MONODROMY OF PRIMITIVE VANISHING CYCLES FOR HYPERSURFACES IN $\mathbb{P}^4$

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Introduction

Let $X \subset \mathbb{P}^N$ be a smooth projective variety over $\mathbb{C}$ of dimension $2n-1$. Let $f : \mathcal{X} \to \mathbb{O}^{sm}$ be the universal family of smooth hyperplane sections of X, where $\mathbb{O}^{sm}$ is the Zariski open subset of the dual projective space corresponding to smooth hyperplane sections.

Denote T to be the total space of the vector bundle $\mathcal{H}^{2n-2}_{van}$ whose fiber is the (middle dimensional) vanishing cohomology $H^{2n-2}_{van}(X_t, \mathbb{Z})$ on a smooth hyperplane section X_t. Topologically, T is an infinite sheeted covering space of the base $\mathbb{O}^{sm}$, with countably many connected components (the classes with different self-intersection number cannot be connected via monodromy). Among all of the connected components, there is a distinguished one T' containing a *vanishing cycle*, i.e., consider a disk Δ transversal to the discriminant locus $\mathbb{O}\backslash\mathbb{O}^{sm}$ with $\Delta^* \subseteq \mathbb{O}^{sm}$, then the restricted family $f|_\Delta$ has local analytic equation $x_1^2 + \cdots + x_{2n-2}^2 = t$ around the node, which is diffeomorphic to the tangent bundle of sphere S^{2n-2} and a vanishing cycle $\alpha_t \in H^{2n-2}_{van}(X_t, \mathbb{Z})$ is the fundamental class of the sphere as the zero section (i.e., $\mathrm{Im}(x_i) = 0$ when $t > 0$). The component T' is well-defined since vanishing cycle is conjugate to each other via monodromy in the universal family of hyperplane sections, see [9], Proposition 3.23.

Our goal is to understand the topology of the covering space $\pi : T' \to \mathbb{O}^{sm}$. In other words, we would like to understand the global monodromy of vanishing cycle on the universal family of hyperplane sections. Classically, only local monodromy around a nodal degeneration is known, which is the Picard-Lefschetz formula. However, no global result is known yet. We will show that the monodromy of π is "complicated enough" to generate $H^{2n-1}_{\mathrm{prim}}(X, \mathbb{Q})$ which we will decribe in the following.

Let $J = F^n H^{2n-1}_{\mathrm{prim}}(X, \mathbb{C})^*/H_{2n-1}(X, \mathbb{Z})_{\mathrm{prim}}$ be the middle dimensional primitive intermediate Jacobian of X. There is a real anlytic map called Topological Abel-Jacobi map [11]

$$\Phi : T \to J. \tag{1}$$

Schnell [8] showed that:

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Theorem 1. Φ *induces map on the level of homology*

$$\Phi_* : H_1(T, \mathbb{Z}) \to H_1(J, \mathbb{Z}) \cong H_{2n-1}(X, \mathbb{Z})_{\text{prim}}$$

has cofinite image.

In fact, Φ_* coincides with what called the *tube mapping* defined in [8], which can be described more geometrically: Choose a base point $t_0 \in \mathbb{O}^{sm}$, let $X_{t_0} = X \bigcap H_{t_0}$ be the corresponding hyperplane section. By Poincare duality, we view a vanishing cohomology class $\alpha_{t_0} \in H^{2n-2}(X_t, \mathbb{Z})$ as a vanishing homology class $\underline{\alpha}_{t_0} \in H_{2n-2}(X_{t_0}, \mathbb{Z})_{\text{van}} := \ker(H_{2n-2}(X_{t_0}, \mathbb{Z}) \to H_{2n-2}(X, \mathbb{Z}))$. Since a loop in T corresponds to a loop in $\mathbb{O}^{sm}$ which fixes $\underline{\alpha}_{t_0}$, Φ_* is equivalent to the map

$$\Phi_* : \{(\alpha_{t_0}, [\gamma]) \in H_{2n-2}(X_{t_0}, \mathbb{Z})_{van} \times \pi_1(\mathbb{O}^{sm}, t_0) | [\gamma] \cdot \underline{\alpha}_{t_0} = \underline{\alpha}_{t_0}\} \to H_{2n-1}(X, \mathbb{Z})_{\text{prim}}$$

$$(\underline{\alpha}_{t_0}, \gamma) \mapsto \Gamma \tag{2}$$

by sending the pair to its "tube" Γ by following $\underline{\alpha}_{t_0}$ along the loop γ. The image is a $(2n-1)$-chain with boundary $\partial\Gamma = \underline{\alpha}_{t_0} - \gamma \cdot \underline{\alpha}_{t_0} = 0$, so it is a $(2n-1)$-cycle. So Schnell's theorem can be regarded as realizing a primitive class as trace of monodromy of vanishing classes on smooth hyperplane sections.

Zhao [11] used Schnell's theorem to show a theorem called *topological Jacobi inversion*:

Corollary 1. *Up to embed X by linear system $|lX_t|$ for some l large enough, the map* (1) *is surjective.*

If we restrict the topological Abel-Jacobi map to the distinguished component T'

$$\Phi' : T' \to J.$$

It also induces the tube mapping

$$\Phi'_* : H_1(T', \mathbb{Z}) \to H_1(J, \mathbb{Z}) \cong H_{2n-1}(X, \mathbb{Z})_{\text{prim}} \tag{3}$$

In the paper, we are going to show that

Theorem 2. *When X is a smooth hypersurface in $\mathbb{P}^4$ of degree $d \geqslant 3$, then Φ'_* has cofinite image.*

To describe a geometric meaning, we state following oberservations:

I. *The vanishing cycle is conjugate to each other*, so we can just fix one vanishing cycle $\alpha \in H^{2n-2}(X_{t_0}, \mathbb{Z})_{\text{van}}$.

II. Choose a Lefschetz pencil $\mathbb{P}^1$ and $U_X \subseteq \mathbb{P}^1$ the locus where hyperplane sections are smooth, then Zariski's lemma states that the map *the map $\pi_1(U_X, t_0) \to \pi_1(\mathbb{O}^{sm}, t_0)$ induced* *by inclusion $U_X \hookrightarrow \mathbb{O}^{sm}$ is surjective.* So we only need to consider those loops contained in the Lefschetz pencil.

