HILBERT SCHEME OF A PAIR OF SKEW LINES ON A CUBIC HYPERSURFACE

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Abstract. We study an irreducible component $H(X)$ of the Hilbert scheme $\mathrm{Hilb}^{2t+2}(X)$ of a smooth cubic hypersurface X containing two disjoint lines. For cubic threefolds, $H(X)$ is always smooth, as shown in [Zha23]. We provide a second proof and generalize this result to higher dimensions. Specifically, for cubic hypersurfaces of dimension at least four, we show $H(X)$ is normal, and it is smooth if and only if X lacks certain "higher triple lines." Using Hilbert-Chow morphism, we describe the birational geometry of $H(X)$.

1. Introduction

Lines in the projective space $\mathbb{P}^n$ are parameterized by the Grassmannian $Gr(2, n+1)$ of two-dimensional linear subspaces in $\mathbb{C}^{n+1}$. Using Grothendieck's language, the Grassmannian $Gr(2, n+1)$ is the Hilbert scheme $\mathrm{Hilb}^{t+1}(\mathbb{P}^n)$ parameterizing universal family of subschemes of $\mathbb{P}^n$ with Hilbert polynomial $t+1$, namely projective lines.

One can consider a pair of lines. If n is at least 3, then a general pair of two lines in $\mathbb{P}^n$ is skew, and they define an irreducible component $H(\mathbb{P}^n)$ of the Hilbert scheme $\mathrm{Hilb}^{2t+2}(\mathbb{P}^n)$. The component $H(\mathbb{P}^n)$ is birational to the symmetric product $\mathrm{Sym}^2 Gr(2, n+1)$ and parameterizes pairs of disjoint lines and their flat limits.

Using deformation theory, Chen, Coskun, and Nollet showed that

Theorem 1.1. [CCN11] *The component $H(\mathbb{P}^n)$ is smooth.*

In this paper, we study a similar question for cubic hypersurfaces in $\mathbb{P}^n$.

1.1. Cubic hypersurfaces.

The study of lines on cubic hypersurfaces dates back to the 19th century, when Cayley and Salmon discovered the 27 lines on a cubic surface [Dol05]. There are 16 lines that skew to a given line, so there are $27 \times 16 = 432$ disjoint pairs of skew lines on a cubic surface.

Starting from dimension 3, lines on a cubic hypersurface vary in a continuous family. They form a subvariety F of Grassmannian and are called the *Fano variety of lines*. For example, when X is a cubic threefold, then F is a surface of general type. Griffiths and Clemens [CG72] studied the Abel-Jacobi map

$$F \times F \to J(X), \tag{1}$$

by integrating a differential 3-form ω against a topological 3-chain bounding a pair of lines L_1 and L_2. Its geometry is used to prove the irrationality of cubic threefolds.

For a cubic fourfold, F is hyperkähler fourfold [BD85] and deformation equivalent to the Hilbert scheme of two points on a K3 surface.

There is a rational map

$$F \times F \dashrightarrow Z \tag{2}$$

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from a pair of two skew lines on cubic fourfold to a hyperkahler 8-fold considered by Voisin [Voi16]. (cf. Section 10.) Since a general pair of lines on X are skew when $\dim(X) \geqslant 3$, both map (1) and (2) are determined by where pairs of skew lines are sent to. This motivates us to study the parameter space of a pair of skew lines on a cubic hypersurface.

Let $X \subseteq \mathbb{P}^n$ be a smooth cubic hypersurface, with $n \geqslant 4$, then a general pair of two lines on X is disjoint and determines an irreducible component $H(X)$ in the Hilbert scheme $\mathrm{Hilb}^{2t+2}(X)$ of X. Then, there is a closed embedding

$$i : H(X) \hookrightarrow H(\mathbb{P}^n).$$

Definition 1.2. We will call the component $H(X)$ the *Hilbert scheme of a pair of skew lines* on X.

In a previous work, we proved that

Theorem 1.3. [Zha23] *When X is a smooth cubic threefold, $H(X)$ is smooth.*

The proof is based on the Abel-Jacobi map and the geometry of the theta divisor of the intermediate Jacobian. In this paper, we will provide a second proof, and the new argument naturally generalizes to higher dimensions.

Theorem 1.4. *Suppose X is a general cubic hypersurface of dimension at least four. The Hilbert scheme of a pair of skew lines $H(X)$ is smooth.*

Our method is to describe the birational geometry of $H(X)$. We relate to the Chow variety $\mathrm{Sym}^2 F$ parameterizing pairs of lines on X with reduced structure. There is a Hilbert-Chow morphism which factors through the Hilbert scheme of two points of F

$$H(X) \xrightarrow{\sigma_2} F^{[2]} \xrightarrow{\sigma_1} \mathrm{Sym}^2 F. \tag{3}$$

The map σ_1 blows up the diagonal, and the second map σ_2 blows up the strict transform of the subvariety D'_F of $\mathrm{Sym}^2 F$ parameterizing pairs of incidental lines.

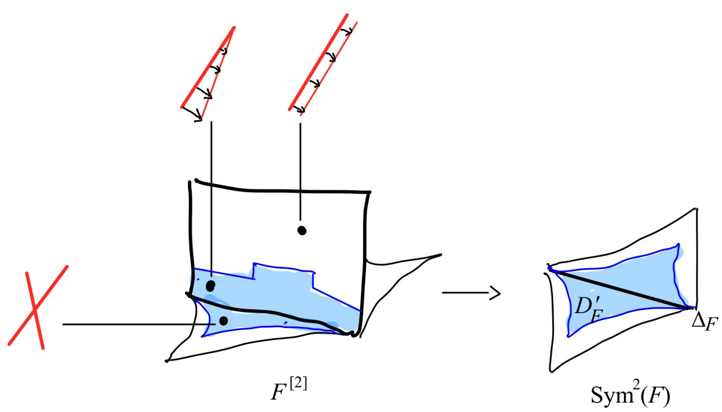

FIGURE 1. $F^{[2]}$ as intermediate Hilbert scheme of a pair of skew lines

The composition (3) is an isomorphism at the locus of pairs of disjoint lines, which is called the type (I) subscheme of X, as in [CCN11]; At a general point of D'_F parameterizing two lines (L_1, L_2) intersecting at a point, the blow-up σ_2 provides collections of $\mathbb{P}^3$ containing the plane $\mathrm{Span}(L_1, L_2)$, hence determines an embedded point supported the intersection $L_1 \cap L_2 = \{x\}$. Such a subscheme is of type (III). Over the diagonal, the exceptional locus

E of σ_1 parameterized a line L together with a normal direction $v \in H^0(N_{L|X})$. Depending on if v comes from a sub-line bundle $\mathcal{O}_L$ or $\mathcal{O}_L(1)$, the infinitesimal data will be different. If $v \in H^0(\mathcal{O}_L)$, then it corresponds to a point on $F^{[2]}$ away from the second blowup center and determines a subscheme of type (II); While if $v \in H^0(\mathcal{O}_L(1))$, it is on the center of σ_2, and the second blowup in the same way provides a $\mathbb{P}^3$ containing the plane $\mathrm{Span}(L, v)$ and determines an embedded point supported on L. This is called a type (IV) subscheme.

In figure 1 we depict the first blowup. The blue region denotes the incidental locus D'_F and its strict transform. The non-equidistribution of normal directions over diagonal Δ_F is a reflection of the fact that there is a distinction of two types of the lines based on splitting data of the normal bundle $N_{L|X}$. We also note the portrait is for $\dim(X) \geqslant 4$. When $\dim(X) = 3$, D'_F does not contain the diagonal.

To understand Theorem 1.3, when $\dim(X) = 3$, the subvariety D'_F is a divisor, so the second blowup σ_2 is an isomorphism. Hence the Hilbert scheme of a pair of skew lines $H(X)$ is always smooth as long as X is smooth. However, when $\dim(X) \geqslant 4$, D'_F has higher codimension, so if its strict transform is singular, the second blow-up will produce singularities on $H(X)$. We need to describe when this will happen.

1.2. **Higher triple lines.** Let $X \subseteq \mathbb{P}^n$ be a cubic hypersurface. A line $L \subseteq X$ is called of the *first type* if there is a codimension-three linear subspace $P^{n-3} \subseteq \mathbb{P}^n$ containing L and is tangent to X at all points on L. It is called a line of the *second type* if there is a codimension-two linear subspace $P^{n-2} \subseteq \mathbb{P}^n$ tangent to X along L. Lines of the second type form a closed subspace in F. They are distinguished based on the first-order data of the cubic equation near L. These notions are introduced and studied in [CG72].

We call a line $L \subseteq X$ on a smooth cubic hypersurface a *higher triple line* if it is the second type, and the second-order data of the cubic equation near L is linearly dependent when restricted to the codimension two space P^{n-2}. The three notions are related by the specialization

$$\text{lines of the first type} \rightsquigarrow \text{lines of the second type} \rightsquigarrow \text{higher triple lines.}$$

Alternatively, a line has second type if there is a linear P^{n-2} and $P^{n-2} \cap X$ is singular at all points at L. When X is futhur a higher triple line, $P^{n-2} \cap X$ has transversal A_2 singularity everywhere on the line.

For example, when X is a cubic threefold, a higher triple line is a triple line, i.e., there is a plane $\mathbb{P}^2$ such that $\mathbb{P}^2 \cap X = 3L$; When X is a cubic fourfold, being a higher triple line is more restrictive, it implies L is contained in a cone of a cuspidal cubic curve as a $\mathbb{P}^3$-section of X (see Figure 2). For special cubic fourfolds, higher triple lines may vary in a continuous family — Fermat cubic fourfold has 45 Eckardt points, and corresponds to (at least) 45 one-parameter families of higher triple lines. However, we show that having a higher triple line is a proper closed condition.

Proposition 1.5. *(cf. Proposition 2.7) A general cubic hypersurface does not contain a higher triple line.*

The second main result of this paper is

Theorem 1.6. *Let X be a smooth cubic hypersurface with $\dim(X) \geqslant 4$. Then, $H(X)$ is normal. Moreover, $H(X)$ is smooth if and only if the X has no higher triple line.*

This recovers Theorem 1.4.

1.3. **Strategy of the proof.** The proof is still based on the description of the Hilbert-Chow morphism (3). In fact, we are oversimplifying the discussion above — To show that the Hilbert-Chow morphism has the factorization (3), we take into consideration the scheme structure of the blowup centers. In other words, we need to know whether the scheme structure of the blowup centers are *reduced.*

By the universal property of blow-up, the birational morphism $H(X) \to \mathrm{Sym}^2 F$ is the restriction of the Hilbert-Chow morphism of the Hilbert scheme of a pair of skew lines on the ambient projective space. To avoid taking about singularities on the diagonal, we can take a branched double cover $\widetilde{H(X)}$ of $H(X)$ by ordering the pair of lines. Now, the Chow variety becomes $F \times F$ and is smooth.

The Hilbert-Chow morphism on the branched double cover of $H(\mathbb{P}^n)$ factors as blowup of the diagonal, and then blow up the strict transform $\tilde{D}$ of the subvariety D parameterizing pairs of incidental lines (cf. [CCN11, p.8], [Zha23, Prop. 3.3]). The factorization is shown in the second column of the following diagram

$$
\begin{array}{ccccc}
\widetilde{H(X)} & \hookrightarrow & \widetilde{H(\mathbb{P}^n)} \cong \mathrm{Bl}_{\tilde{D}}\mathrm{Bl}_{\Delta}\big(Gr(2,n+1) \times Gr(2,n+1)\big) & & \\
\downarrow{\scriptstyle \sigma_2} & & \downarrow & & \\
\mathrm{Bl}_{\Delta_F}(F \times F) & \hookrightarrow & \mathrm{Bl}_{\Delta}\big(Gr(2,n+1) \times Gr(2,n+1)\big) & \hookleftarrow & \tilde{D} \\
\downarrow{\scriptstyle \sigma_1} & & \downarrow & & \downarrow \\
F \times F & \hookrightarrow & Gr(2,n+1) \times Gr(2,n+1) & \hookleftarrow & D.
\end{array}
\tag{4}
$$

In the first column, σ_1 is the blow-up of the diagonal Δ_F of $F \times F$ with reduced structure. σ_2 is to blow up the scheme-theoretic intersection

$$
\tilde{D}_F := \mathrm{Bl}_{\Delta_F}(F \times F) \cap \tilde{D}. \tag{5}
$$

If the the blow-up center $\tilde{D}_F$ is smooth, then $\widetilde{H(X)}$, together with its $\mathbb{Z}_2$ quotient, $H(X)$, will be smooth. So, proving Theorem 1.3 and 1.6 reduces to

Problem 1.7. *Determine when the scheme-theoretical intersection* (5) *is smooth.*

First, note that the intersection (5) is *not transverse* in the sense that the codimension of the intersection is not the sum of the codimensions in the ambient space $\mathrm{Bl}_{\Delta}\big(Gr(2,n+1) \times Gr(2,n+1)\big)$. In fact, $\mathrm{codim}(\tilde{D}) = n-2$, $\mathrm{codim}(\mathrm{Bl}_{\Delta_F}(F \times F)) = 8$, but the scheme (5) has codimension $n+5$. This does not forbid the intersection to be smooth. As an example, two lines intersecting at a point in $\mathbb{P}^3$ are not transverse, but their intersection (a reduced point) is still a smooth variety.

To show the scheme (5) is smooth, it is equivalent to show that the intersection is *clean* (cf. [Li09, Sec. 5.1]), i.e., their set-theoretical intersection is smooth. This amounts to showing that the intersection of the tangent spaces

$$
T_{\mathrm{Bl}_{\Delta_F}(F\times F),p} \cap T_{\tilde{D},p} \tag{6}
$$

has the expected dimension at all points.

