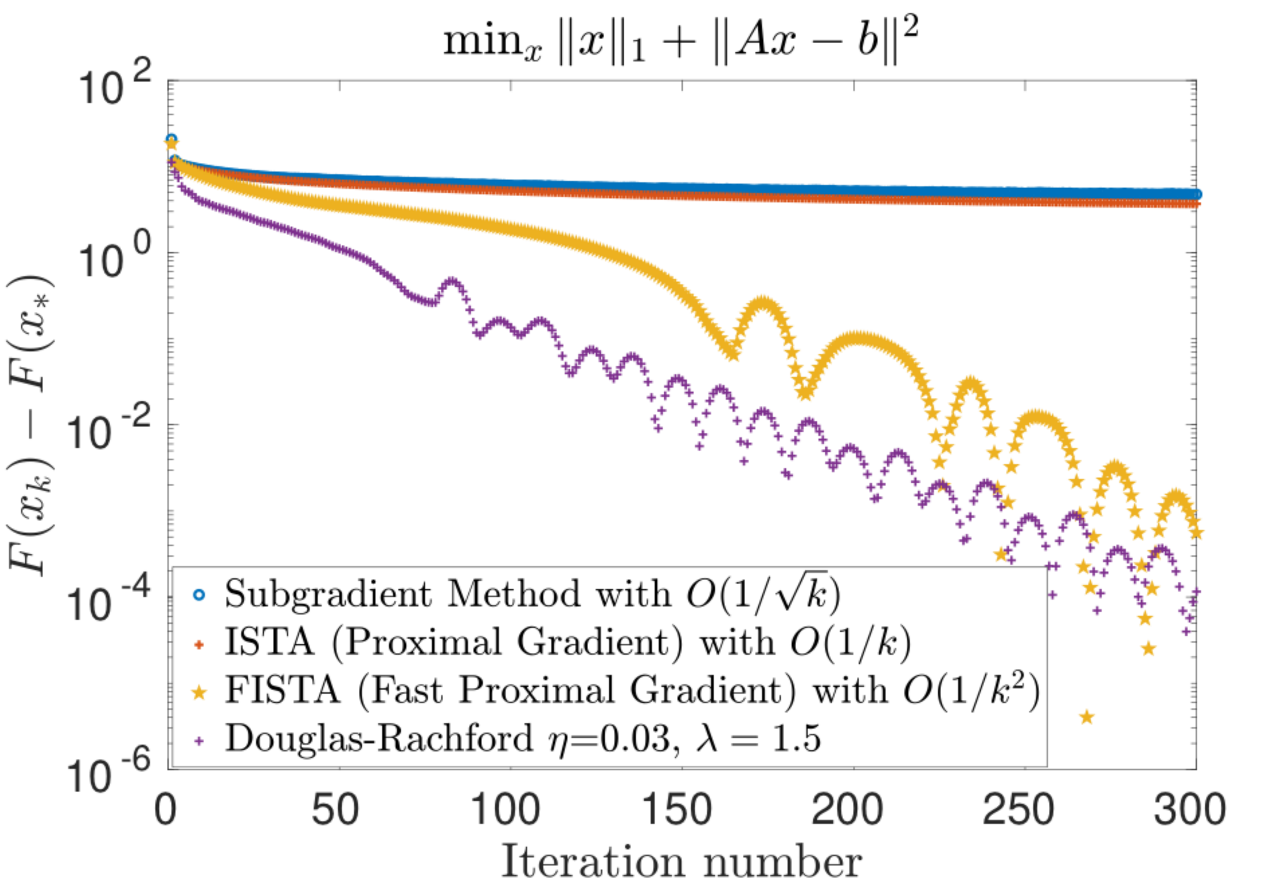
② General Douglas-Rachford Splitting

$$y_{k+1} = \theta y_k + (1-\theta) R_f R_g (y_k) \qquad \theta \in (0,1)$$

③ $\theta = 0$, $y_{k+1} = R_f R_g (y_k)$ is called Peaceman-Rachford splitting



(b) If $\lambda = 2$ in (6.1), then it does not converge. For this example $A \in \mathbb{R}^{40\times 1000}$ thus no strong convexity for $g(\mathbf{x})$.



(c) If tuning parameters, Douglas-Rachford splitting can be faster than FISTA or restarted FISTA for certain accuracy threshold.

(IV)

$\boxed{\text{Lemma}}$ Assume $f(x)$ and $g(x)$ are convex.
and x_* is a minimizer of $f(x)+g(x)$.

Let $y_* \in x_* + \eta \partial g(x_*)$, then

$x_* = \mathrm{Prox}_g^\eta(y_*)$ $(I+\eta\partial g)$ is multi-valued
$(I+\eta\partial g)^{-1}$ is single-valued

$$0 \in \partial f(x_*) + \partial g(x_*) \Longleftrightarrow R_f R_g(y_*) = y_*$$

Remark: y_* is a fixed point of

$$\begin{cases} \text{Peaceman-Rachford} & y_{k+1} = R_f R_g(y_k) \quad \lambda = 2 \\ \text{Douglas-Rachford} & y_{k+1} = \dfrac{I + R_f R_g}{2}(y_k) \quad \lambda = 1 \\ \text{Generalized DR} & y_{k+1} = (1-\lambda) y_k + \lambda \dfrac{I + R_f R_g}{2}(y_k) \quad \lambda \in (0,2) \end{cases}$$