It follows that an equivalent statement of Theorem 2 will be

Theorem 2′. *Let X be a smooth hypersurfaces in $\mathbb{P}^4$ of degree $d \geqslant 3$. Let $\alpha \in H^2_{\mathrm{van}}(X_{t_0}, \mathbb{Z})$ and $\underline{\alpha}$ be its Poincare dual. LetG be the subgroup of $\pi_1(U_X, t_0)$ consisting of $[\gamma]$ satisfying $[\gamma] \cdot \alpha = \alpha$, then the map*

$$\Phi'_* : \{\underline{\alpha}\} \times G \to H_3(X, \mathbb{Z}) \tag{4}$$

as defined in (2) *has cofinite image.*

Our strategy is to first prove the theorem for the case $d = 3$, which is based on the fact that a vanishing cycle on a cubic surface is represented by difference of two disjoint lines $[L_1] - [L_2]$ together with geometry of Abel-Jacobi map on cubic threefolds [1], [4]. The general situation relies on degeneration of the hypersurface of degree d into union of hyperplanes of degree 3 and degree $d-3$ meeting transversely. After birational modification on the total space of the family, we obtain a semistable family where the asymptotic Hodge theory is well understood [7], then the proof follows from the analysis of degeneration of vanishing cycle and its monodromy.

1. Cubic Threefolds

In this section, we will give a proof of Theorem 2. For the purpose of completeness, we will focus on the describing the geometry of component T'.

1.1. **The Component T'.** The first observation is that the covering map $\pi : T' \to \mathbb{O}^{sm}$ is finite. As the smooth hyperplane section of cubic threefold Y is a cubic surface Y_t, whose vanishing cohomology $H^2_{\mathrm{van}}(X_t, \mathbb{Z})$ is concentrated on type $(1,1)$, therefore it is algebraic according to Lefschetz $(1,1)$ theorem. In fact, it is classically known that a vanishing cycle α is represented by a difference of two disjoint lines $[L_1] - [L_2]$, whose monodromy group over the universal parameter space $\mathbb{O}^{sm} \subseteq \mathbb{O} := (\mathbb{P}^4)^*$ is the Weyl group $W(E_6)$, which has order 51840.

Next, let $\mathbb{L} \subset \mathbb{O}$ be a general line and let U be the open subset $\mathbb{L} \bigcap \mathbb{O}^{sm}$. Let $\mathcal{F}$ is the relative Fano scheme of lines on cubic surfaces over $\mathbb{L}$. Explicitly, it is defined by the incidence relation

$$\mathcal{F} = \{(p,t) | L_p \subset Y \cap H_t\} \subset F \times \mathbb{L},$$

where F is the Fano surface of lines on the cubic threefold Y. In other words, we are allowed to restrict the covering map $T' \to \mathbb{O}^{sm}$ to $T'_U \to U$.

Let $\pi : \mathcal{F} \to \mathbb{L}$ be the projection to the sectond factor. Then π

(1) is a ramified covering map whose restriction $\mathcal{F}_U \to U$ is a covering space map of 27-to-1;

(2) has exactly 6 ramification points over each point $t_{node} \in \mathbb{L}\backslash U$ and all ramification points are simple.

We can also consider the fiber product over U

$$p : \mathcal{F}_U \times_U \mathcal{F}_U \to U \tag{5}$$

which parameterizes pair of lines.

Claim. *The set of pairs of disjoint lines form a connected component of the covering space* (5)*.*

Proof. First the universal family of cubic surfaces has monodromy $W(E_6)$ act transitively on ordered 6-tuple $(L_1, ..., L_6)$ of mutually disjoint lines, so given any pair of disjoint line (L_1, L_2) on cubic surface, one complete it to a 6-tuple, so any pair of disjoint line and conjugate to each other under monodromy action in the universal family of cubic surfaces.

Next, Cheng [C20] showed that the monodromy group of lines of cubic surfaces as hyperplane sections of a smooth cubic 3-fold is the full monodromy group $W(E_6)$. Finally, again by Zariski's lemma on fundamental group, one can restrict the universal family to a Lefschetz pencil. □

Now, denote

$$\mathcal{M} \to U$$

the connected component of pair of disjoint lines in (5).

Fiberwise evaluating the pair (L_1, L_2) to the homology class $[L_1]-[L_2]$ on the cubic surface. The class is vanishing since its intersection number with a hyperplane is zero. Moreover, it is known classically it is actually a vanishing cycle. This gives a map

$$\mathcal{M} \to T'_U$$

$$(L_1, L_2) \mapsto [L_1] - [L_2]$$

over U.

Equivalently, we have a diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{6-1} & T'_U \\ \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \pi'} & \\ U & & \end{array}$$

with the horizontal map a covering space map of degree 6, since every local vanishing cycle has exactly 6 ways as difference of disjoint pair of lines. π is a 27×16 sheeted covering map, and π' is 72 sheeted covering map.

To get some understanding of local picture of T'_U, let Δ be a small disk centered at a point $t_i \in \mathbb{L}\backslash U$ and fix a point $t \in \Delta^*$. If $(t, \alpha) \in T'_U$, then according to Picard-Lefschetz formula on ordinary node on even dimensional variety, the local monodromy induced by $\pi_1(\Delta^*, t)$ has order 2. In particular, there is an analytic punctured disk $\Delta^*_\alpha \subset T'_U$ containing the point (t, α), and the projection

$$\pi' : \Delta^*_\alpha \to \Delta^*$$

is given by $z \mapsto z^d$, where d is the order of the local monordromy of α. Let δ be the local vanishing cycle, there are two possibilities:

I. $\langle \alpha, \delta \rangle = 0$, α is invariant under local monodromy and $d = 1$;

II. $\langle \alpha, \delta \rangle \neq 0$, α has nontrivial local monodromy and $d = 2$.

Now $\pi'^{-1}(\Delta^*)$ is disjoint union of finitely many analytic disks which are finite covering space over Δ^* determined by local monodromy of vanishing cycle. We can "fill in the hole" for these analytic disks and do the same thing for each point $t_i \in \mathbb{L}\backslash U$. Therefore we get a proper analytic space $\overline{T'_U}$ with T'_U as an open dense subspace. So we have a diagram extended to

$$\begin{array}{ccc} \bar{\mathcal{M}} & \xrightarrow{6-1} & \overline{T'_U} \\ \downarrow{\scriptstyle \bar{\pi}} & \swarrow{\scriptstyle \bar{\pi}'} & \\ \mathbb{L} & & \end{array}$$

where $\bar{\mathcal{M}}$ is the normalization of closure of $\mathcal{M}$ in $\mathcal{F} \times_{\mathbb{L}} \mathcal{F}$.

Now both $\bar{\pi}$ and $\bar{\pi}'$ are ramified covers, while the horizontal map is still unramified cover between smooth proper curves. We have commutative diagram of Abel-Jacobi maps

$$\begin{array}{ccccc} & & \overline{T'_U} & & \\ & \overset{6-1}{\nearrow} & & \overset{\bar{\Phi}}{\searrow} & \\ \bar{\mathcal{M}} & & & & J \\ & \underset{f}{\searrow} & & \underset{\psi}{\nearrow} & \\ & & F \times F & & \end{array}$$

where $\psi(p, q) = \int_{L_q}^{L_p}$ is the Abel-Jacobi map considered in [4]. f is the natural morphism and it is injective since a pair of disjoint line uniquely determines a hyperplane section. The diagram commutes is due to the fact the topological Abel-Jacobi map (1) coincides Griffiths Abel-Jacobi map, when the vanishing cyles parameterized by T' are all algebraic [11].