We found that this is true for a general cubic hypersurface X. However, for certain special X, the intersection (6) may have a larger dimension at certain points above the diagonal.

Supposedly, the scheme structure at these poins may be non-reduced, but we shall show this will not happen.

The core of this paper is to solve Problem 1.7 by showing

Proposition 1.8. *Let X be a smooth cubic hypersurface with* $\dim(X) \geqslant 3$. *Then, the scheme-theoretical intersection* (5) *is reduced and irreducible. Moreover, the intersection* (5) *is smooth if and only if X has no higher triple lines.*

Our method is rather direct — we compute the rank of the Jacobian matrix explicitly. The computation is based on the analysis of the second-order data of the cubic equation near a given line. The difficulty lies in the locus over the lines of the second type on the diagonal. Computation on the Jacobian matrix excludes the possibility of additional component on the exceptional locus and shows irreducibility. For reducedness, we use Serre's criterion and show the depth of a local ring of an affine chart of (5) is at least one. We reduce the proof to bound the depth of A/I, where A is a regular local ring and I is generated by two elements. The geometry of the higher triple lines on X provides an upper bound of dimensions of singular locus, which controls the height of the prime ideal to localize and contributes to the proof of reducedness.

As a consequence, when $\dim(X) = 3$, $\tilde{D}_F$ has codimension one, and thus a Cartier divisor. Blowing up a Cartier divisor gives the isomorphism $\widetilde{H(X)} \cong \mathrm{Bl}_{\Delta_F}(F \times F)$, hence we recover the smoothness of $H(X)$ (cf. Theorem 1.3), although $\tilde{D}_F$ is singular when the cubic threefold has a triple line. In $\dim(X) \geqslant 4$, $\tilde{D}_F$ has higher codimensions, so it contributes to the singularities of $H(X)$ if it is singular. Therefore, Proposition 1.8 implies Theorem 1.6. The normality of $H(X)$ follows from reducedness of intersection (5).

1.4. **Relations to other work.** In [BB23], the authors demonstrate that for a smooth cubic threefold, triple lines correspond exactly to the singularities of curves of the second type, F_2. We may ask

Question 1.9. When $\dim(X) \geqslant 4$, what is the relationship between singularities of F_2 and higher triple lines? (See also Proposition 7.6 for a comparison.)

Question 1.10. When $\dim(X) = 4$, does cubic fourfolds with a higher triple line form a divisor? Is this a Hassett divisor?

Question 1.11. Is a general cubic fourfold with a higher triple line rational?

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Structure of the Paper. In Section 2, we review results about lines on a cubic hypersurface and introduce the notion of higher triple lines. In Section 3, we review the classical theory on Grassmannian and a particular Schubert subvariety and desingularization. We also study its intersection with the Fano variety of lines for a cubic hypersurface. In Section 4, we study first-order data of the defining equations of (5) in an affine chart and describe the intersection (5) set-theoretically. In Section 5, we compute the rank of the Jacobian matrix of the intersection (5), prove the irreducibility of (5), and derive conditions when the rank is lower than expected. In Section 6, we prove the reducedness of (5). In Section 7, we prove the main theorems and derive a few consequences, including the smoothness of $H(X)$ for cubic threefolds (cf. Theorem 1.3), the normality of $H(X)$ in all dimensions (cf. Theorem 1.6), and characterize the singularities of the intersection (5) and the incidence variety. In Section 8, we prove, using correspondence, that a general cubic hypersurface does not have a higher triple line, leading to the proof of Theorem 1.4. In Section 9, we characterize the singularities of $H(X)$ in terms of the subscheme of type (IV). In Section 10, we discuss the application to the Voisin map.

Notations.

- X, cubic hypersurface in $\mathbb{P}^n$
- F, Fano variety of lines of X
- $H(X)$ and $\widetilde{H(X)}$, Hilbert scheme of a pair of skew lines of X and its branched double cover
- $Gr(2, n+1)$, Grassmannian of lines in $\mathbb{P}^n$
- D and $\tilde{D}$, incidental subvariety of $Gr(2, n+1)^2$ and its strict transform
- D_F, incidence subvariety of $F \times F$
- Δ_F, diagonal of $F \times F$
- $\tilde{D}_F$, the scheme theoretical intersection $\mathrm{Bl}_{\Delta_F}(F \times F) \cap \tilde{D}$ in $\mathrm{Bl}_{\Delta} Gr(2, n+1)^2$
- V_n, the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^{n-2}$

2. Lines on Cubic Hypersurfaces

In this section, we will first review the basic results on Fano varieties of lines. One can refer to [AK77] and [Huy23] for the details. Then we introduce the notion of higher triple lines and discuss a few properties.

Let $X \subseteq \mathbb{P}^n$ be a smooth cubic hypersurface with $n \geqslant 4$. Then the *Fano variety of lines*

$$F = \{L \in Gr(2, n+1) | L \subseteq X\}$$

of X is smooth and has dimension $2n-6$. When $\dim(X) = 3$, F is a surface of general type, when $\dim(X) = 4$, F is a hyperkähler 4-fold, when $\dim(X) \geqslant 5$, F is Fano, i.e., anticanonical bundle is ample.

2.1. Lines of the first and second type.

Definition 2.1. (1) A line $L \subseteq X$ is of the first type if $N_{L|X} \cong \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(1)$.
(2) A line $L \subseteq X$ is of the second type if $N_{L|X} \cong \mathcal{O}(-1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(1)$.

A general line $L \in F$ is of the first type. Equivalently, the dual map along a line of the first and the second type have different degrees, which can tell them apart.

Lemma 2.2. [CG72, Lem. 6.7] [Huy23, Cha. 2, Cor. 2.6] *Let $L \subseteq X$ be a line on a cubic hypersurface. Then (1) L is of the first type iff is a unique $(n-3)$-plane P^{n-3} tangent to X along L. (2) L is of the second type iff is a unique $(n-2)$-plane P^{n-2} tangent to X along L.*

One may interpret Lemma 2.2 as saying X is "flatter" around a line of the second type.

Let L be a line of the second type in X with equation $x_2 = \cdots = x_n = 0$. Then, by change of coordinates, we can express the equation of cubic threefold X as [CG72, 6.10]

$$(7) \qquad F(x_0,\dots,x_n) = x_2x_0^2 + x_3x_1^2 + \sum_{2\leqslant i,j\leqslant n} x_ix_jL_{ij}(x_0,x_1) + C(x_2,\cdots,x_n),$$

where $L_{ij}(x_0,x_1) = L_{ji}(x_0,x_1)$ is a linear homogeneous polynomial, and $C(x_2,\dots,x_n)$ is a homogeneous cubic. Then P^{n-2} in Lemma 2.2 is given by $x_2 = x_3 = 0$.

Lemma 2.3. [CG72, Cor. 7.6] [Huy23, Cha. 2, Ex. 2.14] *The space of lines of the second type $F_2 \subseteq F$ has dimension $\frac{1}{2}\dim(F) = n-3$.*

F_2 is smooth for general X [Huy23, Prop. 2.13], but it may be singular for special X.

Definition 2.4. *A line L of X is called a triple line if there is a plane P^2 containing L, such that either $X \cap P^2 = 3L$, or $P^2 \subseteq X$* [1]*.*

For cubic threefolds, F_2 is a curve in F, and triple lines are precisely the singularities of F_2 [BB23]. In particular, a triple line is of the second type, and there are at most finitely many triple lines [CG72, Lem. 10.15]. In addition, a general cubic threefold does not have a triple line.

However, the properties of the triple line are not parallel in higher dimensions: for cubic fourfold, a triple line could be of the first type and varies in a two-parameter family, even for the general ones [GK24]. We instead study a variant notion whose properties are shared for cubic hypersurfaces in all dimensions. It naturally appears when finding the singularities of the Hilbert scheme of a pair of skew lines $H(X)$, which will be explored in Section 5.

2.2. Higher triple lines.

Definition 2.5. We call a line L on a smooth cubic hypersurface X a *higher triple line* if

- L is of the second type, and
- in terms of the coordinates (7), the columns of the matrix of linear forms

$$(8) \qquad S := \begin{bmatrix} L_{44}(x_0,x_1) & L_{45}(x_0,x_1) & \cdots & L_{4n}(x_0,x_1) \\ L_{54}(x_0,x_1) & L_{55}(x_0,x_1) & \cdots & L_{5n}(x_0,x_1) \\ \vdots & \vdots & \cdots & \vdots \\ L_{n4}(x_0,x_1) & L_{n5}(x_0,x_1) & \cdots & L_{nn}(x_0,x_1) \end{bmatrix}$$

  are linearly dependent over $\mathbb{C}$.

We call the matrix of linear forms S *degenerates* if the above condition is satisfied.

To understand the difference between two notions, suppose L is a line of the second type. Then L being a triple line is equivalent to say the symmetric bilinear form associated with the matrix (8) having a null vector, i.e., there is a non-zero vector v such that $v^TSv = 0$. Note that the vector v corresponds to a point p_v in $\mathbb{P}^n$ with $x_0 = \cdots = x_3 = 0$, and the span of L and p_v is a plane P^2 such that $P^2 \cap X = 3L$ if it is not contained in X. On the other hand, L being a higher triple line implies that $Sv = 0$ and hence $v^TSv = 0$ for some non-zero v, so we observe that

Proposition 2.6. *A higher triple line is a triple line.*

[1] Here, we allow P^2 to be contained in X for convenience. This agrees with the existing definition in [CG72] and [GK24] because neither a smooth cubic threefold nor a general cubic fourfold contains a plane.

For cubic threefolds, the matrix (8) is 1-by-1, and being a higher triple line is exactly the same as a triple line, and equivalent to vanishing of $L_{44}(x_0, x_1)$. Starting from cubic fourfolds, being a higher triple line is a more restrictive condition: the existence of a nonzero $v = [v_1, v_2]^T$ such that $v^T S v = 0$ does not necessarily imply that the matrix S degenerates: For example, the bilinear form associated with 2-by-2 matrix

$$\begin{bmatrix} 0 & L(x_0, x_1) \\ L(x_0, x_1) & 0 \end{bmatrix}$$

has two null vectors $[1, 0]^T$ and $[0, 1]^T$, but the matrix itself can be nondegenerate.

To give a geometrical interpretation, if a line L in X is of the second type, then the intersection $X \cap P^{n-2}$ is singular along the line and has A_1 singularity in the tranversal direction at a general point on L. If L is further a higher triple line, then $X \cap P^{n-2}$ has A_2 singularity in the transversal direction along the entire line L.

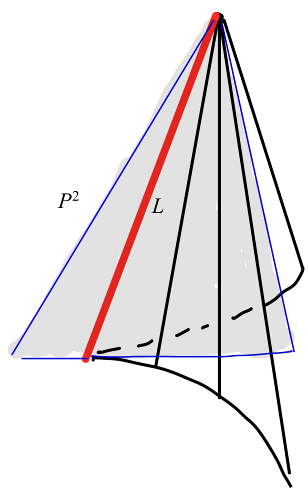

FIGURE 2. $\mathbb{P}^3$-section of cubic fourfold along a higher triple line

For example, a cubic fourfold X with a higher triple line L implies there is a linear $P^3 \cong \mathbb{P}^3$ section of X being a cone over a cuspidal curve.[2] Here, the plane P^2 realizing $P^2 \cap X = 3L$ is the cone of the "horizontal line" meeting the cuspidal curve at 3 times of the cusp point. In fact, this plane P^2 corresponds to a unique vector v up to scaling such that $Sv = 0$. In Section 9, we show how the plane P^2 and P^3 play a role in describing the singularities of $H(X)$.

We have the following dimension count:

Proposition 2.7. *(cf. Proposition 8.4) Having a higher triple line is a condition of codimension at least one in the space of all smooth cubic hypersurfaces. In particular, a general cubic hypersurface has no higher triple line.*

The idea is that, for example, when $\dim(X) = 4$, the spaces of all cones over cuspidal curves form a codimension 8 subspace of the spaces of cubic surfaces $\mathbb{P}(Sym^3\mathbb{C}^4)$, while the dimension of the space $Gr(4, 6)$ of $\mathbb{P}^3$ in $\mathbb{P}^5$ is 8 and is smaller than 9. Hence, by incidental correspondence, a general cubic fourfold does not admit a higher triple line. A complete proof will be given in Section 8.

[2]When $\dim(X) \geqslant 5$, $X \cap P^{n-2}$ is not necessarily a cone.

For special smooth cubic hypersurfaces, we give an upper bound of the dimension of higher triple lines. This generalizes the result on triple lines on cubic threefolds [CG72, Lem. 10.15].

Proposition 2.8. *Let X be a smooth cubic hypersurface of $\mathbb{P}^n$ with $n \geqslant 4$. Then, a general line of the second type (on each irreducible component of F_2) is not a higher triple line.*

Proof. We adapt the proof of [CG72, Lem. 10.15] to higher dimensions. Let C be an irreducible component of F_2, then as a consequence of [CG72, Lem. 7.5] $W = \bigcup_{t \in C} L_t$ is a subvariety of dimension $n-2$ in X. A general point p_0 of W is smooth. Since the dual map $\mathcal{D} : X \to X^\vee$ is finite, its restriction $\mathcal{D}|_W$ to W must have maximal rank at p_0.

Choose $t_0 \in F$ such that $p_0 \in L_{t_0}$. The tangent direction $T_{t_0}C$ corresponds to a tangent direction $v \in T_{p_0}W$ normal to L_0. Hence, if we choose normal coordinates with respect to the line L_{t_0} so that X has equation (7), the codimension-two linear subspace $x_2 = x_3 = 0$ is exactly the tangent space $T_{p_0}W$.

