Today's Plan:

Introduction/Definition of $\begin{cases} \text{Embedded Submanifold in a linear space} \\ \text{Tangent Vector of a manifold} \\ \text{Riemmanian Metric} \end{cases}$

- A real linear space (a.k.a. an abstract vector space over $\mathbb{R}$) is a set $\mathcal{E}$ with addition and scalar multiplication defined and $\mathcal{E}$ is closed under these two operations:

  ① $\forall u, v \in \mathcal{E}$, $u+v \in \mathcal{E}$

  ② $\forall a \in \mathbb{R}$, $\forall u \in \mathcal{E}$, $a \cdot u \in \mathcal{E}$

  Examples: ① $\mathcal{E} = \mathbb{R}^n$

  ② $\mathcal{E} = \mathbb{R}^{n \times p}$

  ③ $\mathcal{E} = L^2(\Omega) = \{f(x) : \int_\Omega |f(x)|^2 dx < \infty\}$

- Let $\mathcal{E}$ be a linear space. Let $\langle u, v \rangle$ be an inner product. With an inner product, we can define distance and open ball, and also open sets.

  $\mathcal{E} = \mathbb{R}^{n \times p}$, $\langle U, V \rangle = \mathrm{tr}(V^T U) = \mathrm{tr}(U^T V) = \sum_{i=1}^{n} \sum_{j=1}^{p} U_{ij} V_{ij}$

Frobenius Norm $\|U\|_F = \sqrt{\langle U, U \rangle} = \sqrt{\mathrm{tr}(U^T U)} = \sqrt{\sigma_1^2 + \cdots + \sigma_p^2}$

- Let $\mathcal{E}$ and $\mathcal{F}$ be two linear spaces with inner products defined. Let $U \subset \mathcal{E}$, $V \subset \mathcal{F}$ be open sets.

For a smooth map $F: U \to V$

its differential at $X_0 \in U$ is $DF(X_0): \mathcal{E} \to \mathcal{F}$ defined by

$$\forall u \in \mathcal{E}, \quad DF(X_0)[u] = \frac{d}{dt} F[X_0 + tu]\Big|_{t=0} = \lim_{t \to 0} \frac{F(X_0+tu) - F(X_0)}{t}$$

Facts : $DF(X_0)$ is a linear map

Example: ① $\mathcal{E} = \mathbb{R}^3$, $\mathcal{F} = \mathbb{R}$, $F: U \subseteq \mathbb{R}^3 \to V \subseteq \mathbb{R}$

$DF(X_0)(u) = \frac{d}{dt} F(X_0 + tu)$ is the directional derivative.

$DF(X_0)$ is the mapping from u to $DF(X_0)[u]$

② $\mathcal{E} = \mathbb{R}^{n \times p}$, $\mathcal{F} = \mathbb{R}^{p \times p}$

$$h: \mathcal{E} \to \mathcal{F}$$
$$X \mapsto X^T X - I_p$$

$$Dh(X)[U] = \lim_{t \to 0} \frac{([X+tU]^T[X+tU] - I) - (X^T X - I)}{t}$$

$$= \lim_{t \to 0} \frac{t U^T X + t X^T U + t^2 U^T U}{t}$$

$$= U^T X + X^T U$$

- For a smooth function $f: \mathcal{E} \to \mathbb{R}$,

the Euclidean gradient $\operatorname{grad} f: \mathcal{E} \to \mathcal{E}$ is defined by

$$\langle \operatorname{grad} f(x), v \rangle = Df(x)[v] \quad \forall x, v \in \mathcal{E}$$

Example: $\mathcal{E} = \mathbb{R}^3$, $\mathcal{F} = \mathbb{R}$, $F: U \subseteq \mathbb{R}^3 \longrightarrow V \subseteq \mathbb{R}$

$DF(x_0)(u)$ is the directional derivative.