Proposition 1. *For Y a cubic threefold, the tube mapping*

$$\Phi'_* : H_1(T'_U, *) \to H_3(Y, \mathbb{Z}). \tag{6}$$

has cofinite image.

Proof. It is shown in [4] that ψ is generically 6-to-1 onto the theta divisor of the intermediate Jacobian J. Moreover, ψ contract the diagonal to $0 \in J$ which turns out to be the isolated triple point singularity of the theta divisor. It follows that $f \circ \psi$ is generically 6-to-1, so $\bar{\Phi}$ is generically one-to-one and therefore a birational morphism from the smooth curve $\overline{T'_U}$ onto its image.

In particular, we conclude that $\bar{\Phi}_*) : H_1(\overline{T'_U}, \mathbb{Z}) \to H_1(J, \mathbb{Z})$ is nonzero, otherwise, the corresponding map on fundamental groups is trivial, so $\bar{\Phi}$ lifts to the universal cover $\mathbb{C}^5$ of J, but any holomorphic map from a proper analytic space to affine space will be constant, which is a contradiction.

Now, since the topological Abel-Jacobi map $\Phi : T'_U \to J$ factors through $\bar{\Phi}$, and the map $\pi_1(T'_U, *) \to \pi_1(\overline{T'_U}, *)$ induced by inclusion $T'_U \hookrightarrow \overline{T'_U}$ is surjective, it follows that the map (6) is nonzero.

The proof will be completed using a lemma below. □

Lemma 1. *Given a smooth variety $X \subset \mathbb{P}^N$ of dimension $2n - 1$. Assume the tube map is nonzero, then the image of the tube map is cofinite.*

Proof. We can choose $W \subset \mathbb{P}^{N+1}$ a smooth variety containing X as a smooth hyperplane section. Choose a general pencil $\mathbb{L}_W$ of hyperplane sections of W passing through $X = W \bigcap H_{v_0}$ and let U_W be the points corresponding to smooth hyperplane section. Then there is an monodromy action

$$\rho : \pi_1(U_W, v_0) \to \mathrm{Aut} H_{2n-1}(X, \mathbb{Q})_{\mathrm{van}}.$$

There is a classical theorem [9] which states that the action ρ is *irreducible.* On the other hand, one can show that the image of tube mapping is invariant under the monodromy action. In particular, $\mathrm{Im}(\Phi'_*) \otimes \mathbb{Q}$ is a ρ-subrepresentation of $H_{2n-1}(X, \mathbb{Q})_{\mathrm{van}}$, so the irreduciblity of ρ together with our assumption implies that $\mathrm{Im}(\Phi'_*) \otimes \mathbb{Q}$ has to be the whole $H_{2n-1}(X, \mathbb{Q})_{\mathrm{van}}$, which implies that $\mathrm{Im}(\Phi'_*) \subseteq H_{2n-1}(X, \mathbb{Z})_{\mathrm{van}}$ is cofinite.

Lastly, let's show that the image of tube mapping is indeed invariant under the monodromy action. Choose a smooth loop $l \subseteq U_W$ based at v_0, then by restricting to a small segment l_i contained in a small open neighborhood U_i of U_W over which the family $\{W \bigcap H_v\}_{v \in U_W}$ is C^∞ trivial, we can fix a uniform Lefschetz pencil for all (2n-2)-folds $W_v = W \bigcap H_v$ for $v \in l_i$ and the family U_{W_v} varies smoothly, so the tube map (4) is locally trivial. It follows that the image of Tube map on U_W is a sub-local system of $H_{2n-1}(W_v, \mathbb{Z})_{\mathrm{van}}$. Finally as we have explained, since the vanishing cycle is conjugate to each other, together with the Zariski's lemma (so it doesn't matter the choice of base point and Lefschetz pencil). So this sub-local system U_W has trivial monodromy. □

2. Degeneration of Dual Varieties

More generally, let X_d be a degree $d \geqslant 2$ smooth hypersurface of $\mathbb{P}^{n+1}$ defined by F_d; X_{d_1}, X_{d_2} be smooth hypersurfaces of degree d_1 and d_2 respectively defined by F_{d_1} and F_{d_2}, with $d = d_1 + d_2$.

We require F_d, F_{d_1}, F_{d_2} to be general, so that their common zero locus is a complete intersection. Besides, by Bertini's theorem, $F_s := sF_d + F_{d_1}F_{d_2}$ is smooth for $s \neq 0$ when $|s|$ is small enough. So for such $s \neq 0$, there is a dual map on smooth hypersurface $X_s := \{F_s = 0\}$

$$\mathcal{D}_s : X_s \mapsto (\mathbb{P}^{n+1})^*$$

$$x \mapsto \big(\frac{\partial F_s}{\partial x_0}(x), ..., \frac{\partial F_s}{\partial x_{n+1}}(x)\big) \tag{7}$$

with $\frac{\partial F_s}{\partial x_j}(x) = s\frac{\partial F_d}{\partial x_j}(x) + F_{d_2}\frac{\partial F_{d_1}}{\partial x_j}(x) + F_{d_1}\frac{\partial F_{d_2}}{\partial x_j}(x)$, $j = 0, ..., n+1$ by direct computation.

The image $(X_s)^*$ is called the dual variety of X_s and it is well know that it is a hypersurface of degree $m = d(d-1)^n$ in the dual space. So this defines a rational section μ on the sheaf $S^m(V^*) \otimes \mathcal{O}_\Delta$ over Δ which has possibly a pole along $s = 0$ where $V = \mathbb{C}^{n+2}$, but by multiplying by a suitable power of s, we can assume the section μ is regular and $\mu(0) \neq 0$. This will not change the defining hypersurface in projective space, so it defines a hypersurface.

Definition. *Define $(X_0)^*$ to be the projective hypersurface $\{\mu = 0\} \subseteq (\mathbb{P}^{n+1})^*$ and call it the dual variety in the limit associated to the family $sF_d + F_{d_1}F_{d_2}$.*

X_0^* is reducible since it contains dual variety of X_{d_1} and dual variety of X_{d_2}. However, since the dual family $\{X_s^*\}_{s\in\Delta}$ is flat, the degree of X_0^* should equal to degree of X_0^*, but a simple count shows that $d(d-1)^n > d_1(d_1-1)^n + d_2(d_2-1)^n$ so there should be more components in X_0^*. In [10], we find other components components explicitly.