By assumption that L_{t_0} is a higher triple line, then the $(n+1) \times (n-1)$ submatrix of the Hessian matrix

$$\frac{\partial^2 F}{\partial x_i \partial x_0}, \frac{\partial^2 F}{\partial x_i \partial x_1}, \frac{\partial^2 F}{\partial x_i \partial x_4}, \cdots, \frac{\partial^2 F}{\partial x_i \partial x_n}, i = 0, 1, \ldots n. \tag{9}$$

is degenerate at p_0. This shows that the restriction of the dual map $\mathcal{D}|_W$ to W is not of maximal rank at the smooth point p_0. This is a contradiction. □

Corollary 2.9. *The dimension of the locus of the higher triple lines is at most*

$$\dim(F_2) - 1 = n - 4.$$

We will show that this bound is sharp in the case below.

2.3. **Eckardt points.**

Definition 2.10. A point $p \in X$ is called an *Eckardt point* if the tangent hyperplane at p intersects X at p with multiplicity 3.

Equivalently, p is an Eckardt point if $T_pX \cap X$ is a cone over a smooth cubic hypersurface Y with $\dim(Y) = \dim(X) - 2$ and the cone point being the Eckardt point p.

There are at most finitely many Eckardt points on a smooth cubic hypersurface [Gam18, Cor. 6.3.4]. As an example, the Fermat cubic $(n-1)$-fold contains $\frac{3n(n+1)}{2}$ Eckardt points and realizes the upper bound [CCS97, Thm. 3.12]. A general cubic hypersurface has no Eckardt point.

For cubic threefold, each Eckardt point is associated with nine triple lines - the tangent hyperplane section is a cone of a cubic curve E. The nine flex points of the cubic curve E correspond to the nine triple lines on X. This can be generalized to higher dimensions, except that higher triple lines can vary in a continuous family. We denote $Hess(Y)$ the Hessian hypersurface associated with hypersurface Y defined by $G = 0$. It is defined by $\det(\frac{\partial^2 G}{\partial x_i \partial x_j}) = 0$ [BFP24].

Proposition 2.11. *Let p be an Eckardt point on X, then the higher triple lines on X passing through p is parameterized by the intersection $Y \cap Hess(Y)$ of Y and its associated Hessian hypersurface. In particular, there is a $(n-4)$-dimensional family of higher triple lines on X through p.*

Proof. Let p be an Eckardt point, then $T_pX \cap X$ is a cone of a smooth cubic hypersurface Y with the cone point p, and satisfies $\dim(Y) = \dim(X) - 2$. Then, every line in the ruling is a line of the second type in X. Moreover, if point $q \in Y$ such that the tangent hyperplane section $T_qY \cap Y$ has an A_2 singularity at q, then the line l_{pq} spanned by q and the cone point p is a higher triple line on X. Such locus on Y is parameterized by the intersection $Y \cap Hess(Y)$ with the Hessian hypersurface. □

3. Incidental Subvariety

3.1. **Grassmannian.** In this section, we review the classical theory of Grassmannian and the singularities of the incidence subvariety D of product of Grassmannians. We will show that $\tilde{D}$ is smooth under strict transform of the blow-up of the diagonal. We will find explicit equations of $\tilde{D}$ in an affine coordinate of blow-up of Grassmannians.

Many of the results are classically known and can be found in [BL00] and [Har95]. Here, we provide a self-contained exposition.

3.1.1. *Lines in $\mathbb{P}^n$.* Let's first consider the Grassmannian $Gr(2, n+1)$ of lines in $\mathbb{P}^n$. Parameterizing lines in $\mathbb{P}^n$ is the same as parameterizing two planes in $\mathbb{C}^{n+1}$ through the origin. So we need two vectors $\vec{v_1} = [a_1, a_2, \ldots, a_{n+1}]^T$ and $\vec{v_2} = [b_1, b_2, \ldots, b_{n+1}]^T$ that are linearly independent. So, it defines a 2-by-$n+1$ matrix of rank 2

$$\begin{bmatrix} a_1 & a_2 & \ldots & a_{n+1} \\ b_1 & b_2 & \ldots & b_{n+1} \end{bmatrix}. \tag{10}$$

Conversely, (1) any 2-by-$n+1$ matrix determines a 2-dimensional subspace of $\mathbb{C}^4$ if it has rank 2; (2) any two of such matrices determine the same subspace if they differ by a $GL_2(\mathbb{C})$ action on the left.

So, to get the one-to-one correspondence between the 2-dimensional subspace of $\mathbb{C}^{n+1}$ and the equivalence classes of 2-by-$n+1$ matrices, we use the Plucker coordinates — consider all of the 2-by-2 minors $x_{ij} = \det \begin{bmatrix} a_i & a_j \\ b_i & b_j \end{bmatrix}$, where $1 \leqslant i < j \leqslant n+1$. Then any two-dimensional subspace corresponds to a point (x_{ij}) with Plucker coordinates in $\mathbb{P}^{(n+1)n/2-1}$. Place the matrix (10) on top of itself produces a degenerate matrix, and the determinant of its 4-by-4 minor produces the Plucker equations

$$x_{ij}x_{st} - x_{is}x_{jt} + x_{it}x_{js} = 0, \ 1 \leqslant i < j < s < t \leqslant n+1. \tag{11}$$

The zero locus defines the Grassmannian $Gr(2, n+1)$. Choosing the affine chart $x_{12} = 1$, we found that the variables x_{st} with $3 \leqslant s < t \leqslant n+1$ depend on other factors, and the equations in (11) without the term x_{12} are redundant. Hence the open subspace $Gr(2, n+1)\backslash\{x_{12} = 0\}$ is isomorphic to $\mathbb{C}^{2(n-1)}$ and has coordinates

$$x_{13}, x_{14}, \ldots, x_{1,n+1}, x_{23}, x_{24}, \ldots, x_{2,n+1}. \tag{12}$$

3.1.2. *Incidence variety.* Let D_L be the Schubert subvariety of $Gr(2, n+1)$ that parameterizes the lines in $\mathbb{P}^{n+1}$ intersecting a given line L. It is normal and irreducible.

We consider a relative version of this cell.

Definition 3.1. Let

$$D \subseteq Gr(2, n+1) \times Gr(2, n+1)$$

be the subvariety parameterizing pairs of incidence lines $D = \{(L_1, L_2) | L_1 \cap L_2 \neq \varnothing\}$.

Then, if we let x_{ij}, y_{ij} be the Plucker coordinates on the first and the second Grassmannian, then D is given by equations in $Gr(2,n+1) \times Gr(2,n+1)$

$$x_{ij}y_{st} - x_{is}y_{jt} + x_{it}y_{js} + x_{st}y_{ij} - x_{jt}y_{is} + x_{js}y_{it} = 0, \ 1 \leqslant i < j < s < t \leqslant n+1. \tag{13}$$

It arises similarly to (11) from the Laplace expansion of a 4-by-4 determinant.

3.1.3. *Singularities.* First of all, D is smooth off the diagonal. To see this, D_L has codimension $n-2$ and is only singular at the point L, and the projection $p : D \to Gr(2,n+1)$ is a fiber bundle with $p^{-1}(L) \cong D_L$.

As an example, we take L as the line with $x_{12} = 1$ and $x_{ij} = 0$ for the other i, j. Then from (13), D_L is given by

$$y_{1s}y_{2t} = y_{1t}y_{2s}, \ s, t \geqslant 3. \tag{14}$$

It has an isolated singularity at the origin. (When $n = 3$, it is an ordinary node.) We note that (14) are homogeneous, and the same equations in the projective space define the projective tangent cone at the origin, which we denote by V_n. In fact, V_n is the Segree embedding

$$\mathbb{P}^1 \times \mathbb{P}^{n-2} \hookrightarrow \mathbb{P}^{2n-3}.$$

For example, V_2 is $\mathbb{P}^1$, V_3 is the quadric surface. In general, V_n is smooth and $\dim(V_n) = \deg(V_n) = n-1$. Moreover, we can regard D_L as the affine cone of V_n, and we have

Lemma 3.2. *The blow-up* $\mathrm{Bl}_L(D_L)$ *of* D_L *at the singularity* L *is smooth, with exceptional divisor* V_n.

Now, we need a parametric version of this result. To find the equation of D near the diagonal, we choose the affine chart $x_{12} = y_{12} = 1$ on both factors. By introducing the diagonal coordinates $u_{ij} = y_{ij} - x_{ij}$ and using (11) and (13), we find that D in the affine chart is given by

$$u_{1s}u_{2t} = u_{1t}u_{2s}, \ s, t \geqslant 3. \tag{15}$$

It is singular along the diagonal Δ given by $u_{ij} = 0$. Consider the blow-up of the diagonal

$$\mathrm{Bl}_\Delta\big(Gr(2,n+1) \times Gr(2,n+1)\big) \to (Gr(2,n+1) \times Gr(2,n+1).$$

Then, in the same way, we have

Proposition 3.3. *The strict transform* $\tilde{D}$ *of* D *is smooth, and has codimension* $n-2$. *The exceptional divisor is a* V_n*-bundle over the diagonal.*

Consequently, Problem 1.7 is to study the intersection of two smooth subvarieties.

Note that we can even include the case when $n = 2$, $V_2 = \mathbb{P}^1$, as the exceptional divisor of the blowup of $Gr(2,3) \cong (\mathbb{P}^2)^*$ at a point. So in small dimensions,

- $n = 2$, $V_2 \cong \mathbb{P}^1$ is a projective line.
- $n = 3$, V_3 is the smooth quadric surface.
- $n = 4$, V_4 is the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$.

3.2. **Fano variety of lines.** In this section, we study the singularities of the incidental subvariety D_F of $F \times F$. We characterize its singularities and prove that the Hilbert scheme of a pair of skew lines $H(X)$ is smooth away from the diagonal (cf. Corollary 3.9).

Definition 3.4. Let D_F be the closure in $F \times F$ of the subspace parameterizing pair of incident lines

$$I := \{(L_1, L_2) \in F \times F | L_1 \cap L_2 \text{ is a point}\}.$$

Proposition 3.5. *D_F is irreducible and smooth away from the diagonal.*

When $\dim(X) = 3, 4$, it is proved by [CG72, Lem. 12.18] and [Gio21, Thm. 4.3.1.2]. Here, we generalize Giovenzana's argument to higher dimensions.

Proof. We define the $\tilde{I}$ to be the set of triples $\{(L_1, L_2, x) \in F \times F \times X | L_1 \cap L_2 = \{x\}\}$. Then, the forgetful map

$$\pi : \tilde{I} \to I$$

is an isomorphism.

$\tilde{I}$ can be regarded as a nested Hilbert scheme, and its tangent space is isomorphic to the fiber products

$$T_{(L_1,L_2,x)}\tilde{I} \cong H^0(N_{L_1|X}) \times_{N_{L_1|X}(x)} T_xX \times_{N_{L_2|X}(x)} H^0(N_{L_2|X}).$$

In other words, there is a diagram below, where ϕ_i is the restriction to the point x. ψ_i is the natural projection.

$$\begin{array}{ccccccc} H^0(N_{L_1|X}) & & & T_xX & & & H^0(N_{L_2|X}) \\ & \overset{\phi_1}{\searrow} & \overset{\psi_1}{\swarrow} & & \overset{\psi_2}{\searrow} & \overset{\phi_2}{\swarrow} & \\ & & N_{L_1|X}(x) & & N_{L_2|X}(x). & & \end{array} \tag{16}$$

Therefore, tangent vectors to $\tilde{I}$ at a pair of incidental lines (L_1, L_2) intersecting at a point x corresponds to two normal vectors $v_i \in H^0(N_{L_i|X})$, with $i = 1, 2$ and a tangent direction $u \in T_xX$ such that $\phi_1(v_i) = \psi_i(u)$ for both $i = 1, 2$.

Now we will show $T_{(L_1,L_2,x)}\tilde{I}$ has the expected rank at all points. Note $\dim H^0(N_{L_i|X}) = 2(n-3)$ no matter the type of the line. $\dim T_xX = n-1$, and $\dim N_{L_i|X}(x) = n-2$.

We will discuss in the following three cases.

(1) Both L_1 and L_2 are lines of the first type. Hence both ϕ_1 and ϕ_2 are surjective and $\dim(\ker(\phi_i)) = n-4$. So the tangent space is parameterized by vectors $u \in T_xX$, and lifts of $\psi_i(u)$. Therefore, the tangent space $T_{(L_1,L_2,x)}\tilde{I}$ has dimension $n-1+2(n-4) = 3n-9$.

(2) One of the lines (say L_1) is of the first type, and the other (say L_2) is of the second type. In this case, ϕ_1 is surjective, and ϕ_2 has a one-dimensional cokernel, which lifts to a subspace V_x of T_xX which is the tangent space of the unique $(n-2)$-plane (cf. Lemma 2.2). Hence, the tangent space $T_{(L_1,L_2,x)}\tilde{I}$ is parameterized by vectors $u \in V_x$, and the lifts of $\psi_i(u)$ and has dimension $(n-2)+(n-4)+(n-3) = 3n-9$.

(3) Both L_1 and L_2 are lines of the second type. Hence both ϕ_1 and ϕ_2 have a one-dimensional cokernel, and their lifts to T_xX are tangent spaces of the $(n-2)$-plane E_i. Then $E_1 \neq E_2$. Otherwise, the plane spanned by L_1 and L_2 will be tangent to X along both L_1 and L_2, contradicting the fact X has degree three. Hence, the tangent space $T_{(L_1,L_2,x)}\tilde{I}$ is parameterized by vectors u in a $(n-3)$-dimensional subspace $E_1 \cap E_2$ of T_xX, and the lifts of $\psi_i(u)$, so it has dimension $(n-3)+2(n-3) = 3n-9$.