Proof:

$$0 \in \partial f(x_*) + \partial g(x_*)$$

$$\Longleftrightarrow 0 \in -\eta \partial f(x_*) - \eta \partial g(x_*)$$

$$\Longleftrightarrow 0 \in x_* - \eta \partial f(x_*) - [\underbrace{x_* + \eta \partial g(x_*)}_{y_* \in}]$$

$$\Longleftrightarrow y_* \in x_* - \eta \partial f(x_*)$$

$$\Longleftrightarrow -y_* \in -x_* + \eta \partial f(x_*)$$

$$\Longleftrightarrow 2x_* - y_* \in (I + \eta \partial f)(x_*)$$

$$\Longleftrightarrow (I + \eta \partial f)^{-1}(2x_* - y_*) = x_*$$

$$\Longleftrightarrow (I + \eta \partial f)^{-1}(R_g(y_*)) = \mathrm{Prox}_g^\eta(y_*)$$

$$\Longleftrightarrow \quad 2(I+\eta\partial f)^{-1}(R_g(y_*)) - [2\mathrm{Prox}_g^\eta(y_*) - y_*] = y_*$$

$$\Longleftrightarrow \quad 2(I+\eta\partial f)^{-1}(R_g(y_*)) - R_g(y_*) = y_*$$

$$\Longleftrightarrow \quad R_f[R_g(y_*)] = y_*$$

$y_{k+1} = \frac{I+R_fR_g}{2}(y_k)$ converges to a fixed point of $S = \frac{I+R_f R_g}{2}$ which is NOT x_* !

$$y_* = \frac{I+R_fR_g}{2}(y_*)$$

$$\Longleftrightarrow \quad 2y_* = y_* + R_fR_g(y_*)$$

$$\Longleftrightarrow \quad y_* = R_f[R_g(y_*)]$$

$$= 2\mathrm{Prox}_f^\eta[R_g(y_*)] - R_g(y_*)$$

$$= 2\mathrm{Prox}_f^\eta[R_g(y_*)] - 2\mathrm{Prox}_g^\eta(y_*) + y_*$$

$$\Longleftrightarrow \quad \underbrace{\mathrm{Prox}_g^\eta(y_*)}_{z} = \mathrm{Prox}_f^\eta[\underbrace{R_g(y_*)}_{2\mathrm{Prox}_g^\eta(y_*) - y_* = 2z - y_*}]$$

$$(I+\eta\partial g)(z) = y_*$$

$$\Longleftrightarrow \quad z = (I+\eta\partial f)^{-1}[2z - (I+\eta\partial g)(z)]$$

$$= (I+\eta\partial f)^{-1}[(I-\eta\partial g)(z)]$$

$$\iff \quad 0 \in \partial f(z) + \partial g(z)$$

$\boxed{\text{Theorem}}$ (Douglas-Rachford splitting)

Assume ① $f(x)$ and $g(x)$ are convex,

② $f(x)+g(x)$ has a minizer

then the iteration

$$\begin{cases} y_{k+1} = \dfrac{I + R_f R_g}{2}(y_k) \\ x_{k+1} = \mathrm{Prox}_g^{\eta}(y_{k+1}) \end{cases}$$

converges, $\forall \eta > 0$

and $\{x_k\}$ converges to one minimizer of $f(x)+g(x)$

Remark: $\{y_k\}$ is an auxiliary variable

Even if x_* is unique, y_* may be non-unique

Remark: $y_{k+1} = \dfrac{I + R_f R_g}{2}(y_k)$

$$= \frac{1}{2} y_k + \frac{1}{2}\left[2\mathrm{Prox}_f\left[R_g(y_k)\right] - R_g(y_k)\right]$$

$$= \mathrm{Prox}_f\left[2\mathrm{Prox}_g(y_k) - y_k\right] - \mathrm{Prox}_g(y_k) + y_k$$

$$= \mathrm{Prox}_f\left[2x_k - y_k\right] - x_k + y_k$$

Example: $\min_x \|x\|_1 + i_{\{Ax=b\}}(x) \iff \min_x \|x\|_1$ s.t. $Ax=b$ $\boxed{A}$

$f(x) + g(x)$

$\text{Prox}_f^\eta = S_\eta(x)$ $\quad S_\eta(x)_i = \begin{cases} x_i - \eta, & x_i > \eta \\ x_i + \eta, & x_i < -\eta \\ 0, & x_i \in [-\eta, \eta] \end{cases}$

$\text{Prox}_g^\eta = P(x)$

$= x + A^T(AA^T)^{-1}(b - Ax)$

$$\begin{cases} x_k = \text{Prox}_g^\eta(y_k) = y_k + A^T(AA^T)^{-1}(b - Ay_k) \\ y_{k+1} = \text{Prox}_f[2x_k - y_k] - x_k + y_k \\ \quad = S_\eta[2x_k - y_k] - x_k + y_k \end{cases}$$

$\boxed{\text{Theorem}}$ (General Douglas-Rackford)

Assume ① $f(x)$ and $g(x)$ are convex,

② $f(x)+g(x)$ has a minizer

then the iteration

$$\begin{cases} y_{k+1} = (1-\lambda)y_k + \lambda \dfrac{I + R_f R_g}{2}(y_k) \\ x_{k+1} = \text{Prox}_g^\eta(y_{k+1}) \end{cases}$$

converges, $\forall \eta > 0$, $\lambda \in (0, 2)$

and $\{x_k\}$ converges to one minimizer of $f(x)+g(x)$

Remark: 1) λ is called Relaxation parameter

2) each λ gives a different algorithm

3) $\lambda = 1$ is Douglas-Rachford

4) $\lambda = 2$: $y_{k+1} = R_f R_g (y_k)$

is Peaceman-Rachford

which may not converge!