Finally we will prove the following:

Lemma 2. *By shrinking Δ to a smaller disk, $\{H_t\}$ is Lefschetz pencil for all X_s with $s \in \Delta$. In other words, we can choose a line $\mathbb{L} \subseteq (\mathbb{P}^{n+1})^*$, such that $\mathbb{L}$ is transverse to all X_s^*. For $s = 0$, this means being transversal to each component of X_0^**

Proof. This argument is based on continuity. First we choose $\mathbb{L}$ to be transverse to X_0^*, then we show that it is transverse to all X_s up to shrinking to a smaller disk.

When $s \neq 0$, the dual variety X_s^* is an irreducible hypersurface of $(\mathbb{P}^{n+1})^*$, defined by a single homogeneous polynomial $\{G_s = 0\}$ varying continuously with respect to the parameter s.

Then the dual variety in the limit $X_0^* := \{z | G_0(z) = 0\}$ is defined by $G_0 := \lim_{s\to 0} G_s$.

By assumption, $\mathbb{L}$ is disjoint from the singularities $\mathrm{Sing}(X_0^*)$, so there is an open neighborhood $\mathcal{U}$ of $\mathrm{Sing}(X_0^*)$ in $(\mathbb{P}^{n+1})^*$ such that $\mathcal{U} \bigcap \mathbb{L} = \varnothing$, so by continuity, we can choose

Δ small enough so that $\mathrm{Sing}(X_s^*) \subseteq \mathcal{U}$ for all $s \in \Delta$. Therefore, $\mathbb{L}$ intersect X_s^* along the smooth locus $(X_s^*)^{sm}$ for each $s \in \Delta$.

Let $\{p_1, ..., p_k\}$ be the set of points of $\mathbb{L} \cap X_0^*$. Let $\sum_{i=0}^{n+1} a_i^j w_i = 0$, $j = 1, ..., n$ be n hyperplanes in $(\mathbb{P}^{n+1})^*$ whose common zero loci is the line $\mathbb{L}$, where $w_0, ..., w_{n+1}$ is the coordinate on the dual space. So by the transversality assumption, the tangent vectors $\big(\partial G_0/\partial w_0(p_i), ..., \partial G_0/\partial w_{n+1}(p_i)\big)$ is not contained in the span of the three hyperplanes. In other words, the matrix $M(s, w)$ is of full rank at $s = 0, w = p_i$, for $i = 1, ..., k$, where $M(s, w)$ is the $(n + 1) \times (n + 2)$ matrix

$$M(s, w) = \begin{bmatrix} a_0^1 & a_1^1 & \cdots & a_{n+1}^1 \\ \cdots & \cdots & \cdots & \cdots \\ a_0^n & a_1^n & \cdots & a_{n+1}^n \\ \frac{\partial G_s}{\partial w_0}(w) & \frac{\partial G_s}{\partial w_1}(w) & \cdots & \frac{\partial G_s}{\partial w_{n+1}}(w) \end{bmatrix}.$$

Again by continuity, $M(s, w)$ will remain to be of maximal rank for $s \in \Delta$ and $w \in \mathcal{U}_i$ for Δ small open neighborhood of 0, and $\mathcal{U}_i$ small open neighborhood of p_i. Finally, we can shrink Δ so that for each $s \in \Delta$, the intersection $\mathbb{L} \bigcap (X_s^*)^{sm}$ is contained in $\bigcup_i \mathcal{U}_i$. Therefore $\mathbb{P}^1$ is a Lefschetz pencil for all $s \in \Delta$. $\square$

3. Deforming of vanishing cycles

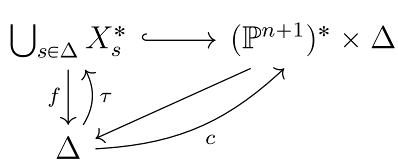
Recall that in the previous section, we associated the family of hypersurfaces of degree d a family of dual varieties

$$f : \bigcup_{s \in \Delta} X_s^* \to \Delta$$

with X_0^* as limit of dual variety nearby, which contains $X_{d_1}^*$ as an irreducible component (where we assume d_1 is the degree at least two factor). Also, we chose a general pencil $\mathbb{L} \subseteq (\mathbb{P}^{n+1})^*$ intersecting transversely to X_s^* for all $s \in \Delta$.

Choose a point $p \in \mathbb{L} \bigcap X_{d_1}^*$, so in particular, p is a smooth point of Y^* and away from other components of X_0^*. By inverse image theorem, up to shrinking to a smaller disk, we can find $\tau(s) \in \mathbb{L}$ varying differentiably with respect to $s \in \Delta$ such that $\tau(s) \in \mathbb{L} \bigcap (X_s^*)^{sm}$ and $\tau(0) = p$.

In other words, τ defines a C^∞ section whose image lies in the smooth part $(X_s^*)^{sm}$ and additionally $\tau(0) \in (X_{d_1}^*)^{sm}$.



This gives a family of hyperplanes $H_{\tau(s)}$ which are tangent to X_s and $H_{\tau(0)}$ is tangent to X_{d_1} and the singularity on the hyperplane section has nondegenerate tangent cone. Choose a constant section c, where $c \in (\mathbb{P}^{n+1})^* \backslash X_0^*$ is close to $\tau(0)$ so that there is a local vanishing cycle $\alpha \in H_{n-1}(X_{d_1} \bigcap H_c, \mathbb{Z})_{van}$ which specializes to the node as H_c specializes to $H_{\tau(0)}$. Now up to shrinking to a smaller disk of Δ containing 0, there is a vanishing cycle $\alpha_s \in H_{n-1}(X_s \bigcap H_c)_{van}$ which specializes to the node as H_c specializes to $H_{\tau(s)}$.

The goal of this section is to prove a following (imprecise) statement:

(8) *The vanishing cycles is on the hyperplane section H_c is a trivial family over Δ.*

Note at the same time the section τ gives a nodal locus via dual correspondence.

We take $\tau(0) \in X_{d_1} \subseteq \mathbb{P}^{n+1}$ which is not on the base locus of Δ-pencil, i.e., not on $X_{d_1} \bigcap X_d$, so if we take a small polydisk D containing $\tau(0)$ so $\tau(s) \in D$ when s is small. Also we require D to stay away from base locus, then D can thought as living in the total space $\mathscr{X}$. We can choose affine coordinate $x_1, ..., x_n, t$ where t corresponds to the pencil $\mathbb{L}$. Recall that $F_s = sF_d + F_{d_1}F_{d_2}$ to be the homogeneous polynomial varying in s, so restriction of F_s to a fixed t is the equation of the hyperplane section $X_s \bigcap H_t$. For each $s \in \Delta$, we denote $\tau(s) = (x_1^s, ..., x_n^s, t^s)$ the nodal locus, i.e., the hyperplane section $X_s \cap H_{t-t_s}$ has an ordinary node at $\tau(s)$. Since $\partial F_s/\partial t(\tau(s)) \neq 0$, the implicit function theorem says that there is a smooth function $f_s(x_1, ..., x_n)$, such that

$$F_s(x_1, ..., x_n, f_s(x_1, ..., x_n)) \equiv 0.$$

Moreover f_s is holomorphic function in $x_1, ..., x_n$ and is analytic with respect to the parameter s. There is a power series expansion

$$f_s(x_1, ..., x_n) = Q_s(x_1 - x_1^s, ..., x_n - x_n^s) + \text{higher powers},$$

where Q_s is a nondegenerate quadric form.