In summary, the tangent space $T_{(L_1,L_2,x)}\tilde{I}$ has constant dimension everywhere away from diagonal Δ_F, hence $\tilde{I}$ and I are smooth. So its closure D_F is irreducible. Since $D_F \backslash I \subseteq \Delta_F$, D_F is smooth away from the diagonal. □

The dimension of the tangent space tells us that

Corollary 3.6. $\dim(D_F) = 3n - 9$. *Hence D_F has codimension $n-3$ in $F \times F$, and codimension $n+5$ in $Gr(2, n+1) \times Gr(2, n+1)$.*

Remark 3.7. One can regard D_F as an irreducible component of $(F \times F) \cap D$ with reduced structure. Note that $\mathrm{Codim}_{F^2}(D_F) < \mathrm{Codim}_{Gr(2,n+1)^2}(D)$ (cf. Proposition 3.3, Remark 3.6), therefore the intersection is not transverse.

Remark 3.8. When $\dim(X) = 3$, the reduced structure of $(F \times F) \cap D$ is $D_F \cup \Delta_F$. The two components intersect along the locus of the lines of the second type on the diagonal.

When $\dim(X) \geqslant 4$, D_F is the reduced structure of $(F \times F) \cap D$. This is because the normal bundle $N_{L|X}$ at a line of the first type has at least one factor of $\mathcal{O}(1)$ and $D_F \supseteq \Delta_F$.

We will describe the singularities of D_F in Section 7.4.

3.3. **Type (III) schemes.** A type (III) subscheme of a cubic hypersurface $X \subseteq \mathbb{P}^n$ is a closed subscheme Z_{III} of X with Hilbert polynomial $2t + 2$ and is a union

$$Z_{III} = L_1 \cup L_2 \cup Z_p$$

consisting of a pair of incidental lines L_1 and L_2 with reduced structure, and Z_p is an embedded point supported at $\{p\} = L_1 \cap L_2$ and is contained in a linear 3-plane contained in T_pX. When $\dim(X) = 3$, one refers to [Zha23, Lem. 4.6].

Since $D_F \backslash \Delta_F$ parameterizes pairs of incidental lines (with an order), the morphism

$$\widetilde{H(X)} \times_{F \times F} (D_F \backslash \Delta_F) \to D_F \backslash \Delta_F,\ Z_{III} \mapsto L_1 \cup L_2$$

as restriction of Hilbert-Chow morphism (**??**) forgets the embedded point Z_p. The fiber is the projective $(n-4)$-space consisting of the set of all linear 3-plane P^3 such that

$$\mathrm{Span}(L_1, L_2) \subseteq P^3 \subseteq T_pX.$$

Corollary 3.9. *The Hilbert scheme of a pair of skew lines $H(X)$ is smooth away from the diagonal, that is, smooth along the locus parameterizing schemes of types (I) and (III).*

Proof. Proposition 3.5 implies that the blow-up center of $\widetilde{H(X)} \to \mathrm{Bl}_{\Delta_F}(F \times F)$ is smooth away from the diagonal. Hence, $\widetilde{H(X)}$ is smooth away from the diagonal. Since the $\mathbb{Z}_2$ action is free away from the diagonal, its image in the quotient $H(X)$ is also smooth. □

4. Set-Theoretical Description of $\mathrm{Bl}_{\Delta_F}(F \times F) \cap \tilde{D}$

Starting from this section, our goal is to study the singularities of $H(X)$ supported on the diagonal of $\mathrm{Sym}^2 F$, which parameterize type (II) and (IV) schemes. As explained in the Introduction, the Hilbert scheme of a pair of skew lines, $H(X)$, up to a double cover, arises from two successive blow-ups, and the second one blows up $\mathrm{Bl}_{\Delta_F}(F \times F)$ on the closed subscheme $\mathrm{Bl}_{\Delta_F}(F \times F) \cap \tilde{D}$. Hence to describe $H(X)$, the main question is

Question 4.1. *Is $\mathrm{Bl}_{\Delta_F}(F \times F) \cap \tilde{D}$ reduced, irreducible? Where are its singularities locate?*

We aim to answer these questions in the following sections.

In this section, we use appropriate coordinates to find the defining equations of the scheme-theoretical intersection, as a preparation for the computation to be carried out in the next section. We use the first-order data to give a set-theoretical description of the exceptional locus.

4.1. **First order data.** Recall that F is the Fano variety of lines on cubic hypersurface X, with $\dim(X) = n-1 \geqslant 3$. Then $F \subseteq Gr(2, n+1)$ is a closed subvariety, and local equations can be written with respect to the standard form of the cubic equation (7) at a given line L. In the local coordinates (12), suppose the line L is of the first type at the origin $x_{ij} = 0$, then F has equations [CG72, (6.14)]

$$\begin{cases} x_{13} + \cdots = 0. \\ x_{14} + x_{23} + \cdots = 0. \\ x_{15} + x_{24} + \cdots = 0. \\ x_{25} + \cdots = 0. \end{cases} \tag{17}$$

If the line L is of the second type at $x_{ij} = 0$, then F has equations [CG72, (6.15)]

$$\begin{cases} x_{13} + \cdots = 0. \\ x_{14} + \cdots = 0. \\ x_{23} + \cdots = 0. \\ x_{24} + \cdots = 0. \end{cases} \tag{18}$$

Here, $\cdots$ denotes a polynomial that involves terms with orders at least 2. In this section, we only need the first-order information. One should also note that the first-order data are equivalent to the information from the normal bundle $N_{L|X}$ (cf. Definition 2.1). We prefer to work with explicit local equations because the computation to be carried out in the next section is based on them.

4.1.1. *Diagonal coordinates.* Now, let x_{ij} and y_{ij} be the coordinates on the first and second factors of Grassmannian. In the affine chart $x_{12} = y_{12} = 1$, we introduce diagonal coordinates $u_{ij} = y_{ij} - x_{ij}$ as in Section 3.1. Then $u_{ij} = 0$ defines the diagonal. Now, in the new coordinates $(\underline{u_{ij}}, \underline{x_{ij}}) = (u_{13}, \ldots, u_{2,n+1}, x_{13}, \ldots, x_{2,n+1})$, a line of the first type L on the diagonal of $F \times F \subseteq Gr(2, n+1) \times Gr(n+1)$ has equations (17) together with

$$\begin{cases} u_{13} + \cdots = 0, \\ u_{14} + u_{23} + \cdots = 0, \\ u_{15} + u_{24} + \cdots = 0, \\ u_{25} + \cdots = 0. \end{cases} \tag{19}$$

4.2. **Set-theoretical fiber.** Note that to find the set-theoretical fiber of the intersection $\mathrm{Bl}_\Delta Gr(2, n+1)^2 \cap \tilde{D}$ over the line $(L, L) \in \Delta_F$, we just need to pull back the equations to the blow-up $\mathrm{Bl}_\Delta Gr(2, n+1)^2$, intersect with local equations of $\tilde{D}$, and set $x_{ij} = 0$. The higher-order terms do not contribute.

Therefore, the fiber over a line of the first type is given by linear parts of (19) together with (15) in the blowup coordinate $u_{ij}\lambda_{st} = u_{st}\lambda_{ij}$ (cf. (25)). Direct computation shows it cuts the fiber V_n of $\tilde{D}$ by

$$\lambda_{1i} = \lambda_{2i} = 0, \ i = 3, 4, 5,$$

which is isomorphic to V_{n-3}.

Similarly, when we consider diagonal coordinates $F \times F$ near a line of the second type, the equation is given by (18) together with

$$\begin{cases} u_{13} + \cdots = 0, \\ u_{14} + \cdots = 0, \\ u_{23} + \cdots = 0, \\ u_{24} + \cdots = 0. \end{cases} \tag{20}$$

Hence, the fiber of the intersection over a line of second type $L \in \Delta_F$ cuts V_n by

$$\lambda_{13} = \lambda_{14} = \lambda_{23} = \lambda_{24} = 0,$$

which is isomorphic to V_{n-2}.

Notation 4.2. Let $(\tilde{D}_F) := \mathrm{Bl}_{\Delta_F}(F \times F) \cap \tilde{D}$ denote the intersection. Let

$$\Delta(F_2) := \{(L, L) \in D_F \subseteq F \times F \mid L \in F_2\}$$

be the diagonal embedding of the locus of lines of the second type.

To summarize, we proved

Proposition 4.3. *The restriction of $(\tilde{D}_F)_{red}$ to the exceptional locus over the diagonal $(\tilde{D}_F)_{red}|_{\Delta_F}$ is a V_{n-3}-bundle over the lines of the first type $\Delta_F \backslash \Delta(F_2)$ and a V_{n-2}-bundle over the lines of the second type $\Delta(F_2)$.*

Proof. This is based on the above analysis. The only exception is when $n = 4$ and X is a cubic threefold, where $(F \times F) \cap D$ is reducible. When we pullback the diagonal equations (19) to the blowup, there is another irreducible component $E \subseteq \mathrm{Bl}_{\Delta_F}(F \times F)$, which is the entire exceptional divisor. Remove that, we get the intersection $\mathrm{Bl}_{\Delta_F}(F \times F) \cap \tilde{D}$ whose restriction to the diagonal is only supported over lines of the second type. □

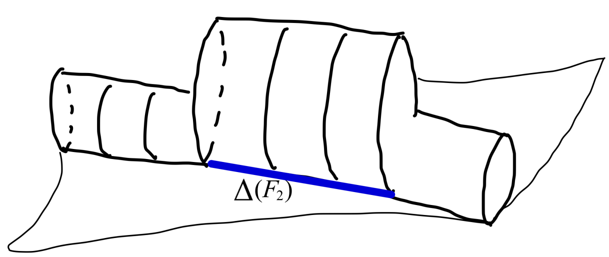

FIGURE 3. Set-theoretical Picture of $\mathrm{Bl}_{\Delta_F}(F \times F) \cap \tilde{D}$

Example 4.4. When X is a cubic threefold, $(\tilde{D}_F)_{red}|_{\Delta_F}$ is a $\mathbb{P}^1$-bundle over $\Delta(F_2)$.

Example 4.5. When X is a cubic fourfold, $(\tilde{D}_F)_{red}|_{\Delta_F}$ is a $\mathbb{P}^1$-bundle over the locus of the lines of the first type $\Delta_F \backslash \Delta(F_2)$, and a quadric surface bundle over the second type $\Delta(F_2)$.

4.3. **Type (II) and type (IV) schemes.** In this section, we give a modular interpretation of $\tilde{D}_F$ on the diagonal.

4.3.1. *Type (II) scheme.* A type (II) subscheme of a cubic hypersurface X is a closed subscheme Z_{II} of X that arises from a line L and a subline bundle

$$\mathcal{O}_L \subseteq N_{L|X}.$$

The scheme Z_{II} is the infinitesimal neighborhood of L in the quadric surface "spanned" by L and the normal direction $\mathcal{O}_L$. (When $\dim(X) = 3$, one refers to [Zha23, Prop. 4.9].)

We note that the second blow-up $\widetilde{H(X)} \to \mathrm{Bl}_{\Delta_F}(F \times F)$ is an isomorphism on the complement of $\tilde{D}_F$. Over the diagonal, this is precisely the Type (II) locus. Hence, we have

Lemma 4.6. *$H(X)$ is smooth on the locus that parameterizes type (II) schemes.*

4.3.2. *Type (IV) scheme.* A type (IV) subscheme of X is a closed subscheme Z_{IV} with Hilbert polynomial $2t + 2$ supported on a line L, and admits a primary decomposition

$$Z_{IV} = Z_{L,P^2} \cup Z_p, \tag{21}$$

where

- Z_{L,P^2} is the first-order infinitesimal neighborhood of L in a 2-plane P^2 which is tangent to X at all points on L;
- Z_p is an embedded point whose ideal is the square of I_{p,P^3}, the ideal of a reduced point $p \in L$ in a 3-plane P^3 tangent to X at p.

One can filtrate the data of type (IV) scheme (21) as follows: (1) the plane P^2 corresponds to a subline bundle

$$\mathcal{O}_L(1) \subseteq N_{L|X}, \tag{22}$$

and (2) an embedded point corresponds to a 3-plane such that $P^2 \subseteq P^3 \subset T_pX$.

This filtration corresponds to the factorization of the Hilbert-Chow morphism

$$\widetilde{H(X)} \xrightarrow{\sigma_2} \mathrm{Bl}_{\Delta_F}(F \times F) \xrightarrow{\sigma_1} F \times F,$$
$$Z_{IV} \mapsto (p, L, P^2) \mapsto 2L.$$

For a type (IV) scheme, the intermediate step σ_2 forgets the embedded point but remembers the normal direction $\mathcal{O}_L(1)$ (and the reduced point). This data is captured at the intersection $(\tilde{D}_F)|_{\Delta_F}$ on the diagonal.

To give a set-theoretical interpretation, Proposition 4.3 says that for each line L of the first type, there are

$$\mathbb{P}^1 \times \mathbb{P}^{n-5}$$

many "intermediate" type (IV) schemes - the $\mathbb{P}^1$ factor parameterizes $p \in L$, while the second factor parameterizes the embeddings (22). When L is a line of the second type, the second factor is switched to $\mathbb{P}^{n-4}$ due to the change of the normal bundle (see Definition 2.1).