$\Rightarrow \operatorname{grad} F(x_0) = \nabla F(x_0)$

Example: $\mathcal{E} = \mathbb{R}^{n \times p}$,

$$f(X) = \langle X, X \rangle = \|X\|_F^2 = \sum_{i=1}^{n} \sum_{j=1}^{p} X_{ij}^2$$

$$Df(X)[V] = \lim_{t \to 0} \frac{f(X+tV) - f(X)}{t}$$

$$= \lim_{t \to 0} \sum_{i=1}^{n} \sum_{j=1}^{p} \frac{|X_{ij} + tV_{ij}|^2 - |X_{ij}|^2}{t}$$

$$= 2 \sum_{i=1}^{n} \sum_{j=1}^{p} X_{ij} \cdot V_{ij} = \langle 2X, V \rangle$$

$$\Rightarrow \operatorname{grad} f(X) = 2X$$

- Rank of a linear map

Let $\mathcal{E}$ & $\mathcal{F}$ be finite dimensional linear spaces

For a linear map from $\mathcal{E}$ to $\mathcal{F}$, it can be expressed as

$$L_A: \mathbb{R}^n \to \mathbb{R}^m$$
$$x \longmapsto Ax$$

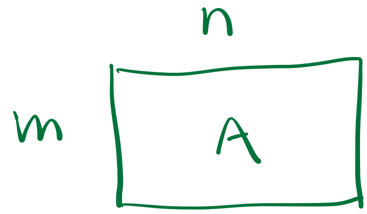
$$m \, \overset{n}{\boxed{\quad A \quad}}$$

under some bases for $\mathcal{E}$ & $\mathcal{F}$.

The matrix A is different
But $\operatorname{Rank}(A)$ is the same
for different bases.

- A surjective linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ has rank m.

$$L_A : \mathbb{R}^n \to \mathbb{R}^m$$
$$x \longmapsto Ax$$



Surjectivity $\Rightarrow \mathbb{R}^m = \{Ax : x \in \mathbb{R}^n\} =$ Col Space of A

$\Rightarrow$ dim of Col Space is m

- Rank of the differential

Example: $h : \mathbb{R}^{n\times p} \longrightarrow \mathbb{R}^{p\times p}$ $\quad p \le n$

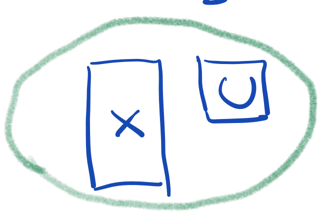
$$X \longmapsto X^T X - I_p$$

$$Dh(X) : \mathbb{R}^{n\times p} \longrightarrow \mathbb{R}^{p\times p}$$

$$Dh(X)[V] = V^T X + X^T V, \quad \forall V \in \mathbb{R}^{n\times p}$$

$$Dh(X)[\tfrac{1}{2} X U] = (\tfrac{1}{2} X U)^T X + X^T (\tfrac{1}{2} X U)$$

$U \in \mathbb{R}^{p\times p}$



$$= \tfrac{1}{2} U^T X^T X + \tfrac{1}{2} X^T X U = U$$

If $X^T X = I$, $U^T = U$

$\forall U \in S_{p\times p} = \{Y \in \mathbb{R}^{p\times p} : Y^T = Y\}$

$\forall X \in St(n,p) = \{X \in \mathbb{R}^{n\times p} : X^T X = I_p\}$, $p \le n$

$\Rightarrow$ $Dh(X) : \mathbb{R}^{n\times p} \longrightarrow \mathbb{R}^{p\times p}$

can be regarded as a surjective linear map to $S_{p\times p}$

$\Rightarrow$ $Dh(X) : \mathbb{R}^{n\times p} \longrightarrow S_{p\times p} \subseteq \mathbb{R}^{p\times p}$

is surjective if $X \in St(n,p)$

$\Rightarrow$ Rank of $Dh(X)$ is $\frac{p(p+1)}{2}$ if $X \in St(n,p)$

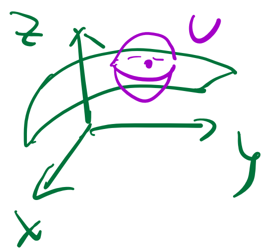
In other words, "Col Space" of $Dh(X)$ is $S_{p\times p}$.

- The generic definition of a manifold and embedded submanifold can be very abstract. To have an easier and quicker understanding of an embedded submanifold in a linear space, we consider the following definition/theorem/fact:

<u>Definition</u> (or Theorem) Let $\mathcal{E}$ be a linear space of dim d and M be an non-empty subset.
Let $\mathcal{F}$ be another linear space.