Now by a parametric version of holomorphic Morse lemma, we have

Claim. *There is an analytic change of coordinate $x_1', ..., x_n'$ such that*

$$f_s(x_1', ..., x_n') = x_1'^2 + \cdots + x_n'^2.$$

Moreover, the change of coordinate depends analytically with respect to the parameter s.

This implies the following result, which is a precise statement of (8):

Corollary 2. *There is an analytic isomorphism*

$$D \xrightarrow{\cong} \{x_1'^2 + \cdots + x_n'^2 = t\} \times \Delta$$

preserving projection to Δ.

Before we end the section, we prove a lemma which will be used later.

Lemma 3. *In the fixed t-pencil $\mathbb{L}$ in the Lemma 2, there exists a connected analytic open subset obtained $U \subseteq \mathbb{L}$ by removing finitely many closed disks from $\mathbb{L}$, such that*
(i) c (defined ealier in this section) is contained in U, and
(ii) for all $s \in \Delta^*$, and $t \in U$, $H_t \cap X_s$ is smooth, and
(iii) for each $t \in U$ $H_t \bigcap X_{d_i}$ and $H_t \bigcap X_{d_1} \bigcap X_{d_2}$ is smooth.

Proof. Since the interesection points of the pencil and the dual variety $\mathbb{L} \bigcap X_s^*$ varies continuously, so for each $z_i \in \mathbb{L} \bigcap X_0^*$, there is a small disk $D_{z_i} \subseteq \mathbb{L}$ centered at z_i, such that the intersection of $\bigcup_{s \in \Delta} \mathbb{L} \bigcap X_s^* \subset \bigcup_i D_{z_i}$. □

4. Proof of Theorem 2 for quartic threefold

In this section, we will prove Theorem 2 for quartic by degenerating it into a union of cubic threefold Y and a hyperplane P in $\mathbb{P}^4$, where Y and P intersects transversely. More precisely, let F_X, F_Y and F_P be general homogeneous polynomials of degree 4, 3 and 1 respectively, and the one dimensional of quartic is

$$\mathscr{X} = \{sF_X + F_Y F_P = 0\} \subset \Delta \times \mathbb{P}^4, \tag{9}$$

with $s \in \Delta$ a small disk centered at $0 \in \mathbb{C}$, with special fiber $Y \cup P$ and general fiber a smooth quartic 3-fold. By Bertini's theorem, we can choose the disk small enough, so that $s = 0$ is the only singular fiber.

X_s is used to denote the quartic threefold given by the equation $\{F_s = F_X + F_Y F_P = 0\}$. Also, by Lemma 3, we have an open subset $U \subseteq \mathbb{L}$ such that it contains a base point c where the vanishing cycle on the hyperplane section $X_s \bigcap H_c$ deforms trivially as s varies in Δ. Moreover, $H_t \bigcap X_s$ is smooth for all $t \in U$ and $s \in \Delta*$ and all $H_t \bigcap Y$ and $H_t \bigcap P \bigcap Y$ are smooth.

According to Clemens's Degeneration on Kahler Manifolds [2], there is a deformation retract of X_s onto $Y \bigcup P$ which induces diffeomorphism of $Y \backslash (Y \bigcap P)$ into a smooth submanifold X_s' of X_s (and $P \backslash (Y \bigcap P)$ into a smooth submanifold X_s'' of X_s and disjoint from X_s'). So in order to guarantee that the 3-cycles in image of the tube mapping T can be deform to nearby quartic, we have to make sure both the vanishing cycle α and those 3-cycles transported along loops γ which fix α are all supported in $Y \backslash (Y \bigcap P)$.

Now, by Lemma (1), the proof of theorem 2 reduces to prove

Proposition 2. *The tube map of quartic threefold X is nonzero.*

4.1. Terminology.

In this section, we will introduce some terminology on the tube mapping on an open submanifold of a (possibly singular) variety.

Let M be a n-dimensional smooth subvariety of $\mathbb{P}^N$, and let $\mathbb{L} = \mathbb{P}^1$ be a pencil of hyperplanes in $\mathbb{P}^N$ in general position. Denote $U \subseteq \mathbb{L}$ be the set of points corresponding to the hyperplanes which are not tangent to M. Consider the incidence variety

$$\tilde{M} := \{(x,t) \in M \times U | x \in M \bigcap H_t\},$$

together with projections

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\sigma} & M \\ \downarrow{\scriptstyle \pi} & & \\ U & & \end{array}$$

π is proper submersion, so is locally trivial thanks to the Ehresmann's theorem.

Let $t_0 \in U$ be a fixed base point and $\alpha \in H_{n-1}(M \bigcap H_{t_0}, \mathbb{Z})_{van}$ a vanishing cohomology on the hyperplane section and $\gamma : [0,1] \to U$ with $\gamma(0) = \gamma(1) = t_0$ be a loop based at t_0 satisfying $[\gamma] \cdot \alpha = \alpha$ in $H_{n-1}(M \bigcap H_{t_0}, \mathbb{Z})$ as monodromy action. So the trace of α along γ defines is an integral n-cycle $A_{\alpha,\gamma}$ on $\tilde{M}$

Definition. *Call $A_{\alpha,\gamma} \in H_n(\tilde{M}, \mathbb{Z})$ the tube mapping associated to the pair (α, γ).*

So $\sigma(A_{\alpha,\gamma})$ defines a primitive class in $H_n(M, \mathbb{Z})_{prim}$ which is the Tube mapping in the sense of [8].

More generally, we can consider tube mapping supported in an open submanifold if Ehresmann's theorem still holds. More precisely, let $M^\circ \subseteq M$ be an open submanifold and $\tilde{M}^\circ$ the set of pairs (x,t) with $x \in M^\circ \bigcap H_t$. Assume that the restriction of π to $\pi|_U : \tilde{M}^\circ \to U$ is $\mathbb{C}^\infty$ locally trivial, i.e., for each $t \in U$, there is an open subset $V \subseteq U$ and a fiber preserving diffeomorphism

$$\begin{array}{ccc} \pi^{-1}(V) \bigcap \tilde{M}^\circ & \xrightarrow{\cong} & (\pi^{-1}(t) \bigcap \tilde{M}^\circ) \times V \\ \downarrow{\scriptstyle \pi} & \swarrow & \\ V & & \end{array}$$

Assume additionally that the monodromy action identity $[\gamma] \cdot \alpha = \alpha$ holds in $H_{n-1}(M^\circ \bigcap H_{t_0}, \mathbb{Z})$. Then the trace of α along γ is a n-cycle on $\tilde{M}^\circ$.