Note that when $n = 4$ and X is a cubic threefold, the fiber is empty over a line a first type. This just means that there is no type (IV) scheme supported on a line of the first type, as observed in [Zha23, Lemma 4.8].

5. Tangent Space of $\mathrm{Bl}_{\Delta_F}(F \times F) \cap \tilde{D}$

In Section 4.3, we describe the locus $\mathrm{Bl}_{\Delta_F}(F \times F) \cap \tilde{D}$ and the exceptional fiber set theoretically based on the first-order data of a line in X.

In this section, we will study the scheme-theoretical information of $\mathrm{Bl}_{\Delta_F}(F \times F) \cap \tilde{D}$. We will show that it is irreducible and find the condition when it is smooth. This is based on the study of the second-order data of F around a line L. We particularly focus on the case where L is a line of the second type.

5.1. **Equations on the blow-up.** Here, we rewrite the diagonal equations (20) of $F \times F$ at a line of second type by including the second-order data.

$$(23)\qquad \begin{cases} u_{13} + h_{13}(u_{13}, \dots, u_{2,n+1}, x_{13}, \dots, x_{2,n+1}) = 0. \\ u_{14} + h_{14}(u_{13}, \dots, u_{2,n+1}, x_{13}, \dots, x_{2,n+1}) = 0. \\ u_{23} + h_{23}(u_{13}, \dots, u_{2,n+1}, x_{13}, \dots, x_{2,n+1}) = 0. \\ u_{24} + h_{24}(u_{13}, \dots, u_{2,n+1}, x_{13}, \dots, x_{2,n+1}) = 0. \end{cases}$$

where h_{ij} has degree starting from 2 and equals to the difference $\phi_{ij}(y_{st}) - \phi_{ij}(x_{st})$ of second order terms of (18). (Also see (36) for explicit description.) To express h_{ij} in terms of diagonal coordinates u_{st} and x_{st}, we need the following lemma.

Lemma 5.1. *Let $f(x_1, \dots, x_m)$ be a homogeneous polynomial of degree d. Let $u_i = y_i - x_i$, then*

$$(24)\qquad f(y_1, \dots, y_m) - f(x_1, \dots, x_m)$$

is a homogeneous polynomial of degree d in $u_1, \dots, u_m, x_1, \dots, x_m$.

For what follows, the only thing we need from the above is the degree-two terms.

Example 5.2. Direct computations show that

$$y_i^2 - x_i^2 = u_i(u_i + 2x_i),$$

and

$$\begin{aligned} y_i y_j - x_i x_j &= (y_i y_j - y_i x_j) + (y_i x_j - x_i x_j) \\ &= (u_i + x_i)u_j + u_i x_j \\ &= u_j x_i + u_i x_j + u_i u_j. \end{aligned}$$

Corollary 5.3. $h_{ij} = \sum u_{ab} p_{ab} + \sum u_{ab} u_{cd} q_{abcd}$*, where p_{ab} is a polynomial of degree at least one and only involves $x_{13}, \dots, x_{2,n+1}$.*

5.1.1. *Blowup coordinates.* We blowup the diagonal on $\mathbb{C}^{2n-2} \times \mathbb{C}^{2n-2}$ with coordinate (u_{ij}, x_{ij}), the equation of the blow-up is

$$(25)\quad (u_{13}, \dots, u_{2,n+1}, x_{13}, \dots, x_{1,n+1}; \lambda_{13}, \dots, \lambda_{2,n+1}) \in \mathbb{C}^{2n-2} \times \mathbb{C}^{2n-2} \times \mathbb{P}^{2n-3}, u_{ij}\lambda_{st} = u_{st}\lambda_{ij}.$$

Choose an affine chart $\lambda_{2,n+1} = 1$, and set $u = u_{2,n+1}$, then we have

$$(26)\qquad u_{ij} = \lambda_{ij} u.$$

Pullback the equation (23) to the blow-up and applying the substitution (26), we find by Corollary 5.3 that u can be factored out. Hence we set $\bar{h}_{ij} = h_{ij}/u$, and we obtain the defining equations of the strict transform $\mathrm{Bl}_{\Delta_F}(F \times F)$ of $F \times F$ over a line of the second type on the diagonal

$$(27)\qquad \begin{cases} \lambda_{13} + \bar{h}_{13}(u\lambda_{13}, \dots, u\lambda_{2,n}, u, x_{13}, \dots, x_{2,n+1}) = 0, \\ \lambda_{14} + \bar{h}_{14}(u\lambda_{13}, \dots, u\lambda_{2,n}, u, x_{13}, \dots, x_{2,n+1}) = 0, \\ \lambda_{23} + \bar{h}_{23}(u\lambda_{13}, \dots, u\lambda_{2,n}, u, x_{13}, \dots, x_{2,n+1}) = 0, \\ \lambda_{24} + \bar{h}_{24}(u\lambda_{13}, \dots, u\lambda_{2,n}, u, x_{13}, \dots, x_{2,n+1}) = 0. \end{cases}$$

To compute $\bar{h}_{ij}$, we need to use Example (5.2). If the second-order term of one of the equations in (18) includes $x_{ab}x_{cd}$, then the corresponding h_{ij} contains the term $u_{ab}x_{cd} + u_{cd}x_{ab} + u_{ab}u_{cd}$. Substituting using (26), $\bar{h}_{ij}$ includes the term

$$\lambda_{ab}x_{cd} + \lambda_{cd}x_{ab} + \lambda_{ab}\lambda_{cd}u. \tag{28}$$

These calculations will be used in Section 5.4 to determine the rank of the Jacobian matrix.

5.2. **Jacobian matrix.** To summarize the equations we found, we look at the defining equations of the intersection $\mathrm{Bl}_{\Delta_F}(F \times F) \cap \tilde{D}$, restricted to the preimage of the diagonal under the blow-up map. We have three sets of equations

$$\lambda_s + \bar{h}_s(u\lambda_{13}, \dots, u\lambda_{2,n}, u, x_{13}, \dots, x_{2,n+1}) = 0, \tag{29}$$

$$x_s + \phi_s(x_{13}, \dots, x_{2,n+1}) = 0. \tag{30}$$

for subscript $s = 13, 14, 23, 24$, defining $\mathrm{Bl}_{\Delta_F}(F \times F)$, together with the pullback of (15)

$$\lambda_{1i} - \lambda_{1,n+1}\lambda_{2i} = 0,\ i = 3, \dots, n \tag{31}$$

defining $\tilde{D}$.

Take the partial derivatives of these equations and evaluate at $u = x_{13} = \cdots = x_{2,n+1} = 0$ on the diagonal at the given line L of the second type. The parameters λ_{ij} are allowed to take any value on the fiber, which is described in Proposition 4.3. We obtain the Jacobian matrix (32).

(32)

$$\begin{array}{ccccccc|cccc|ccccc}
\lambda_{13} & \lambda_{14} & \lambda_{23} & \lambda_{24} & \lambda_{15} & \cdots & \lambda_{1n} & x_{13} & x_{14} & x_{23} & x_{24} & \lambda_{1,n+1} & u & x_{15} & x_{25} & \cdots \\
1 & 0 & 0 & 0 & 0 & \cdots & 0 & * & * & * & * & 0 & \frac{\partial \bar{h}_{13}}{\partial u} & \frac{\partial \bar{h}_{13}}{\partial x_{15}} & \frac{\partial \bar{h}_{13}}{\partial x_{25}} & \cdots \\
0 & 1 & 0 & 0 & 0 & \cdots & 0 & * & * & * & * & 0 & \frac{\partial \bar{h}_{14}}{\partial u} & \frac{\partial \bar{h}_{14}}{\partial x_{15}} & \frac{\partial \bar{h}_{14}}{\partial x_{25}} & \cdots \\
0 & 0 & 1 & 0 & 0 & \cdots & 0 & * & * & * & * & 0 & \frac{\partial \bar{h}_{23}}{\partial u} & \frac{\partial \bar{h}_{23}}{\partial x_{15}} & \frac{\partial \bar{h}_{23}}{\partial x_{25}} & \cdots \\
0 & 0 & 0 & 1 & 0 & \cdots & 0 & * & * & * & * & 0 & \frac{\partial \bar{h}_{24}}{\partial u} & \frac{\partial \bar{h}_{24}}{\partial x_{15}} & \frac{\partial \bar{h}_{24}}{\partial x_{25}} & \cdots \\
\hline
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
\hline
1 & 0 & -\lambda_{1,n+1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & -\lambda_{23} & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & -\lambda_{1,n+1} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & -\lambda_{24} & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & -\lambda_{25} & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & & \ddots & & 0 & 0 & 0 & 0 & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & -\lambda_{2n} & 0 & 0 & 0 & \cdots
\end{array}$$

It is a matrix with $n+6$ rows and $4n-4$ columns. Note that here we discarded the $\lambda_{25}, \dots, \lambda_{2n}$ columns since they do not affect the rank computation. Also, when $n = 4$, the columns 5 through n-th should be deleted.

The rows of the Jacobian matrix span the cotangent space at a point p on the underlying reduced scheme of the intersection $\mathrm{Bl}_{\Delta_F}(F \times F) \cap \tilde{D}$.

Proposition 5.4. *The Jacobian matrix* (32) *has rank at least* $n+4$ *and at most* $n+5$.

Proof. The first eight rows and the last $n-4$ rows are clearly linearly independent. Their span has dimension $n+4$ and is contained in the cotangent space. The rank of the matrix is at most the codimension of the intersection $\tilde{D}_F$ in $\mathrm{Bl}_\Delta(Gr(2,n+1)^2)$, which is at most $n+5$ (cf. Corollary 3.6, Proposition 4.3). □

5.2.1. *Non-transversal intersection.* From another point of view, the equations (29), (30) and the last $n-4$ equations in (31) has linearly independent tangent vectors, and therefore they intersect transversely at $u = x_{ij} = 0$. As a result, the variety they define is smooth and has codimension equal to the number of equations.

However, upon adding the two additional equations

$$\lambda_{13} - \lambda_{1,n+1}\lambda_{23} = 0 \text{ and } \lambda_{14} - \lambda_{1,n+1}\lambda_{24} = 0,$$

the intersection is no longer transverse — the dimension of variety decreases only by 1 (this already occurs before the blow-up, as observed in Remark 3.7). Therefore the Jacobian matrix has rank at most $n+5$ and cannot attain full row rank.

Meanwhile, the dimension of the tangent space decreases by either 1 or 0. In the former case, the variety defined by (29), (30), and (31) remains smooth locally, and the scheme structure is reduced (cf. [Li09, Lem. 5.1]). In the latter case, the underlying variety is singular, and the scheme may have a non-reduced structure. Therefore, we need to analyze when the second case occurs. This depends on the second-order data.

Before proceeding with the row reduction computation, let us mention a similar setup to compute Jacobian matrix over a line of the first type, which turns out to be completely determined by the first-order data.

5.2.2. *Line of the 1st type.* We can also analyze the Jacobian matrix for $\mathrm{Bl}_{\Delta_F}(F \times F) \cap \tilde{D}$ above a line of the first type on the diagonal. Similarly, equations (17), the pullback of diagonal equations (19), and the last $n-5$ equations of (31) intersect transversely. The remaining three equations

$$\lambda_{1i} - \lambda_{1,n+1}\lambda_{2i} = 0, \text{ with } i = 3, 4, 5$$

further cut down the dimension of the variety by 2. It suffices to understand how the normal vector associated with the above three polynomials linearly depends on the normal vectors associated with the pullback of diagonal equations (19). The direct calculation shows that the relevant variables are $\lambda_{13}, \lambda_{14}, \lambda_{15}$, and $\lambda_{23}, \lambda_{24}, \lambda_{25}$, and the corresponding 7 by 6 matrix always has the maximal rank. Hence, the whole Jacobian matrix always has the expected rank.

Proposition 5.5. *The Jacobian matrix at point of* $\mathrm{Bl}_{\Delta_F}(F \times F) \cap \tilde{D}$ *over a line of the first type over the diagonal always has expected rank* $n+5$.

Let us highlight a direct consequence of what we know about the Jacobian matrix so far.

5.2.3. *Irreducibility.*

Lemma 5.6. *The scheme-theoretical intersection* $\tilde{D}_F = \mathrm{Bl}_{\Delta_F}(F \times F) \cap \tilde{D}$ *is irreducible.*

Proof. First, the strict transform of an incidental variety D_F is an irreducible component of the intersection $\mathrm{Bl}_{\Delta_F}(F \times F) \cap \tilde{D}$. Suppose that there is a second component, say D'. Then, it must be supported on the exceptional locus over the diagonal, and its intersection with the strict transform is a singular point. By Proposition 5.5, it is not supported over lines of the first type. Hence, it has to be supported on the locus above the lines of the second type. However, by Lemma 2.3 and Proposition 4.3, such locus has dimension

$$\dim(F_2) + \dim(V_{n-2}) = (n-3) + (n-3) = 2n-6.$$

Hence, the codimension of D' is at least $\dim(\mathrm{Bl}_\Delta Gr(2, n+1)^2) - (2n-6) = 2n+2$, which exceeds $n+5$, the rank of the Jacobian matrix (cf. Proposition 5.4), which is a contradiction. □

5.2.4. *Row Reductions.* Now we come back to the Jacobian matrix (32) for a line of the second type, and we want to determine when its rank drops. This reduces to understanding the linear relation between the 9th and 10th row of the Jacobian matrix (32) and the first four rows. By subtracting the first row from the 9th row and plus $\lambda_{1,n+1}$ times the third row, we can eliminate the left-hand-side entries of the 9th row. The same process can be applied to the 10th row. Thus, the condition for a rank drop is determined by the vanishing of the remaining entries in the 9th and 10th rows.

