Part II: Nonsmooth Convex Optimization

A quick preview of what we plan to cover:

① The main examples:

1) "Lasso" (Tibshirani 1996)

Least absolute shrinkage & selection operator

$$\min_{x \in \mathbb{R}^n} \frac{\lambda}{2} \|Ax - b\|^2 + \|x\|_1$$

2) "Basis Pursuit" (Chen, Donoho, Saunders, 1998)

$$\min_{x \in \mathbb{R}^n} \|x\|_1$$

such that $Ax = b$

$$\begin{array}{cc} & n \\ m & \boxed{\quad A \quad} \end{array} \boxed{x} = \boxed{b}$$

minimizing $\|x\|_1$ gives a sparse x (less nonzero entries of x)

② Subgradient

$f(x) = |x|$ does not have a gradient

but it has a subgradient $\partial f(x)$

$$\partial f(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \\ [-1, 1], & x = 0 \end{cases}$$

③ Algorithms & Convergence

Subgradient descent

$$x_{k+1} = x_k - \eta g_k, \quad g_k \in \partial f(x_k)$$

④ Composite optimization for $\min_x f(x)+g(x)$

1) Forward-Backward Splitting

option for reading: F-B-F splitting

2) ADMM

3) Douglas-Rachford

More on Convex Functions

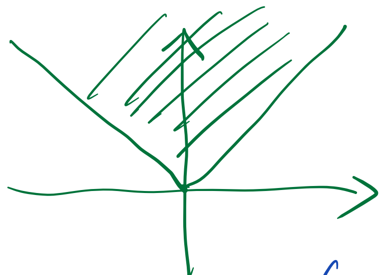
<u>Def</u> $f(x): \mathbb{R}^n \to \mathbb{R}$ is convex if

$$f[(1-a)x+ay] \le (1-a)f(x)+af(y), \quad 0<a<1$$

<u>Def</u> Epigraph of $f(x)$ is

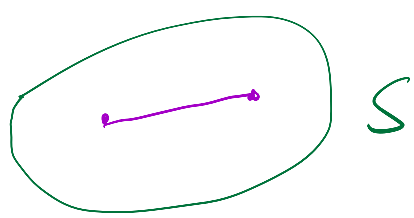
$$\{(x,y)\in \mathbb{R}^n \times \mathbb{R} : f(x)\le y\}$$

$f(x)=|x|$



<u>Def</u> A set S is convex if

$$\forall x,y\in S, \quad (1-a)x+ay \in S, \quad \forall a\in(0,1)$$



Theorem For $f: \mathbb{R}^n \to \mathbb{R}$,

$f(x)$ is convex $\Longleftrightarrow$ Epigraph of $f(x)$ is convex.

Theorem If $f(x): \mathbb{R}^n \to \mathbb{R}$

and $f_i(x): \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \dots, N$ are convex on $\mathbb{R}^n$,

then

① $g(x) = f(Ax + b)$ is convex

② $g(x) = \sum_{i=1}^{N} a_i f_i(x)$ is convex, $a_i \geq 0$

③ $g(x) = \max_i f_i(x)$ is convex

Theorem If $f(x): \mathbb{R}^n \to \mathbb{R}$ is convex,

then $f(x)$ is $\begin{cases} \text{locally Lip-continuous} \\ \text{continuous on } \mathbb{R}^n \end{cases}$

$\forall x_0 \in \mathbb{R}^n$, there is a ball centered at x_0 with a radius $\delta > 0$

$$B_\delta(x_0) = \{ x \in \mathbb{R}^n, \ \|x - x_0\| < \delta \}$$

s.t. $\forall x \in B_\delta(x_0)$, $\exists L$ s.t.

$$|f(x) - f(x_0)| \leq L \|x - x_0\|$$

Subgradients

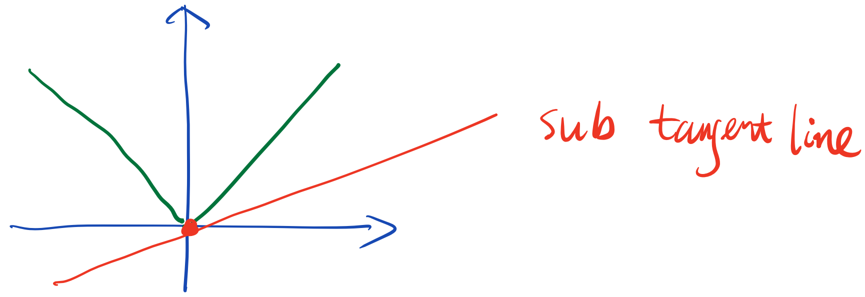
Definition: For $f: \mathbb{R}^n \to \mathbb{R}$, a vector $g \in \mathbb{R}^n$ is a subgradient of $f(x)$ at x if

$$f(y) \geq f(x) + \langle g, y-x \rangle, \quad \forall y \in \mathbb{R}^n$$

Example: $f(x) = |x|$

at $x=0$

$g \in [-1, 1]$



Def: The set of all subgradients of $f(x)$ at x is called the subdifferential, denoted by $\partial f(x)$.

$$\partial f(x) = \{ g \in \mathbb{R}^n : f(y) \geq f(x) + \langle g, y-x \rangle, \forall y \}$$