Definition. *We call $A_{\alpha,\gamma}$ the tube mapping associated to the pair (α, γ) supported in M°.*

In what follows, we will show for cubic 3-fold and hypersurface 3-fold of higher degree, such open submanifold M° exist (and in fact diffeomorphic to each other) which support certain amount of 3-cycles arising from such pairs (α, γ).

4.2. **Proof of Proposition 2.** The proof breaks up into several steps.

Step 1. Vanishing cycle on affine complement. Let $Y_t = Y \bigcap H_t$ for $t \in \mathbb{L}$, and $V_t = Y_t \backslash (Y \bigcap P)$ the affine complement. Denote $U_Y \subseteq \mathbb{L}$ the set of points where hyperplane sections on Y is smooth. We first claim that

Lemma 4. *For any $t \in U_Y$, the vanishing homology on cubic surface $H_2(Y_t, \mathbb{Z})_{van}$ is isomorphic to the image of $H_2(V_t, \mathbb{Z}) \to H_2(Y_t, \mathbb{Z})$ induced by inclusion $V_t \hookrightarrow Y_t$.*

Proof. This is a special case of Prop. 7.3 of [6]. Write $P_t = H_t \bigcap P$ the projective 2-plane. By definition, the vanishing homology $H_2(Y_t, \mathbb{Z})_{van}$ is the kernel of $H_2(Y_t, \mathbb{Z}) \to H_2(H_2, \mathbb{Z}) = \mathbb{Z}$ induced by inclusion, which is identified with kernel of

$$H_2(Y_t, \mathbb{Z}) \to H_0(Y_t \bigcap P_t, \mathbb{Z}),\ \alpha \mapsto \alpha \cap P_t \tag{10}$$

by the intersection pairing on H_t. Now (10) fits into the exact sequence

$$H_2(V_t, \mathbb{Z}) \to H_2(Y_t, \mathbb{Z}) \to H_0(Y_t \bigcap P_t, \mathbb{Z}),$$

where the last map factors through Thom isomorphism

$$H_2(Y_t, \mathbb{Z}) \to H_2(Y_t, V_t, \mathbb{Z}) \cong H_0(Y_t \bigcap P_t, \mathbb{Z}).$$

So by exactness, the lemma is proved. □

It follows that one can represent a vanishing cycle α by a cycle supported in the affine complement V_t, and therefore any open subspace of V_t which is deformation equivalent to V_t.

Step 2. 3-cycles away from hyperplane. Let $\mathcal{U}(P)$ be a tubular open neighborhood of $Y \bigcap P$ in Y and for $t \in U$ denote the $Y'_t := Y_t \backslash \mathcal{U}(P)$ the submanifold with boundary. The following is a consequence of a theorem which will be stated in the Appendix.

Lemma 5. *The family $\{Y'_t\}_{t \in U}$ is C^∞-locally trivial. Namely, for each $t \in U'_Y$, there is a neighborhood $\mathcal{V}$ of t such that there is a fiber preserving diffeomorphism $\pi^{-1}(\mathcal{V}) \cong V_t \times \mathcal{V}$ preserves inclusion into $Y_t \times \mathcal{V}$.*

This lemma tells us that it makes sense to talk about monodromy of homology on Y'_t over the base U. We are going to show that the monodromy of vanishing cycle on open part Recall $U_Y \subseteq \mathbb{L}$ is the set of points where hyperplane sections on Y is smooth, and $U \subseteq \mathbb{L}$ is obtained by finitely many small disks centered at $\mathbb{L} \bigcap X_0^*$, so in particular $U \subseteq U_Y$. Choose a base point $t_0 \in U$ (in particular, we choose $t_0 = c$). Our main proposition in this section will be

Proposition 3. *There are finitely many loops $l_1, ..., l_n \in U$ based at t_0 which generate the fundamental group $\pi_1(U_Y, t_0)$. Moreover, for any vanishing cycle $\alpha \in H_2(Y_{t_0}, \mathbb{Z})_{van}$ supported in Y'_{t_0}, the trace of α transported along any (composite of) $l_i, i = 1, ..., n$ is a 3-cycle in $Y \backslash \mathcal{U}(P)$.*

Proof. Denote $p_1, ..., p_n \in \mathbb{L}\backslash U_Y$ be the points corresponding Y'_{p_i} being homotopic to complement of a smooth cubic curve in a singular cubic surface, and $q_1, ..., q_m \in \mathbb{L}$ be the points corresponding to Y'_{q_j} homotopic to complement of a singular cubic curve in a smooth cubic surface. Now, let the loop l_i based at 0 be defined as straight line towards p_i, go around anticlockwise, and go back and stay in U. Then loops $l_1, ..., l_n$ is a generating set of 3rd primitive homology on the 3-fold under tube map. Moreover the closed region bounded by any composite of these loops does not contain the point q_j, so does not deposit monodromy on 1st homology of the cubic curve. It follows from the exact sequence

$$H_1(C_{t_0}, \mathbb{Z}) \to H_2(Y'_{t_0}, \mathbb{Z}) \to H_2(Y_{t_0}, \mathbb{Z}) \to 0,$$

that the monodromy on the open Y'_{t_0} coincides with the monodromy of the compact cubic surface. In other words, we have a commutative diagram □

$$\begin{array}{ccc} H_2(Y_{t_0}, \mathbb{Z}) & \xrightarrow{\gamma*} & H_2(Y_{t_0}, \mathbb{Z}) \\ \uparrow & & \uparrow \\ H_2(Y'_{t_0}, \mathbb{Z}) & \xrightarrow{\gamma*} & H_2(Y'_{t_0}, \mathbb{Z}) \end{array}$$

Step 3. Deformation of 3-cycle to nearby quartic. Based on two steps discussed above, we have

• an analytic open subset $U \subseteq \mathbb{L}$ such that all $t \in U$ corresponds to hyperplane H_t intersecting transversely with X_s for all $s \in \Delta$ (when $s = 0$, this implies H_t is transverse to both Y and P, moreover $H_t \bigcap P$ is plane transverse to the cubic surface $Y \bigcap H_t$);

• a base point $t_0 \in U$ ($t_0 = c$) and a local vanishing cycle $\alpha \in H_2(Y \bigcap H_{t_0}, \mathbb{Z})_{van}$ supported in the open part Y'_{t_0}, and a continous family of local vanishing cycles $\alpha_s \in H_2(X_s \bigcap H_{t_0}, \mathbb{Z})_{van}$;

• a loop $\gamma \subseteq U$ based at t_0 such that the monodromy action $[\gamma] \cdot \alpha = \alpha$ in $H_2(Y'_{t_0}, \mathbb{Z})$. So the associated tube mapping class $A_{\alpha,\gamma}$ is supported in $Y\backslash\mathcal{U}(P)$, whose image is nonzero in $H_3(Y, \mathbb{Z})$.