Proposition 5.7. *The Jacobian matrix* (32) *has lower rank* $n+4$ *if and only if*

$$\lambda_{23} = \lambda_{24} = 0, \tag{33}$$

together with

$$\begin{cases} \lambda_{1,n+1}\frac{\partial \bar{h}_{2j}}{\partial u} - \frac{\partial \bar{h}_{1j}}{\partial u} = 0, \\ \lambda_{1,n+1}\frac{\partial \bar{h}_{2j}}{\partial x_{i,k+1}} - \frac{\partial \bar{h}_{1j}}{\partial x_{i,k+1}} = 0, \end{cases} \tag{34}$$

for $i = 1, 2$, $j = 3, 4$, *and* $4 \leqslant k \leqslant n$. *All expressions are interpreted as being evaluated* $u = x_{ij} = 0$.

5.3. **Second order data.** To solve the above equations, we need to work out the computations based on the equations of $\bar{h}_{ij}$. In light of Lemma 5.1 and (28), this amounts to understanding the second-order terms of ϕ_{ij} in (30). Let's find the explicit equations for ϕ_{ij}.

Recall that if L is a line of the second type in X, then by changing the coordinates we can assume L is defined by equation $x_2 = \cdots = x_n = 0$ and the cubic hypersurface X as (7)

$$F(x_0, \ldots, x_n) = x_2x_0^2 + x_3x_1^2 + \sum_{2\leqslant i,j\leqslant n} x_ix_jL_{ij}(x_0, x_1) + C(x_2, \cdots, x_n), \tag{35}$$

where $L_{ij}(x_0, x_1) = L_{ji}(x_0, x_1)$ is a linear homogeneous polynomial, and $C(x_2, \ldots, x_n)$ is a homogeneous cubic.

In the Plucker coordinates, recall we choose the affine chart $x_{12} = 1$, and the coordinates $(x_{13}, x_{14}, \ldots, x_{1,n+1}, x_{23}, x_{24}, \ldots, x_{2,n+1})$ corresponds to the $\mathbb{C}^2$ spanned by row vectors of the matrix

$$\begin{bmatrix} 1 & 0 & x_{13} & \ldots & x_{1,n+1} \\ 0 & 1 & x_{23} & \ldots & x_{2,n+1} \end{bmatrix}.$$

Therefore, for a line L' to be contained in X and near L, it satisfies the equations

$$F(\lambda[1 : 0 : x_{13} : \ldots : x_{1,n+1}] + \mu[0 : 1 : x_{23} : \ldots : x_{2,n+1}]) = 0$$

for all $[\lambda : \mu] \in \mathbb{P}^1$. Use (7), the above equation gives

$$0 = \lambda^2(\lambda x_{13} + \mu x_{23}) + \mu^2(\lambda x_{14} + \mu x_{24}) + \sum_{2\leqslant i,j\leqslant n+1} (\lambda x_{1,i+1} + \mu x_{2,i+1})(\lambda x_{1,j+1} + \mu x_{2,j+1}) L_{ij}(\lambda, \mu) + \text{third order terms.}$$

Expanding in terms of $\lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3$, we get four equations as in (18), we write the second-order terms below.

$$\begin{aligned}
\phi_{13} &= \sum_{4\leqslant i,j\leqslant n} a^0_{ij} x_{1,i+1} x_{1,j+1}, \\
\phi_{23} &= \sum_{4\leqslant i,j\leqslant n} a^1_{ij} x_{1,i+1} x_{1,j+1} + \sum_{4\leqslant i,j\leqslant n} a^0_{ij} (x_{1,i+1} x_{2,j+1} + x_{1,j+1} x_{2,i+1}), \\
\phi_{14} &= \sum_{4\leqslant i,j\leqslant n} a^0_{ij} x_{2,i+1} x_{2,j+1} + \sum_{4\leqslant i,j\leqslant n} a^1_{ij} (x_{1,j+1} x_{2,i+1} + x_{1,i+1} x_{2,j+1}), \\
\phi_{24} &= \sum_{4\leqslant i,j\leqslant n} a^1_{ij} x_{2,i+1} x_{2,j+1}.
\end{aligned} \tag{36}$$

Here $L_{ij}(x_0, x_1) = a^0_{ij} x_0 + a^1_{ij} x_1$. Note that for convenience, we drop the third-order term because they do not affect the computation of the tangent space, and we drop the terms that involve x_{13}, x_{14}, x_{23} and x_{24} for the reason below.

5.4. **Solving the equations.** Now we look at the conditions given by equations (33) and (34).

First, using (31) and $\lambda_{23} = \lambda_{24} = 0$, we obtain $\lambda_{13} = \lambda_{14} = 0$ in the blow-up coordinates. So, the terms in ϕ_{ij} that involve x_{13}, x_{14}, x_{23} and x_{24} are irrelevant.

Second, note ϕ_{24} does not depend on x_{1j}, so in particular, $\frac{\partial \bar{h}_{24}}{\partial x_{1j}} = 0$, and the equations $\lambda_{1,n+1} \frac{\partial \bar{h}_{24}}{\partial x_{1,k+1}} - \frac{\partial \bar{h}_{14}}{\partial x_{1,k+1}} = 0$ from (34) reduces to

$$\frac{\partial \bar{h}_{14}}{\partial x_{1,k+1}}|_{u=x_{ij}=0} = 0, \ k = 4, \ldots, n \tag{37}$$

Use (28) and (36), we find

$$\bar{h}_{14} = \sum_{4\leqslant i,j\leqslant n} a^0_{ij}(2x_{2,i+1}\lambda_{2,j+1} + u\lambda_{2,i+1}\lambda_{2,j+1}) + 2a^1_{ij}(x_{1,j+1}\lambda_{2,i+1} + x_{2,i+1}\lambda_{1,j+1} + u\lambda_{1,i+1}\lambda_{2,j+1}).$$

Hence (37) reduces to

$$2 \sum_{4\leqslant i\leqslant n} a^1_{ik} \lambda_{2,i+1} = 0, \ k = 4, \ldots, n \tag{38}$$

Similarly, ϕ_{13} does not depend on x_{2j}, so in particular, $\frac{\partial \bar{h}_{13}}{\partial x_{2j}} = 0$, and the equations $\lambda_{1,n+1} \frac{\partial \bar{h}_{23}}{\partial x_{2,k+1}} - \frac{\partial \bar{h}_{13}}{\partial x_{2,k+1}}$ from (34) reduces to

$$\lambda_{1,n+1} \frac{\partial \bar{h}_{23}}{\partial x_{2,k+1}}|_{u=x_{ij}=0} = 0, \ k = 4, \ldots, n \tag{39}$$

By a similar computation, we find

$$2\lambda_{1,n+1}\sum_{4\leqslant i\leqslant n} a^0_{ik}\lambda_{1,i+1}=0,\ \ k=4,\dots,n \tag{40}$$

Now, use equations (31) and assume that $\lambda_{1,n+1}\neq 0$, the equations (38) and (40) become

$$\begin{cases}\sum_{4\leqslant i\leqslant n} a^1_{ik}\lambda_{2,i+1}=0\\ \sum_{4\leqslant i\leqslant n} a^0_{ik}\lambda_{2,i+1}=0\end{cases} \tag{41}$$

for all $k=4,\dots,n$.

In addition, equations $\lambda_{1,n+1}\frac{\partial\bar{h}_{24}}{\partial x_{2,k+1}}-\frac{\partial\bar{h}_{14}}{\partial x_{2,k+1}}=0$ from (34) reduce to

$$\lambda_{1,n+1}(2\sum_{4\leqslant i\leqslant n} a^1_{ik}\lambda_{2,i+1})-(2\sum_{4\leqslant i\leqslant n} a^0_{ik}\lambda_{2,i+1}+2\sum_{4\leqslant i\leqslant n} a^1_{ik}\lambda_{1,i+1})=0,\ \ k=4,\dots,n. \tag{42}$$

Together with (31) and (38), they implies (41) under the assumption of $\lambda_{1,n+1}=0$.

One can check the rest of the equations in (34) are dependent on the equations above and do not provide new relations.

In summary, our computation shows that

Proposition 5.8. *The Jacobian matrix* (32) *at point of* $\mathrm{Bl}_{\Delta_F}(F\times F)\cap\tilde{D}$ *over a line of the second type on the diagonal has lower rank* $n+4$ *if and only if the two symmetric matrices*

$$A^p:=\begin{bmatrix} a^p_{44} & a^p_{45} & \cdots & a^p_{4n}\\ a^p_{54} & a^p_{55} & \cdots & a^p_{5n}\\ \vdots & \vdots & \cdots & \vdots\\ a^p_{n4} & a^p_{n5} & \cdots & a^p_{nn}\end{bmatrix},\ \ p=0,1 \tag{43}$$

are degenerate, and the kernel of the two matrices has a nontrivial intersection, which is equivalent to the matrix of linear forms (8)

$$S=A^0x_0+A^1x_1$$

being degenerate.

When $n=4$ and X is cubic threefold, this is exactly

$$a^0_{44}=a^1_{44}=0,$$

and equivalent to L being a triple line on X (cf. Definition 2.4).

Remark 5.9. Suppose $v=[\lambda_{25},\dots,\lambda_{2,n+1}]^T$ is a non-zero vector that lies in the common kernel of A^p. Then $Sv=0$ and it corresponds to a point $p_v\in\mathbb{P}^n$ "at infinity" with coordinates $x_i=0$ for $0\leqslant i\leqslant 3$. The span of p_v and L determines a plane $P^2(v)$. If $P^2(v)$ is not contained in X, then it makes L a triple line — $P^2(v)\cap X=3L$.

5.5. **Summary.** To summarize our computation carried out in this section, we proved:

Theorem 5.10. *Let* $X\subseteq\mathbb{P}^n$ *be a smooth cubic hypersurface with* $n\geqslant 4$. *Let* p *be a point on* $Bl_\Delta(F\times F)\cap\tilde{D}$ *over a line* (L,L) *on the diagonal.*

- *If* L *is of the first type, then the tangent space at* p *always has expected dimension* $3n-9$*;*

- *If L is of the second type, then the tangent space at p has expected dimension $3n-9$ if and only if the matrix of linear forms* (8) *is non-degenerate, i.e., L is not a higher triple line.*

Proof. The argument about lines of the first (resp. second) type follows from Proposition 5.5 (resp. 5.8). ☐

Remark 5.11. One can also show that the intersection $\tilde{D}_F \cap E$ is smooth if and only if X has no higher triple line, where E is the exceptional divisor of $\mathrm{Bl}_{\Delta_F}(F \times F)$. This can be obtained by keeping track of the computation of rank of the Jacobian matrix (32) with an additional condition $u = 0$.

We also provide an upper bound of the dimension of singularities of $\mathrm{Bl}_{\Delta_F}(F \times F) \cap \tilde{D}$.

Lemma 5.12. *Let X be a smooth cubic hypersurface in $\mathbb{P}^n$ with $n \geqslant 4$. The singular locus of $\tilde{D}_F := \mathrm{Bl}_{\Delta_F}(F \times F) \cap \tilde{D}$ has dimension at most $2n-7$.*

Proof. By Theorem 5.10, the singular locus of $\tilde{D}_F$ is over the locus of higher triple lines on the diagonal. By Corollary 2.9, the locus of lines of higher triple lines has dimension at most $n-4$. Since the fiber dimension is at most $n-3$ (cf. Proposition 4.3), we have

$$\dim(Sing(\tilde{D}_F)) \leqslant (n-4) + (n-3) = 2n-7.$$

☐

The bound is sharp when $\dim(X) = 3$, where the singular locus consists of finitely many $\mathbb{P}^1$ over finitely many triple lines. When $\dim(X) \geqslant 4$, we don't know if the bound is still sharp, but for the purpose of proving the reducedness below, it is enough.

6. Reducedness of $\mathrm{Bl}_{\Delta_F}(F \times F) \cap \tilde{D}$

In this section, we study the reducedness properties of the intersection $\mathrm{Bl}_{\Delta_F}(F \times F) \cap \tilde{D}$. In particular, Theorem 5.10 tells us where the intersection is smooth by computing the rank of the Jacobian matrix. On the locus where the Jacobian matrix has lower rank, the scheme may have an embedded component. However, we will show this cannot happen.

Lemma 6.1. *Let A be a regular local ring over a field k, and $I \trianglelefteq A$ be an ideal generated by two elements $I = (f, g)$. Then, the depth of the quotient ring $\mathrm{depth}(A/I)$ is at least $\dim(A) - 2$.*

Proof. By assumption, A is a UFD. Hence, we can write $f = hf_1$ and $g = hg_1$, where h is the greatest common divisor of f and g.

We have a free A-module resolution of A/I

$$A \xrightarrow{\begin{bmatrix} g_1 \\ -f_1 \end{bmatrix}} A^2 \xrightarrow{[f\ g]} A \to A/(f,g).$$

Hence, the projective dimension of A/I is at most 2. Therefore, by Auslander–Buchsbaum formula [Eis95, Theorem 19.9],

$$\mathrm{depth}(A/I) = \mathrm{depth}(A) - \mathrm{pd}(A/I).$$

Since A is regular, $\dim(A) = \mathrm{depth}(A)$, the claim follows. ☐

In what follows, we apply the lemma to a local ring localized at a regular affine subvariety of $\mathrm{Bl}_{\Delta_F}(F\times F)\cap\tilde{D}$ containing singular locus. For example, when $\dim(X)=3$, $\mathrm{Bl}_{\Delta_F}(F\times F)\cap\tilde{D}$ is 4-dimensional and its singular locus is disjoint union $\sqcup_i\mathbb{P}^1$ indexed by finitely triple lines, so A is a 3-dimensional regular local ring, arising from the localization of coordinate ring of an affine chart of $\mathrm{Bl}_{\Delta_F}(F\times F)\cap\tilde{D}$ at one of the $\mathbb{P}^1$. The lemma ensures A/I has depth is at least one, allowing us to conclude reducednss using Serre's criterion.