If $\forall x \in M$, there exist an open set U in $\mathcal{E}$, $x \in U$

and a smooth $h: U \to \mathcal{F}$ such that



① $\forall y \in U$, $h(y) = 0 \Leftrightarrow y \in M$

② $\text{Rank}[Dh(x)] = k$, $\forall x \in M$

then M is an embedded submanifold of $\mathcal{E}$ of dimension $d-k$

Remark : $m \overset{d}{\boxed{A}}$ $\quad L_A : \mathbb{R}^d \longrightarrow \mathbb{R}^m$

$$x \longmapsto Ax$$

$$\text{Nullity}(A) + \text{Rank}(A) = d$$

$$Dh(x) : \mathbb{R}^d \to \mathbb{R}^k$$

$$\text{Rank}[Dh(x)] = k \Rightarrow \underbrace{\dim\left(\text{Ker}[Dh(x)]\right)}_{\text{dimension of submanifold}} = d-k$$

Remark : This is a straightforward extension of definition of surfaces

Example : Any surface S described by $h(x,y,z)=0$

such as $h(x,y,z) = x^2+y^2-z$

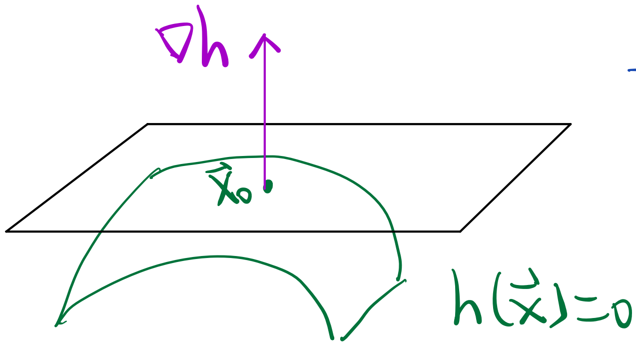
$$Dh(\vec{x})[\vec{u}] = \begin{pmatrix} h_x \\ h_y \\ h_z \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\forall \vec{x} \in S, \begin{pmatrix} h_x \\ h_y \\ h_z \end{pmatrix} \neq \vec{0}, \text{ So } \text{Rank}[Dh(\vec{x})] = 1$$

$Dh(\vec{x}) : \mathbb{R}^3 \to \mathbb{R}$ matrix is $[h_x \ h_y \ h_z]$

Kernel or Null space of $Dh(\vec{x}_0)$ is

$$\text{Ker}[Dh(\vec{x}_0)] = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathbb{R}^3 : \begin{pmatrix} h_x \\ h_y \\ h_z \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0 \right\}$$



The tangent plane at $\vec{x}_0$ has equation

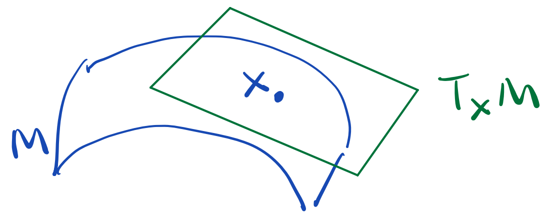
$$\begin{pmatrix} h_x(\vec{x}_0) \\ h_y(\vec{x}_0) \\ h_z(\vec{x}_0) \end{pmatrix} \cdot \begin{pmatrix} u-x_0 \\ v-y_0 \\ w-z_0 \end{pmatrix} = 0$$

So the tangent plane of surfaces can be understood as

$$\mathrm{Ker}[Dh(\vec{x}_0)] + \vec{x}_0 \quad \text{or simply} \quad \mathrm{Ker}[Dh(\vec{x}_0)]$$

Definition/Theorem Let M be an embedd submanifold of $\mathcal{E}$.

At $x \in M$, the tangent space is $T_xM = \mathrm{Ker}[Dh(\vec{x})]$



Remark:

① $T_xM = \mathrm{Ker}[Dh(\vec{x})]$ is a linear space

② $\mathrm{Ker}[Dh(\vec{x})] + \vec{x}$ is NOT a linear space

Example: Stiefel Manifold

$$\mathrm{St}(n,p) = \{X \in \mathbb{R}^{n \times p} : X^TX = I_p\}, \ p \le n \qquad \boxed{X}$$

Special Cases:

$p=1$ $\quad \mathrm{St}(n,1) = S^{n-1} = \{x \in \mathbb{R}^n : x^Tx = 1\}$

$p=n$ $\quad \mathrm{St}(n,n) = O(n) = \{X \in \mathbb{R}^{n \times n}, X^TX = I\}$ Orthogonal Group

Want to $\begin{cases} \text{① Verify it's an embedded submanifold in } \mathbb{R}^{n\times p} \\ \text{② Calculate its dimension} \\ \text{③ Derive its tangent space} \end{cases}$

$$\mathcal{E} = \mathbb{R}^{n\times p}, \quad M = \mathrm{St}(n,p) \subset \mathcal{E}, \quad \mathcal{F} = S_{p\times p}$$