Our goal is to produce a 3-cycle A_s in the nearby fiber X_s which is obtained by the tube mapping of the pair (α_s, γ), and A_s specializes to $A_0 := A_{\alpha,\gamma}$. In this section, we will deal with construction of A_s. In the next section, we will construct the family of vanishing cycles α_s on the quartic $X_s \bigcap H_{t_0}$.

Consider the total space of the family of quartic threefolds over a small disk.

$$\mathscr{X} = \{(x, s) \in \mathbb{P}^4 \times \Delta | (sF_X + F_Y F_H)(x) = 0\} \to \Delta, \ (x, s) \mapsto s.$$

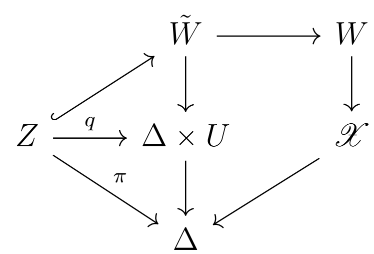
The total space is singular along $\{s = 0, F_X = 0, F_Y = 0, F_H = 0\}$, which is a smooth curve E of genus 19 in $Y \bigcap P$.

For the reason of Hodge theory, we want a smooth total space carrying the degeneration, and the special fiber should be normal crossing divisor, so the information of weight filtration will be related to the geometry of special fiber, so we need to resolve the total space.

The singularity is a nondegenerate node along a transversal hyperplane, so we can produce a small resolution on the total space by blowing up a threefold in W which contains E. A good thing for small resolution is that the special fiber will be normal crossing with *two* components (comparatively, blowup along E will produces an extra component), so the weight filtration of the limiting mixed Hodge structure is easy to describe. Since our proof of Proposition 2 relies heavily on the knowledge of the weight filtration, we prefer small resolution in this situation.

To produce such a small resolution, for example, we can blowup P in $\mathscr{X}$. As P being a divisor, the blowup does not change the $\mathscr{X}$ outside the singular locus E. In fact it has an effect of replacing E by a $\mathbb{P}^1$-bundle over E. We denote the new total space as W, with projection $W \to \Delta$. So fiber over Δ^* stay the same as smooth quartic threefold X_s, while the central fiber is isomorphic to $Y \bigcup \tilde{P}$, where $\tilde{P}$ is the blowup of P along the curve E. So we write $W = \bigcup_{s\in\Delta} \tilde{X}_s$.

Now consider the $\tilde{W} = \{\big((x,s),t\big) \in W \times U | x \in \tilde{X}_s \bigcap H_t\}$ which blows up the base locus on the pencil $\mathbb{P}^1$. So there is a commutative diagram



By taking an open neighborhood $\mathcal{U}$ of $Y \bigcap \tilde{P}$ away from the total space $\tilde{W}$, the map $\tilde{W}\backslash\mathcal{U} \to \Delta \times U$ is a submersion. By composing with projection $\Delta \times U \to \Delta$ and up to shrinking to a smaller disk of Δ containing origin, the fiber of $\tilde{W}\backslash\mathcal{U} \to \Delta$ has two disjoint components $\tilde{W}_s'$ and $\tilde{W}_s''$. On the special fiber, $\tilde{W}_0'$ (resp. $\tilde{W}_0''$) is blowup along base locus of open submanifold of Y (resp. $\tilde{P}$) away from $Y \bigcap \tilde{P}$. We denote Z the union $\bigcup_{s\in\Delta} \tilde{W}_s'$ and denote q the projection of Z to $\Delta \times U$ and π to Δ, respectively.

Claim. *Both q and π are C^∞-locally trivial.*

Proof. By considering the closure $\bar{Z}$ of Z inside $\tilde{W}$ and extend the two projections to $\bar{Z} \bigcap \pi^{-1}(\Delta)$ are both proper and submersive along the boundary and interior, so it satisfies the assumption of Theorem 4 in the appendix, so both q and π are C^∞-locally trivial. □

Proposition 4. *Up to shrinking to a smaller disk of Δ, there is a fiber preserving diffeomorphism*

$$\psi : Z \xrightarrow{\cong} Z_0 \times \Delta$$

such that $\pi_{Z_0} \circ \psi(Z_s \bigcap H_t) \subseteq Z_0 \bigcap H_t$ for all $t \in U$. In other words, the ψ is a trivialization which preserves hyperplane sections.

Proof. By shrinking U to a smaller open neighborhood U' whose closure is countained in U, we can find finitely many open covering $U_1, ..., U_N$ of U' and a smaller open disk $\Delta' \subseteq \Delta$ containing origin such that the q map over $\Delta' \times U_i$ is a trivial family. In other words, there is a diagram

$$\begin{array}{ccc} Z|_{U_i} & \xrightarrow[\cong]{\psi_i} & (\tilde{W}_0' \bigcap H_{t_i}) \times \Delta' \times U_i \\ & \searrow^{q} & \downarrow \\ & & \Delta' \times U_i \\ & \searrow^{\pi} & \downarrow \\ & & \Delta' \end{array}$$

where $t_i \in U_i$ and $q^{-1}(U_t) = Z|_{U_t}$.

Denote ψ_i the trivialization on $Z|_{U_i}$. Note that it preserves hyperplane sections. To construct a trivialization globally on Z, we need to use *partition of unity*. To be more precise, let (x_1, x_2) be a real coordinate on Δ' and $\partial/\partial x_j$, $j = 1, 2$ a constant real vector field on Δ'. Pullback to the product $(\tilde{W}_0' \bigcap H_t) \times \Delta' \times U_i$ and then pushforward via ψ_i^{-1}. Now we get a vector field v_j^i on $Z|_{U_i}$ whose horizontal part is $\partial/\partial x_j$. Now choose a partition of unity of Z with respect to the open covering $Z|_{U_i}$, we get smooth functions f_i supported in $\{Z|_{U_i}\}$ such that $\sum f_i \equiv 1$. It follows that $v_j := \sum_i f_i v_j^i$ defines a vector field on Z globally with constant horizontal part $\partial/\partial x_j$. Let ϕ_v denote the one parameter group of diffeomorphism generated by a vector field v. This induces a desired fiber preserving diffeomorphism

$$\psi : Z_0 \times \Delta' \cong Z$$

$$(z, ax_1 + bx_2) \mapsto \phi_{av_1+bv_2}(z)$$

□

Proposition above allows us to define a family of 3 cycles A_λ on $Z_s \subseteq \tilde{W}_s$ via the following way: Denote $\psi_s = \psi(\cdot, s)$ for $s \in \Delta$ the diffeomorphism. Let α be a vanishing cycle supported on $Z_0 \bigcap H_{t_0}$ and $\gamma(0) = \gamma(1) = t_0$ be a loop on U satisfying $[\gamma] \cdot \alpha = \alpha$ and $A_0 = A$ the 3-cycle as tube mapping associated to the pair (α, γ) supported on the open submanifold $Y \backslash \mathcal{U}(P)$. Define $A_s := \psi_s(A)$ the 3-cycle on $Z_s \subseteq X_s$.