Proposition 6.2. *Let X be a smooth cubic hypersurface $\mathbb{P}^n$ with $n\geqslant 4$. Then the scheme-theoretical intersection $\mathrm{Bl}_{\Delta_F}(F\times F)\cap\tilde{D}$ is reduced.*

Proof. By Theorem 5.10, it suffices to study a neighborhood on the exceptional locus over (L,L) on the diagonal, where L is a line of the second type.

In an affine chart, the ideal J of the intersection $\mathrm{Bl}_{\Delta_F}(F\times F)\cap\tilde{D}$ is generated by the 8 functions (29) and (30) cutting out $\mathrm{Bl}_{\Delta_F}(F\times F)$, together with the $n-2$ functions (31) cutting out $\tilde{D}$. Our goal is to show the reducedness of R/J, where R is the polynomial ring with variables u, x_{ij}, and λ_{ij}.

Recall that according to the computation of the Jacobian matrix (32), the equations (29), (30) and the last $n-4$ equations in (31) intersect transversely. Let J_1 denote the ideal generated by these functions. Hence $R':=R/J_1$ is a regular ring, which has dimension

$$\dim(R')=\dim\mathrm{Bl}_{\Delta_F}(Gr(2,n+1)^2)-8-(n-4)=3n-8.$$

Moreover, $R\to R/J$ factors through $R'\to R'/J_2$, where J_2 is the ideal generated by the remaining two functions

$$f:=\lambda_{13}-\lambda_{1,n+1}\lambda_{23}, g:=\lambda_{14}-\lambda_{1,n+1}\lambda_{24}.$$

Now, we have the isomorphism $R/J\cong R'/J_2$, so after localization, we are in the situation in Lemma 6.1.

To show the reducedness of R'/J_2, we need to show Serre's criterion [Eis95, Ex. 11.10]

$$(\mathcal{R}_0)+(\mathcal{S}_1).$$

First, Lemma 5.12 implies that $\mathrm{Spec}(R'/J_2)$ is smooth in codimension one, so in particular, it satisfies $(\mathcal{R}_0)$. For $(\mathcal{S}_1)$, it suffices to check localization at prime ideals whose support is contained in the singular locus.

Specifically, let $\mathfrak{P}$ be a prime ideal of R' containing J_2. Now apply Lemma 6.1 to the regular local ring $R'_{\mathfrak{P}}$ and the ideal $(J_2)_{\mathfrak{P}}$, so we obtain

$$\mathrm{depth}(R'_{\mathfrak{P}}/(J_2)_{\mathfrak{P}})\geqslant\dim(R'_{\mathfrak{P}})-2. \tag{44}$$

It takes the minimal value when the prime ideal $\mathfrak{P}$ corresponds to an irreducible component of the singular locus of $\mathrm{Spec}(R'/J_2)$. According to Lemma 5.12, the dimension of the singular locus is bounded above by $2n-7$. Therefore,

$$\dim(R'_{\mathfrak{P}})=\mathrm{height}(\mathfrak{p})\geqslant(3n-8)-(2n-7)=n-1.$$

Hence by (44), we find that $\mathrm{depth}(R'_{\mathfrak{P}}/(J_2)_{\mathfrak{P}})\geqslant n-3\geqslant 1$, as long as $n\geqslant 4$. Therefore, R'/J_2 satisfies Serre's condition $(\mathcal{S}_1)$ and is reduced. ☐

7. Main Theorems and Applications

In this section, we mention some applications as consequences of results in previous sections. In particular, Theorem 1.3 and 1.6 from the introduction will be proved.

7.1. **Cubic Threefolds.** We note that when X is a cubic threefold, $\tilde{D}_F$ has codimension one in $\mathrm{Bl}_{\Delta_F}(F \times F)$. Hence, by Proposition 6.2 and Lemma 5.6, it is a divisor and, moreover, a Cartier divisor due to the smoothness of $\mathrm{Bl}_{\Delta_F}(F \times F)$. Since blow-up at a Cartier divisor is an isomorphism, we have $\widetilde{H(X)} \cong \mathrm{Bl}_{\Delta_F}(F \times F)$. Passing to the $\mathbb{Z}_2$ quotient, we recover the main theorem in [Zha23].

Corollary 7.1. *([Zha23]) Let X be a smooth cubic threefold. Then, the Hilbert scheme of a pair of skew lines $H(X)$ is smooth and isomorphic to $\mathrm{Bl}_{\Delta_F}\mathrm{Sym}^2 F$.*

For higher dimensions, $\tilde{D}_F$ has codimension at least two, so the singularities of $\tilde{D}_F$ will contribute to the singularities of the second blow-up. However, we will see that $H(X)$ is still normal.

7.2. **Smoothness Criterion of Hilbert scheme.**

Theorem 7.2. *(cf. Theorem 1.6) Let X be a cubic hypersurface with $\dim(X) \geqslant 4$. Then,*
  (1) $H(X)$ is normal;
  (2) $H(X)$ is smooth if and only if the X has no higher triple line.

Proof. For (1), by passing to the $\mathbb{Z}_2$ quotient, Proposition 6.2 implies that the birational morphism
$$H(X) \to \mathrm{Bl}_{\Delta_F}\mathrm{Sym}^2 F$$
is a blow-up of a smooth scheme along a reduced subscheme. The ideal sheaf of a reduced scheme is integrally closed [HS06, Rmk. 1.1.3 (4)]. Therefore, by [HS06, Prop. 5.2.4], the corresponding Rees algebra is integrally closed. Hence, the blow-up is normal.

For (2), according to the analysis in the Introduction, to show the smoothness of $\widetilde{H(X)}$, it suffices to show that $\mathrm{Bl}_{\Delta_F}(F \times F) \cap \tilde{D}$ is set-theoretically smooth. By Proposition 3.5, it is always smooth away from the diagonal, as long as X is smooth. By Theorem 5.10, when X has no higher triple lines, the second blow-up center $\mathrm{Bl}_{\Delta_F}(F \times F) \cap \tilde{D}$ is smooth everywhere. Hence, the blow-up $\widetilde{H(X)}$ is smooth. Now, the double cover $\widetilde{H(X)} \to H(X)$ is branched along the strict transform $\tilde{E}$ of the exceptional divisor E of $\mathrm{Bl}_{\Delta_F}(F \times F)$. On the other hand, $\tilde{E} \to E$ is blow-up of the intersection $\tilde{D}_F \cap E$, which is again smooth when X has no higher triple line (cf. Remark 5.11), hence $\tilde{E}$ is a smooth divisor and the $\mathbb{Z}_2$-quotient $H(X)$ is smooth as well.

Conversely, when X has a higher triple line, blowup center $\widetilde{H(X)} \to \mathrm{Bl}_{\Delta_F}(F \times F)$ is reduced and singular of codimension at least two, so $\widetilde{H(X)}$ and its $\mathbb{Z}_2$ quotient is singular as well. ☐

7.3. **Hypersurface singularities.** Recall that a variety Y has hypersurface singularities if $\dim(T_p Y) \leqslant \dim(Y) + 1$ at any point $p \in Y$. Equivalently, Y can be locally analytically embedded in $\mathbb{C}^{\dim(Y)+1}$ as a hypersurface.

Corollary 7.3. *Let X be a smooth cubic hypersurface in $\mathbb{P}^n$ with $n \geqslant 4$. Then the scheme-theoretical intersection $\tilde{D}_F = \mathrm{Bl}_{\Delta_F}(F \times F) \cap \tilde{D}$ is reduced, irreducible, and has hypersurface singularities.*

Proof. The reducedness and irreducibility follow from Proposition 6.2 and Lemma 5.6. Finally, Proposition 5.4 tells us that the rank of the Jacobian matrix can drop by at most one.

Hence, the tangent space has dimension at most $\dim(\tilde{D}_F) + 1$, showing D_F has hypersurface singularities. $\square$

Remark 7.4. Having hypersurface singularities implies that $\tilde{D}_F$ is a local complete intersection and is Cohen-Macaulay. By Lemma 5.12, $\tilde{D}_F$ is smooth in codimension $(2n-6) \geqslant 2$, therefore $\tilde{D}_F$ is normal.

Remark 7.5. According to [Kol13, Lem. 10.21], Proposition 6.2 implies that the union

$$\mathrm{Bl}_{\Delta_F}(F \times F) \cup \tilde{D},$$

with the reduced scheme structure is *seminormal*.

7.4. **Singularities of incidental subvariety.** Recall that $D_F \subseteq F \times F$ is the incidental subvariety (cf. Definition 3.4). Now, as a consequence of the reducedness and irreducibility (cf. Corollary 7.3), we have a fibered diagram.

$$\begin{array}{ccc} \tilde{D}_F & \hookrightarrow & \mathrm{Bl}_{\Delta_F}(F \times F) \\ \downarrow{\scriptstyle \sigma_F} & & \downarrow{\scriptstyle \sigma} \\ D_F & \hookrightarrow & F \times F. \end{array}$$

In other words, $\tilde{D}_F$ is the strict transform of D_F. Moreover, σ_F is a desingularization if X has no higher triple line. We can use the birational morphism $\sigma_F : \tilde{D}_F \to D_F$ to characterize the singularities of D_F.

Proposition 7.6. *Let $(D_F)^{\mathrm{sing}}$ denote the singular locus of D_F.*

- *When $\dim(X) = 3$, $(D_F)^{\mathrm{sing}}$ is the locus of triple lines on the diagonal;*
- *When $\dim(X) = 4$, $(D_F)^{\mathrm{sing}} = \Delta(F_2)$ is the locus of the lines of the second type;*
- *When $\dim(X) \geqslant 5$, $(D_F)^{\mathrm{sing}} = \Delta_F$ is the whole diagonal.*

Proof. When $\dim(X) = 3$, it follows from [CG72, Lem. 12.18], [BB23]. When $\dim(X) = 4$, it is proved in [Gio21, Thm. 4.3.1.2].

We assume $\dim(X) = n-1 \geqslant 4$. Hence $\Delta_F \subseteq D_F$ by Remark 3.8. The fiber of $\tilde{D}_F \to D_F$ over a line of the first type is V_{n-3}, the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^{n-5} \hookrightarrow \mathbb{P}^{2n-9}$ (cf. Proposition 4.3). It is a nondegenerate variety, so the tangent space of D at (L, L) normal to the diagonal is the ambient vector space $\mathbb{C}^{2n-8}$. Since $2n - 8 + \dim(\Delta_F) \geqslant \dim(D_F)$ (cf. Corollary 3.6), and the equality holds iff $n = 5$. We conclude that D_F is smooth at (L, L) when $\dim(X) = n - 1 = 4$ and singular otherwise.

Similarly, when $L \in F_2$ is a line of the second type, the fiber of $\tilde{D}_F \to D_F$ at (L, L) is the Segre embedding V_{n-2}, which spans the ambient space $\mathbb{C}^{2n-6}$. When L is a smooth point of F_2, such space is the tangent space of D at (L, L) normal to the diagonal. Therefore, $\dim(T_{(L,L)}D_F) = 2n - 6 + \dim(\Delta_F) > \dim(D_F)$, and D_F is singular there. $\square$

8. Dimension Count

This section is a continuation of Section 2. We will bound the dimension of the space of cubic hypersurfaces with a higher triple line and prove Proposition 2.7, which indicates the main Theorem 1.4.

8.1. **Transversal A_2 Singularities.** We want to understand the geometric meaning of the degeneracy condition of the matrix of linear forms (8).

The linear dependence of the columns implies that we can choose new coordinates $x_4, \dots, x_n$ such that the two matrices (43) have the last row being zero (hence the last column is zero as well). In other words, in equation (7) of the cubic hypersurface, there is no term that involves the monomials

$$x_p x_i x_n,\ p = 0, 1, \text{ and } i = 4, \dots, n.$$

This implies that the codimension two linear section of the cubic hypersurface X given by $P^{n-2} = \{x_2 = x_3 = 0\}$ is a cubic $(n-3)$-fold with equation

$$\sum_{4 \leqslant i,j \leqslant n-1} x_i x_j L_{ij}(x_0, x_1) + C(x_4, \cdots, x_n) = 0. \tag{45}$$

Now, the codimension-two subvariety $Y = X \cap P^{n-2}$ is singular along the line L_{x_0,x_1} and has *transversal A_2 singularities* along the line.

For example, when $\dim(X) = 3$, $P^2 \cap X = 3L$ is a triple line; when $\dim(X) = 4$, $P^3 \cap X$ is a cone over a cuspidal plane curve (cf. figure 2).

To provide a dimension count of cubic hypersurfaces with a higher triple line, we first need to count the dimension of the cubics of the form (45). We first need a linear algebra argument:

Lemma 8.1. *Suppose that the matrix S degenerates (cf. Definition 2.5), then* $\det(S) = 0$ *as a homogeneous polynomial in x_0, x_1.*

Proof. By changing the coordinates described above, one can assume that the matrix of the linear forms (8) has vanishing last row and column. Therefore, $\det(S) = 0$. Alternatively, a coordinate-free proof can be given using Cramer's rule [Lan02, XIII, Thm. 4.4]. □

It is not clear to us if the converse holds, but Lemma 8.1 at least provides an estimation of the number of relations. For example, if S has size $(n-3) \times (n-3)$, then $\det(S)$ is a homogeneous polynomial of degree $n-3$, therefore $\det(S) = 0$ provides $n-2$ conditions.