$f(x) = |x|$

$$\partial f(x) = \{1\}, \quad x > 0$$

$$\partial f(x) = \{-1\}, \quad x < 0$$

$$\partial f(0) = [-1, 1]$$

Theorem $f(x) : \mathbb{R}^n \to \mathbb{R}$

$f(x)$ is convex $\Leftrightarrow$ $\partial f(x)$ is not empty at any x

Proof: "$\Leftarrow$" $\quad z = (1-a)x + ay, \quad a \in (0,1)$

$g \in \partial f(z)$

$$f(x) \geq f(z) + \langle g, x-z \rangle = f(z) - a \langle g, y-x \rangle$$

$$f(y) \geq f(z) + \langle g, y-z \rangle = f(z) + (1-a) \langle g, y-x \rangle$$

$$(1-a)f(x) + af(y) \geq (1-a)f(z) + af(z)$$
$$= f(z)$$
$$= f((1-a)x + ay)$$

" $\Rightarrow$ " skipped.

Example: $f(x) = \|x\|_1 = \sum_{i=1}^{n} |x_i|$

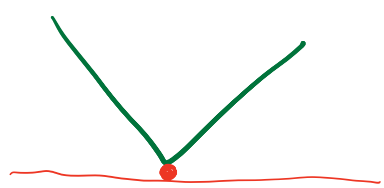
$$g \in \partial f(x) \Leftrightarrow g_i \begin{cases} = 1, & x_i > 0 \\ = -1, & x_i < 0 \\ \in [-1, 1], & x_i = 0. \end{cases}$$

$x \in \mathbb{R}^n$
$g \in \mathbb{R}^n$

Theorem

$$x_* \text{ minimizes } f(x) \Leftrightarrow \vec{0} \in \partial f(x_*)$$

Proof: x_* is a minimizer

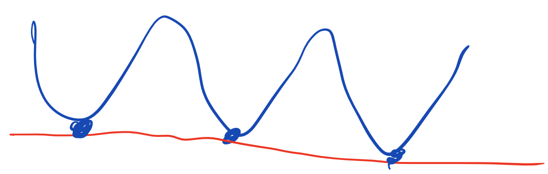


$$\Updownarrow$$

$$\boxed{\begin{aligned} f(x) &\geq f(x_*) \\ &= f(x_*) + \langle \vec{0}, x - x_* \rangle \end{aligned}}$$

$$\Updownarrow$$

$$\vec{0} \in \partial f(x_*)$$



Theorem $f(x): \mathbb{R}^n \to \mathbb{R}$ is convex

① $\nabla f(x)$ exists $\Rightarrow \partial f(x) = \{\nabla f(x)\}$

② $\partial f(x)$ has only one element $\Rightarrow \nabla f(x)$ exists

Theorem: ① $\partial[af](x) = a[\partial f(x)]$

② $\partial[f+g] = \partial f + \partial g$

Two Simple Algorithms for $\min_x f(x)$ $\quad x_{k+1} = x_k - \eta \nabla f(x_k)$

$0 \in \partial f(x_*)$ with a convex non-differentiable $f(x)$:

① Subgradient Method (Forward Euler for ODE)

$$x_{k+1} = x_k - \eta_k g_k, \quad g_k \in \partial f(x_k)$$

② Proximal Point Method (Backward Euler)

$$x_{k+1} = x_k - \eta_k g_{k+1}, \quad g_{k+1} \in \partial f(x_{k+1})$$

Need to solve:

$$x_{k+1} + \eta\, g_{k+1} = x_k$$

$$(I + \eta \partial f)(x_{k+1}) = x_k$$

$$x_{k+1} = (I + \eta \partial f)^{-1}(x_k)$$

Proximal Operator

Def $\quad \mathrm{Prox}_{\gamma f}(x) = \underset{u}{\operatorname{argmin}} \left[\gamma f(u) + \frac{1}{2}\|u-x\|^2 \right]$

Claim $\quad \mathrm{Prox}_{\gamma f}(x) = (I + \gamma \partial f)^{-1}(x)$

Proof: $\begin{cases} \frac{1}{2}\|u-x\|^2 \text{ is convex w.r.t. } u \\ \eta f(u) \text{ is convex} \end{cases}$

$\Rightarrow \eta f(u) + \frac{1}{2}\|u-x\|^2$ is convex

Let $\quad u_* = \underset{u}{\operatorname{argmin}} \left[\eta f(u) + \frac{1}{2}\|u-x\|^2 \right]$

Then $\quad 0 \in \partial[\eta f](u_*) + (u_* - x)$

$\iff \quad x \in u_* + \eta \cdot \partial f(u_*)$

$\iff \quad u_* = (I + \eta \partial f)^{-1}(x)$

If there is another y s.t.

$x \in y + \eta \partial f(y)$

then $\quad 0 \in \partial[\eta f](y) + (y - x)$

$\Rightarrow y$ minimizes $g(u) = \eta f(u) + \frac{1}{2}\|u-x\|^2$

$g(u)$ is strongly convex (because $\frac{1}{2}\|u-x\|^2$ is)

$\Rightarrow$ unique minimizer $\Rightarrow y = u^*$.

Example: $f(x) = |x|$

$$(I + \gamma \partial f)^{-1}(x) = \begin{cases} x - \gamma, & \text{if } x > \gamma \\ x + \gamma, & \text{if } x < -\gamma \\ 0, & \text{if } x \in [-\gamma, \gamma] \end{cases}$$

Shrinkage operator