Symmetric matrices

$$h: \mathcal{E} \longrightarrow \mathcal{F}$$
$$X \longmapsto X^TX - I_p$$

$h(X) = 0 \Leftrightarrow X \in \mathrm{St}(n,p)$

$$Dh(X): \mathcal{E} \longrightarrow \mathcal{F}$$
$$V \longmapsto V^T X + X^T V$$

At $X \in St(n,p)$, $Dh(X)(\frac{1}{2}XU) = U$, $\forall U \in \mathcal{F}$

$\Rightarrow$ $Dh(X)$ is surjective, $X \in St(n,p)$

$\Rightarrow$ $Dh(X)$ has Rank $\frac{p(p+1)}{2}$ at $X \in St(n,p)$

Also, $h(X) = 0 \Leftrightarrow X \in M$ ($Ker(h) = M$)

$\Rightarrow$ M is an embedded submanifold of dim $np - \frac{p(p+1)}{2}$ in $\mathcal{E} = \mathbb{R}^{n \times p}$

(Orthogonal Group $O(n) = \{X \in \mathbb{R}^{n \times n} : X^T X = I\}$ is an embedded submanifold of dim $\frac{n(n-1)}{2}$ in $\mathbb{R}^{n \times n}$)

$$Dh(X): \mathcal{E} \longrightarrow \mathcal{F}$$
$$V \longmapsto V^T X + X^T V$$

$\Rightarrow$ Tangent Space is $T_X M = \{V \in \mathbb{R}^{n \times p} : V^T X + X^T V = 0\}$

Sphere S^{n-1}, $T_X S^{n-1} = \{V \in \mathbb{R}^n : V^T X + X^T V = 0\}$
$= \{V \in \mathbb{R}^n : V^T X = 0\}$

Remark: For an abstract manifold in a generic definition

① Tangent Space $T_X M$ is always a linear space of the same dim as M.

② The map $h(x)$ may not exist.

③ When we have such a map, $\begin{cases} \mathrm{Ker}(h) = M \\ \mathrm{Ker}(Dh(x)) = T_x M \end{cases}$

Definition of Metric and Riemannian Manifold

① If each tangent space $T_x M$ has an inner product, we call it metric, denoted as g_x

1) $\forall u, v \in T_x M$, $g_x(u,v) = g_x(v,u)$

2) $\forall a, b \in \mathbb{R}$, $g_x(au+bv, w) = a g_x(u,w) + b g_x(v,w)$

3) $g_x(u,u) \geq 0$ & $g_x(u,u) = 0 \Leftrightarrow u = 0$

② g_x is smooth w.r.t. x, we say it's Riemmanian

③ A smooth manifold M with a Riemannian metric g is called Riemmanian manifold (M, g)

④ If $M \subset \mathcal{E}$ is an embedded submanifold in $\mathcal{E}$ and g is the inner product in $\mathcal{E}$,

(g does not depend on x)

then we say M is a Riemmanian submanifold of $\mathcal{E}$.

Example: ① $M = S^{n-1}$, $g_x(u, v) = u^T v$

Then (S^{n-1}, g) is a Riemmanian submanifold of $\mathbb{R}^n$

② $M = S^{n-1}$, $g_x(u, v) = u^T(xx^T + I)v$

(S^{n-1}, g) is a Riemmanian manifold

Layers of concepts for an abstract set M

① A not too crazy set (second countable Hausdorff)

② A topology (has a notion of open sets)

③ A topological manifold (Locally Euclidean)

④ A smooth manifold (transition map is smooth)

⑤ A Riemannian manifold (M, g)

⑥ A pseudo-Riemmanian manifold (M, g)

pseudo-Riemmanian metric, denoted as g_x

1) $\forall\, u, v \in T_x M$, $g_x(u, v) = g_x(v, u)$

2) $\forall\, a, b \in \mathbb{R}$, $g_x(au + bv, w) = a g_x(u, w) + b g_x(v, w)$

Remove $\boxed{3)\ \ g_x(u, u) \geq 0 \ \&\ g_x(u, u) = 0 \Leftrightarrow u = 0}$

Example: General Relativity

Lorentzian Manifold of SpaceTime

$$g\left[\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ t_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ t_2 \end{pmatrix}\right] = x_1x_2 + y_1y_2 + z_1z_2 - t_1t_2$$