Corollary 3. *The 3-cycles A_s on $Z_s \subseteq X_s$ are Tube mapping associated to a pair (α, γ).*

Now we are ready to finish the proof of Proposition 2.

Proof of Proposition 2. As we have shown above, there is a 3-cycle A_s in the quartic 3-fold X_s for $s \in \Delta$ as tube mapping of a pair (α_s, γ), where $\alpha_s \in H_2(X_s \bigcap H_{t_0}, \mathbb{Z})_{van}$ and $\gamma \subseteq U$ is a loop based at t_0 which fixes α via monodromy action. So it suffices to show that A_s is not a zero class in $H_3(X_s, \mathbb{Z})$ for some $s \neq 0$.

Recall in the beginning of Step 3, we produce a small resolution on the total space of the quartic family (9) and get a family

$$h : W \to \Delta \tag{11}$$

with W smooth and general fiber being X_s and special fiber being $Y \bigcup \tilde{P}$.

This is a semistable degeneration and there is an associated limiting mixed Hodge structure H^3_{lim} with W_3 part contributed by the image of $H^3(Y \bigcup \tilde{P})$.

Another way to describe the $W_3 H^3_{\text{lim}} = H^3(Y \bigcup \tilde{P})$ is by considering the invariant sections on a local system: Denote $h' : W^* \to \Delta^*$ the restriction of (11). Then the invariant sections on $j_* R^3 h'_* \underline{\mathbb{Z}}$ are identified with points on $i^* j_* R^3 h'_* \underline{\mathbb{Z}}$, which are precisely $W_3 H^3_{\text{lim}}$.

On the other hand, $H^3(Y \bigcup \tilde{P})$ fits into an exact sequence

$$0 \to H^2(Y \bigcap \tilde{P}, \mathbb{Z})_{van} \to H^3(Y \bigcup \tilde{P}, \mathbb{Z}) \to H^3(Y, \mathbb{Z})_{prim} \oplus H^3(\tilde{P}, \mathbb{Z})_{prim} \to 0.$$

Since by our construction, the 3-cycle A_s specializes to A_0 contained in $Y \backslash \mathcal{U}(P)$ via a family of 3-cycles A_s defined in the Corollary 3 above, so A_0 is a primitive cohomololgy class on Y and nonzero, in particular $A_0 \in W_3 H^3_{\text{lim}} = i^* j_* R^3 h'_* \underline{\mathbb{Z}}$ according to the exact sequence above, so the 3-cycles A_s defines a section in

$$\eta : \Delta \to j_* R^3 h'_* \underline{\mathbb{Z}}$$

with $\eta(0) \neq 0$. It follows that $\eta(s)$ is not zero for $s \neq 0$ close enough to 0. In particular, for such s, A_s is not a zero class in $H_3(X_s, \mathbb{Z})$. □

5. Proof of Theorem 2

Let X_d be a smooth hypersurface of degree $d \geqslant 4$ in $\mathbb{P}^4$. Then we claim:

Theorem 3. *The tube mapping of vanishing cycles on X_d*

$$\Phi_* : H_1(T'_{\mathbb{Z}}, \mathbb{Z}) \to H_3(X_d, \mathbb{Z})$$

is of full rank.

Proof. By degenerating X_d to X_{d_1} and a smooth hyperplane P intersecting transversely. The initial step is true for $d = 3$ as proved in an earlier section. The induction step is to repeat same proof as for quartic. □

Appendix

Theorem 4. *(Ehresmann's theorem for manifolds with boundary) Let M be a smooth manifold possibly with boundary, and B is a smooth manifold such that $\partial B = \varnothing$. Let $\pi : M \to B$ be a proper smooth map such that the restriction to the interior $\pi|_{M^\circ}$ and restriction to the boundary $\pi|_{\partial M}$ are submersive, then M is locally trivial over B, that is, for each $b \in B$, there exists an open neighborhood U of b such that there is a diffeomorphism*

$$\Psi : \pi^{-1}(U) \cong M_0 \times U$$

such that $\pi = p_2 \circ \Psi$, where $M_0 = \pi^{-1}(b)$ and p_2 is the projection to the second factor.

References

[1] Arnaud Beauville. *Les singularités du diviseur de la jacobienne intermédiaire de 4 l'hypersurface cubique dans P* . In Algebraic threefolds (Varenna, 1981), volume 947 of Lecture Notes in Math., pages 190–208. Springer, Berlin-New York, 1982.

[2] Clemens, Herbert. *Degeneration of Kähler manifolds.* Duke Math. J. 44 (1977), no. 2, 215–290.

[3] Clemens, Herbert *The Néron model for families of intermediate Jacobians acquiring "algebraic" singularities.* Inst. Hautes Études Sci. Publ. Math. No. 58 (1983), 5–18.

[4] Clemens, Herbert; Griffiths, Phillip A. *The intermediate Jacobian of the cubic threefold.* Ann. of Math. (2) 95 (1972), 281–356.

[5] Cheng, Yiran. Hyperplane Sections of Hypersurfaces. (arxiv:2001.10983)

[6] Griffiths, Phillip A. *On the periods of certain rational integrals. I.* Ann. of Math. (2) 90 (1969), 460-495.

[7] Morrison, David R. *The Clemens-Schmid exact sequence and applications.* Topics in transcendental algebraic geometry (Princeton, N.J., 1981/1982), 101–119, Ann. of Math. Stud., 106, Princeton Univ. Press, Princeton, NJ, 1984.

[8] Schnell, Christian *Primitive cohomology and the tube mapping.* Math. Z. 268 (2011), no. 3-4, 1069–1089.

[9] Voisin, Claire Hodge theory and complex algebraic geometry. II. Translated from the French by Leila Schneps. Cambridge Studies in Advanced Mathematics, 77. Cambridge University Press, Cambridge, 2003.

[10] Zhang, Yilong, Degeneration of dual varieties, in preparation

[11] Zhao, Xiaolei. *Topological Abel-Jacobi Mapping and Jacobi Inversion.* Thesis (Ph.D.)-The University of Michigan. 2015.