Lemma 8.2. *The space of cubic hypersurfaces in $\mathbb{P}^{n-2}$ with transversal A_2 singularities along a line forms a subspace of codimension at least* $2n-1$.

Proof. Let $Y \in \mathbb{P}(Sym^3\mathbb{C}^{n-1})$ be a cubic hypersurface in $\mathbb{P}^{n-2}$. We require Y to have transversal A_2 singularities along a line. Then it imposes the following conditions

(1) Y contains the line $L = \{x_2 = \cdots = x_{n-2} = 0\}$ requires the vanishing of the monomials $x_0^3, x_0^2x_1, x_0x_1^2, x_1^3$ in the defining equation of Y, which gives four conditions.
(2) Y is singular along L forces the vanishing of the terms $l_{ij}(x_2, \dots, x_{n-2})x_0^i x_1^j$, with $i + j = 2$ and l_{ij} a linear form. Hence, it gives $3(n-3)$ conditions.
(3) The matrix of linear forms S degenerates implies $\det(S) = 0$ (cf. Lemma 8.1), which provides another $n-2$ conditions.

Therefore, (1), (2), and (3) together give $4 + 3(n-3) + (n-2) = 4n-7$ conditions. We allow the line L to move in $\mathbb{P}^{n-2}$. It is parameterized by $Gr(2, n-1)$, which has dimension $2(n-3)$, so the space of cubic hypersurfaces that have transversal A_2 singularities along a line is a closed subspace of $\mathbb{P}(Sym^3\mathbb{C}^{n-1})$ with codimension at least

$$(4n-7) - 2(n-3) = 2n-1.$$

□

Example 8.3.
- In $\mathbb{P}^2$, a cubic curve with transversal A_2 along a line is three times of a line $3L$, nonreduced. They form a codimension 7 subspace of all cubic curves.
- In $\mathbb{P}^3$, a cubic surface with transversal A_2 along a line is a cone over a cuspidal cubic curve, which is non-normal (cf. Remark 9.4). Such cubic surfaces form a codimension 9 subspace of all cubic surfaces.
- In $\mathbb{P}^4$, a cubic threefold with transversal A_2 singularities along a line is normal (if there are no other singularities). Such cubics form a subspace of codimension (at least) 11 of all cubic threefolds.

8.2. **Correspondence.** Now we use intersection correspondence to bound the dimension of the cubic hypersurface with a higher triple line.

Proposition 8.4. *The set of smooth cubic hypersurfaces in $\mathbb{P}^n$, with $n \geqslant 4$, which have higher triple lines, forms a closed subspace of codimension at least one.*

Proof. Let W be the space of all cubic $(n-3)$-fold in $\mathbb{P}^n$. Then $\pi : W \to Gr(n-1, n+1)$ is a $\mathbb{P}(Sym^3\mathbb{C}^{n-1})$-bundle, with fiber over $P^{n-2} \subseteq \mathbb{P}^n$ being the spaces of cubic hypersurfaces in P^{n-2}. Then, by Lemma 8.2, and the bundle structure, the locus $\mathcal{C}$ consisting of cubic $(n-3)$-fold in $\mathbb{P}^n$ with transversal A_2 singularities along a line has codimension at least $2n-1$.

We consider the incidence map

$$\phi : \mathbb{P}(Sym^3\mathbb{C}^{n+1}) \times Gr(n-1, n+1) \to W,$$
$$(X, P^{n-2}) \mapsto X \cap P^{n-2},$$

by intersecting a cubic hypersurface X and a codimension-two plane P^{n-2} in $\mathbb{P}^n$.

Then ϕ is subjective. We use the fact that the space of $\mathbb{P}^{n-2}$ in $\mathbb{P}^n$ has dimension $\dim Gr(n-1, n+1) = 2(n-1)$. Therefore, $pr_1(\phi^{-1}(\mathcal{C}))$ has codimension at least

$$(2n-1) - 2(n-1) = 1$$

in the space of all cubic hypersurfaces in $\mathbb{P}^n$. Hence, the claim follows. □

Corollary 8.5. *(cf. Theorem 1.4) Let X be a general cubic hypersurface with* $\dim(X) \geqslant 3$. *The Hilbert scheme of a pair of skew lines $H(X)$ is smooth.*

Proof. This follows from Theorem 7.2 and Proposition 8.4. □

9. Interpreting Singularities of the Hilbert Scheme

In this section, we provide a modular meaning of the singularities of $\tilde{D}_F$ and $H(X)$.

To summarize what we have proved, by descending the first column of (4) to the $\mathbb{Z}_2$ quotient, $H(X)$ arises from successive blowup

$$\mathrm{Bl}_{\tilde{D}'_F}\mathrm{Bl}_{\Delta_F}\mathrm{Sym}^2F \to \mathrm{Bl}_{\Delta_F}\mathrm{Sym}^2F \to \mathrm{Sym}^2F$$

along the diagonal Δ_F, and $\tilde{D}'_F$, which is the $\mathbb{Z}_2$ quotient of $\tilde{D}_F$.

From Proposition 5.8, we see that the singular locus of $\tilde{D}_F$ is on the diagonal fixed by $\mathbb{Z}_2$ action, hence we can identify the singular locus of $\tilde{D}'_F$ and $\tilde{D}_F$, which is fibered over locus of higher triple lines on the diagonal Δ_F and the fiber is at least a copy of $\mathbb{P}^1$. Then, according to the description of type (IV) schemes in Section 4.3, we have

Proposition 9.1. *When* $\dim(X) \geqslant 4$, *each singularity of $H(X)$ corresponds to a triple*

$$(p, L, P^3)$$

where L is a higher triple line L, $p \in L$ is a point, and P^3 is a linear 3-dimensional subspace of P^n containing L and a normal direction $v \in H^0(\mathcal{O}_L(1))$, with $\mathcal{O}_L(1) \subseteq N_{L|X}$. This data determines a type (IV) subscheme Z_{IV} supported on L, and an embedded point supported on p and contained in P^3.

Here, the 3-plane P^3 determines the normal direction of the embedded point to P^2 and vice versa. More generally, there is a morphism

$$\pi : H(X) \to Gr(4, n+1)$$

by assigning each $Z \in H(X)$ to the unique 3-plane $\pi(Z) \cong \mathbb{P}^3$ containing Z.

One may ask the following question.

Question 9.2. *Suppose $Z_{IV} \in H(X)$ is a singular point. How to describe the 3-plane $\pi(Z_{IV})$ containing Z_{IV}?*

If $Z_{IV} \in H(X)$ is a singularity, then the 3-plane $\pi(Z_{IV})$. Suppose it is not contained in X (this is the case when $\dim(X) \leqslant 4$), then the intersection $\pi(Z_{IV}) \cap X$ is a cubic surface, which also contains Z_{IV} as a closed subscheme. We denote it by C. We observe that

Proposition 9.3. *The cubic surface C is either*

- *a cone of a planer cuspidal cubic curve, or*
- *the union of a plane and quadric cone meeting tangentially along L.*

Proof. Let P^2 be the unique plane as in Remark 5.9, and P^{n-2} be the $(n-2)$ plane tangent to X along L (cf. Lemma 2.2). Then, any linear subspace P^k such that

$$P^2 \subseteq P^k \subseteq P^{n-2}$$

satisfies that $P^k \cap X$ is singular along the line L and has transversal A_2 singularities along L. In particular, take $P^k = \pi(Z)$, then $\pi(Z) \cap X$ has equation

$$l(x_0, x_1)x_2^2 + c(x_2, x_3) = 0,$$

where l is a linear form and c is a cubic form. Then if $x_2 \nmid c(x_2, x_3)$, the affine curve $x_2^2 + c(x_2, x_3) = 0$ has a cusp at $(0, 0)$ and C is a cone over it. Otherwise, the curve is the union of a line and conic tangent at a point, so its cone C is the union of a plane and quadric cone meeting along a line. ☐

In the first case, there is a unique P^2 such that $P^2 \cap X = 3L$ (cf. Remark 5.9), while in the second case, P^2 is contained in X. In either case, L is a triple line by Definition 2.4.

Remark 9.4. There is a hierarchy of cubic surfaces based on singularities and codimension of parameter space (cf. [LLSvS17, p.13], [GG24, App.]), which is roughly

$$\text{normal w/} \begin{cases} ADE \text{ singualarities} \\ \text{elliptic singularity} \end{cases} \rightsquigarrow \text{non-normal, integral} \begin{cases} X_6 \\ X_7 \\ X_8 \\ X_9 \end{cases} \rightsquigarrow \text{non-integral cubic surfaces.}$$

Here we use the notations in [LLSvS17], and the subscribe denotes the codimension of the parameter space. The cone of the cuspidal curve is X_9.

10. Voisin's Map

In this section, we assume $\dim(X) = 4$. When X does not contain a plane, Voisin [Voi16, Prop. 4.8] constructed a dominant rational map

$$\psi : F \times F \dashrightarrow Z. \tag{46}$$

In [Che21], the author proposed a resolution by blowing up the incidental subvariety D_F (cf. Definition 3.4). Let $\widetilde{F \times F}$ be the blow-up of $F \times F$ along the incidental subvariety.

Lemma 10.1. [Che21] *The rational map* (46) *extends to a regular morphism*

$$\tilde{\psi} : \widetilde{F \times F} \to Z.$$

Here, we remark that D_F is reduced and singular (cf. Proposition 7.6). So, the total space $\widetilde{F \times F}$ is also singular.

Proposition 10.2. *Let X be a smooth cubic hypersurface. We consider the two different birational morphisms The two step blowups*

$$\sigma : \mathrm{Bl}_{\tilde{D}_F}\mathrm{Bl}_{\Delta_F}(F \times F) \xrightarrow{\sigma_2} \mathrm{Bl}_{\Delta_F}(F \times F) \xrightarrow{\sigma_1} F \times F,$$

and

$$\sigma' : \mathrm{Bl}_{D_F}(F \times F) \to F \times F.$$

Then when $\dim(X) = 3$ *or* $\dim(X) \geqslant 5$*, the birational morphism σ factors through σ'. When* $\dim(X) = 4$*, σ does not factor through σ'.*

Proof. By universal property of the blowup. σ factors through σ' if and only if the total transform of the blowup center D_F of σ'

$$\sigma^*(D_F)$$

is a Cartier divisor.

According to Proposition 7.6, when $\dim(X) \geqslant 5$, D_F is singular along the diagonal. Since Δ_F is the blowup center of σ_1, we note $\sigma_1^*(I_{D_F}) = I_{\tilde{D}_F} \cdot I_E^2$, where E is the exceptional divisor. Since $\tilde{D}_F$ is the blowup center of σ_2,

$$\sigma^*(I_{D_F}) = \sigma_2^*(I_{\tilde{D}_F} \cdot I_E^2) = \sigma_2^*(I_{\tilde{D}_F}) \cdot \sigma_2^*(I_E)^2,$$

is product of ideal sheaf of Cartier divisors, and hence Cartier. When $\dim(X) = 3$, one can show similarly that $\sigma^*(I_{D_F}) = I_{\tilde{D}_F}$ is an ideal sheaf of a Weil divisor, hence Cartier (cf. Corollary 7.1).

When $\dim(X) = 4$, by Proposition 7.6, D_F is smooth at a general point of diagonal, and at a line of second type, the tangent cone has multiplicity 2, so

$$I_{\tilde{D}_F} \cdot I_E^2 \subseteq \sigma_1^*(I_{D_F}) \subseteq I_{\tilde{D}_F} \cdot I_E \tag{47}$$

However, the initial ideal of $I_{\tilde{D}_F}$ at a line of second type is generated by

$$u_{13}, u_{14}, u_{23}, u_{24}, u_{15}u_{26} - u_{16}u_{25}$$

The pullback of these generators $\sigma_1^*(u_{ij} = u\lambda_{ij})$ (cf. (26)) and factor out a generator u of I_E, we have $\sigma_2^*(I_{\tilde{D}_F})/u$ near a line of second type is generated by

$$\lambda_{13}, \lambda_{14}, \lambda_{23}, \lambda_{24}, u(\lambda_{15}\lambda_{26} - \lambda_{16}\lambda_{25}).$$

The first four generators and $\lambda_{15}\lambda_{26} - \lambda_{16}\lambda_{25}$ cuts out the strict transform $\tilde{D}_F$, while the first four generators and u cuts out the entire $\mathbb{P}^3 \subseteq \mathrm{Bl}_{\Delta_F}(F \times F)$ over $L \in F_2$. Hence both of the inclusions (47) are proper, and $\sigma_1^*(I_{D_F})$ has an embedded component G, which is a $\mathbb{P}^3$-bundle over F_2. Furthur pullback to the second blowup, then embedded component still has codimension 3, hence $\sigma^*(I_{D_F})$ is not an ideal sheaf of a carrier divisor. □

Proposition 10.3. *Voisin's rational map extends to a third blowup*

$$\mathrm{Bl}_{\tilde{G}}\widetilde{H(X)} \to Z,$$

where $\tilde{G}$ is the strict transform of G under the blowup σ_2.

Proof. This is by universal property of the blowup. By the discussion above, $\sigma^*(I_{D_F})$ is ideal sheaf of $\tilde{G}$ up to factors of invertible sheaves, so blowup $\tilde{G}$, the pullback $(\sigma \circ \sigma_3)^*(I_{D_F})$ becomes Cartier. Hence it factors through Chen's blowup. Then by composing with $\tilde{\psi}$, we have the extension of Voisin map $\mathrm{Bl}_{\tilde{G}}\widetilde{H(X)} \to Z$. □

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